# HL IB Maths 2025 syllabus Revision Notes





## Number & Algebra

Number Toolkit
Exponents & Logs
Sequences & Series
Proof and Reasoning
Binomial Theorem

## Functions

Linear Functions and Graphs
Quadratic Functions and Graphs
Functions Toolkit
Further functions
Transformations

## Geometry and Trig

Geometry Toolkit
3D Shapes
Trigonometry
Further Trigonometry
Trig functions and graphs
Trig equations and Identities

## Statistics and Probability

Statistics Toolkit
Correlation
Probability and Distributions

## Calculus

Differentiation
Integration
Optimisation
Kinematics

www.thrivingscholars.com

# IB Maths DP

## 1. Number & Algebra

CONTENTS

1.1 Number & Algebra Toolkit
- 1.1.1 Standard Form
- 1.1.2 Laws of Indices
- 1.1.3 Partial Fractions

1.2 Exponentials & Logs
- 1.2.1 Introduction to Logarithms
- 1.2.2 Laws of Logarithms
- 1.2.3 Solving Exponential Equations

1.3 Sequences & Series
- 1.3.1 Language of Sequences & Series
- 1.3.2 Arithmetic Sequences & Series
- 1.3.3 Geometric Sequences & Series
- 1.3.4 Applications of Sequences & Series
- 1.3.5 Compound Interest & Depreciation

1.4 Simple Proof & Reasoning
- 1.4.1 Proof

1.5 Further Proof & Reasoning
- 1.5.1 Proof by Induction
- 1.5.2 Proof by Contradiction

1.6 Binomial Theorem
- 1.6.1 Binomial Theorem
- 1.6.2 Extension of The Binomial Theorem

1.7 Permutations & Combinations
- 1.7.1 Counting Principles
- 1.7.2 Permutations & Combinations

1.8 Complex Numbers
- 1.8.1 Intro to Complex Numbers
- 1.8.2 Modulus & Argument
- 1.8.3 Introduction to Argand Diagrams

1.9 Further Complex Numbers
- 1.9.1 Geometry of Complex Numbers
- 1.9.2 Forms of Complex Numbers

1.9.3 Complex Roots of Polynomials
1.9.4 De Moivre's Theorem
1.9.5 Roots of Complex Numbers
1.10 Systems of Linear Equations
1.10.1 Systems of Linear Equations
1.10.2 Algebraic Solutions

www.thrivingscholars.com

www.thrivingscholars.com





# Standard Form

**Standard form** (sometimes called **scientific notation** or **standard index form**) gives us a way of writing very big and very small numbers using powers of 10.

## Why use standard form?

- Some numbers are too big or too small to write easily or for your calculator to display at all
  - Imagine the number $50^{50}$, the answer would take 84 digits to write out
  - Try typing $50^{50}$ into your calculator, you will see it displayed in **standard form**
- Writing very big or very small numbers in standard form allows us to:
  - Write them more neatly
  - Compare them more easily
  - Carry out calculations more easily
- Exam questions could ask for your answer to be written in standard form

## How is standard form written?

- In standard form numbers are always written in the form $a \times 10^k$ where $a$ and $k$ satisfy the following conditions:
  - $1 \leq a < 10$
    - So there is one non – zero digit before the decimal point
  - $k \in \mathbb{Z}$
    - So $k$ must be an integer
  - $k > 0$ for large numbers
    - How many times $a$ is multiplied by 10
  - $k < 0$ for small numbers
    - How many times $a$ is divided by 10

## How are calculations carried out with standard form?

- Your GDC will display large and small numbers in standard form when it is in normal mode
  - Your GDC may display standard form as *a*E*n*
    - For example, $2.1 \times 10^{-5}$ will be displayed as $2.1\text{E}-5$
    - If so, be careful to **rewrite the answer given in the correct form**, you will not get marks for copying directly from your GDC
- Your GDC will be able to carry out calculations in standard form
  - If you put your GDC into scientific mode it will automatically convert numbers into standard form
    - Beware that your GDC may have more than one mode when in scientific mode
    - This relates to the number of significant figures the answer will be displayed in
    - Your GDC may add extra zeros to fill spaces if working with a high number of significant figures, you do not need to write these in your answer

www.thrivingscholars.com



- To add or subtract numbers written in the form $a \times 10^k$ without your GDC you will need to write them in full form first
- To multiply or divide numbers written in the form $a \times 10^k$ without your GDC you can either write them in full form first or use the laws of indices

### Exam Tip

- Your GDC will give very big or very small answers in standard form and will have a setting which will allow you to carry out calculations in scientific notation
- Make sure you are familiar with the form that your GDC gives answers in as it may be different to the form you are required to use in the exam

www.thrivingscholars.com

## Worked Example

Calculate the following, giving your answer in the form $a \times 10^k$, where $1 \leq a < 10$ and $k \in \mathbb{Z}$.

i)

$3780 \times 200$

Using GDC: Choose scientific mode.

Input directly into GDC as ordinary numbers.

$3780 \times 200 = 7.56 \times 10^5$

GDC will automatically give answer in standard form.

Without GDC:

Calculate the value:

$3780 \times 200 = 756000$

Convert to standard form:

$756000 = 7.56 \times 10^5$

$\boxed{7.56 \times 10^5}$

ii) $(7 \times 10^5) - (5 \times 10^4)$

www.thrivingscholars.com

Using GDC: Choose scientific mode.

Input directly into GDC

$7 \times 10^5 - 5 \times 10^4 = 6.5 \times 10^5$

This may be displayed as 6.5E5

Without GDC:

Convert to ordinary numbers:

$7 \times 10^5 = 700000$

$5 \times 10^4 = 50000$

Carry out the calculation:

$700000 - 50000 = 650000$

Convert to standard form:

$650000 = 6.5 \times 10^5$

$\boxed{6.5 \times 10^5}$

iii)

$(3.6 \times 10^{-3})(1.1 \times 10^{-5})$

www.thrivingscholars.com



Input directly into GDC:
(Choose scientific mode).

$(3.6 \times 10^{-3})(1.1 \times 10^{-5}) = 3.96 \times 10^{-8}$

$$\boxed{3.96 \times 10^{-8}}$$

Note:

$10^{-3} \times 10^{-5} = 10^{-8}$

$(3.6 \times 10^{-3})(1.1 \times 10^{-5}) = 3.96 \times 10^{-8}$

$3.6 \times 1.1 = 3.96$

www.thrivingscholars.com



# Laws of Indices

## What are the laws of indices?

- Laws of indices (or index laws) allow you to simplify and manipulate expressions involving exponents
  - An exponent is a power that a number (called the base) is raised to
  - Laws of indices can be used when the numbers are written with the same base
- The index laws you need to know are:
  - $(xy)^m = x^m y^m$
  - $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$
  - $x^m \times x^n = x^{m+n}$
  - $x^m \div x^n = x^{m-n}$
  - $(x^m)^n = x^{mn}$
  - $x^1 = x$
  - $x^0 = 1$
  - $\frac{1}{x^m} = x^{-m}$
  - $x^{\frac{1}{n}} = \sqrt[n]{x}$
  - $x^{\frac{m}{n}} = \sqrt[n]{x^m}$
- These laws are **not in the formula booklet** so you must remember them

## How are laws of indices used?

- You will need to be able to carry out multiple calculations with the laws of indices
  - Take your time and apply each law individually
  - Work with numbers first and then with algebra
- Index laws only work with terms that have the same base, make sure you **change the base** of the term before using any of the index laws
  - Changing the base means rewriting the number as an exponent with the base you need
  - For example, $9^4 = (3^2)^4 = 3^{2\times 4} = 3^8$
  - Using the above can them help with problems like $9^4 \div 3^7 = 3^8 \div 3^7 = 3^1 = 3$

### Exam Tip

- Index laws are rarely a question on their own in the exam but are often needed to help you solve other problems, especially when working with logarithms or polynomials
- Look out for times when the laws of indices can be applied to help you solve a problem algebraically

www.thrivingscholars.com

## Worked Example

Simplify the following equations:

i)

$$\frac{(3x^2)(2x^3y^2)}{(6x^2y)}.$$

Apply each law separately:

$3 \times 2 = 6$

$$\frac{(3x^2)(2x^3y^2)}{6x^2y}$$

expand numerator

$x^2 \times x^3 = x^5$

$$\frac{(6x^2)(x^3y^2)}{6x^2y}$$

cancelling

$$\frac{6x^5y^2}{6x^2y}$$

$x^5 \div x^2 = x^{5-2} = x^3$

$y^2 \div y = y^{2-1} = y$

$$x^3y$$

$$\frac{(3x^2)(2x^3y^2)}{6x^2y} = x^3y$$

ii)

$$(4x^2y^{-4})^3(2x^3y^{-1})^{-2}.$$

www.thrivingscholars.com

$$(4x^2y^{-4})^3(2x^3y^{-1})^{-2}$$

Rewrite as a fraction

$$\frac{(4x^2y^{-4})^3}{(2x^3y^{-1})^2}$$

expand numerator and denominator

$$\frac{64\,x^6y^{-12}}{4\,x^6y^{-2}}$$

cancelling

$$\frac{\cancel{64}\,\cancel{x^6}\,y^2}{\cancel{4}\,\cancel{x^6}\,y^{12}}$$

The negative exponents can be rewritten as their reciprocals

$$16y^{-10}$$

$$\frac{16}{y^{10}}$$

www.thrivingscholars.com



# Partial Fractions

## What are partial fractions?

- Partial fractions allow us to simplify rational expressions into the sum of two or more fractions with constant numerators and linear denominators
  - This allows for integration of rational functions
- The method of partial fractions is essentially the reverse of adding or subtracting fractions
  - When adding fractions, a common denominator is required
  - In partial fractions the common denominator is split into parts (factors)
- If we have a rational function with a quadratic on the denominator partial fractions can be used to rewrite it as the sum of two rational functions with linear denominators
  - This works if the non-linear denominator can be **factorised** into two distinct factors
    - For example: $\frac{ax+b}{(cx+d)(ex+f)} = \frac{A}{cx+d} + \frac{B}{ex+f}$
- If we have a rational function with a linear numerator and denominator partial fractions can be used to rewrite it as the sum of a constant and a fraction with a linear denominator
  - The linear denominator does not need to be factorised
  - For example: $\frac{ax+b}{cx+d} = A + \frac{B}{cx+d}$

## How do I find partial fractions if the denominator is a quadratic?

- STEP 1
  **Factorise** the denominator into the product of two linear factors
  - Check the numerator and cancel out any common factors
    - e.g. $\frac{5x+5}{x^2+x-6} = \frac{5x+5}{(x+3)(x-2)}$
- STEP 2
  **Split** the fraction into a **sum** of two fractions with single **linear denominators** each having unknown **constant numerators**
  - Use A and B to represent the unknown numerators
    - e.g. $\frac{5x+5}{(x+3)(x-2)} \equiv \frac{A}{x+3} + \frac{B}{x-2}$
- STEP 3
  Multiply through by the denominator to eliminate fractions
  - Eliminate fractions by cancelling all common expressions
    - e.g. $5x+5 \equiv A(x-2) + B(x+3)$
- STEP 4
  Substitute values into the identity and solve for the unknown constants
  - Use the root of each **linear factor** as a value of to find the unknowns
    - e.g. Let $x = 2$: $5(2)+5 \equiv A((2)-2) + B((2)+3)$ etc
  - An alternative method is **comparing coefficients**
    - e.g. $5x+5 \equiv (A+B)x + (-2A+3B)$
- STEP 5
  Write the **original** as partial fractions

www.thrivingscholars.com



- Substitute the values you found for *A* and *B* into your expression from STEP 2
  - e.g. $\frac{5x + 5}{x^2 + x - 6} = \frac{2}{x + 3} + \frac{3}{x - 2}$

## How do I find partial fractions if the numerator and denominator are both linear?

- If the denominator is not a quadratic expression you will be given the form in which the partial fractions should be expressed
- For example express $\frac{12x - 2}{3x - 1}$ in the form $A + \frac{B}{3x - 1}$
- STEP 1
  Multiply through by the denominator to eliminate fractions
  - e.g. $12x - 2 \equiv A(3x - 1) + B$
- STEP 2
  Expand the expression on the right-hand side and **compare coefficients**
  - Compare the coefficients of *x* and solve for the first unknown
    - e.g. $12x = 3Ax$
    - therefore $A = 4$
  - Compare the constant coefficients and solve for the second unknown
    - e.g. $-2 = -A + B = -4 + B$
    - therefore $B = 2$
- STEP 3
  Write the **original** as partial fractions
  - $\frac{12x - 2}{3x - 1} = 4 + \frac{2}{3x - 1}$

## How do I find partial fractions if the denominator has a squared linear term?

- A **squared linear factor** in the denominator actually represents two factors rather than one
- This must be taken into account when the rational function is split into partial fractions
  - For the squared linear denominator $(ax + b)^2$ there will be two factors: $(ax + b)$ and $(ax + b)^2$
  - So the rational expression $\frac{p}{(ax + b)^2}$ becomes $\frac{A}{ax + b} + \frac{B}{(ax + b)^2}$
- In IB you will be given the form into which you should split the partial fractions
  - Put the rational expression equal to the given form and then continue with the steps above
- There is more than one way of finding the missing values when working with partial fractions
  - Substituting values is usually quickest, however you should look at the number of times a bracket is repeated to help you decide which method to use

www.thrivingscholars.com

### Exam Tip

- An exam question will often have partial fractions as part (a) and then integration or using the binomial theorem as part (b)
  - Make sure you use your partial fractions found in part (a) to answer the next part of the question

www.thrivingscholars.com

## Worked Example

a)

Express $\frac{2x - 13}{x^2 - x - 2}$ in partial fractions.

$\frac{2x - 13}{x^2 - x - 2} = \frac{2x - 13}{(x + 1)(x - 2)}$

↖ The denominator is a quadratic so factorise first.

$\frac{2x - 13}{(x + 1)(x - 2)} \equiv \frac{A}{x + 1} + \frac{B}{x - 2}$

Multiply through by the denominator to eliminate fractions

$2x - 13 \equiv A(x - 2) + B(x + 1)$

Choose values of $x$ to substitute into the identity that will eliminate each constant:

Let $x = 2$: $2(2) - 13 = A((2) - 2) + B((2) + 1)$

$x - 2 = 0$, $x = 2$

$-9 = 3B$

$B = -3$

Let $x = -1$: $2(-1) - 13 = A((-1) - 2) + B((-1) + 1)$

$x + 1 = 0$, $x = -1$

$-15 = -3A$

$A = 5$

$$\frac{2x - 13}{x^2 - x - 2} = \frac{5}{x + 1} - \frac{3}{x - 2}$$

b)

Express $\frac{x(3x - 13)}{(x + 1)(x - 3)^2}$ in the form $\frac{A}{(x + 1)} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2}$.

www.thrivingscholars.com

Multiply through by the denominator:

$$\frac{x(3x-13)}{(x+1)(x-3)^2} = \frac{A(x-3)^2 + B(x+1)(x-3) + C(x+1)}{(x+1)(x-3)^2}$$

Eliminate fractions and expand:

$$x(3x-13) = A(x^2-6x+9) + B(x^2-2x-3) + Cx + C$$

$$3x^2 - 13x = (A+B)x^2 + (-6A-2B+C)x + 9A - 3B + C$$

($A+B$: coefficient of $x^2$; $-6A-2B+C$: coefficient of $x$)

Compare coefficients:

$A + B = 3$ ① (coefficients of $x^2$)

$-6A - 2B + C = -13$ ② (coefficients of $x$)

$9A - 3B + C = 0$ ③ (constant terms)

Rearrange ① and substitute into ② and ③

$A = 3 - B \Rightarrow -6(3-B) - 2B + C = -13$

$-18 + 6B - 2B + C = -13$

$4B + C = 5$ ②

$\Rightarrow 9(3-B) - 3B + C = 0$

$27 - 9B - 3B + C = 0$

$12B - C = 27$ ③

Solving ② and ③: $4B + C = 5$

$12B - C = 27$

$B = 2,\ C = -3$

Substitute into ①: $A = 3 - B = 3 - 2 = 1$

$$\frac{x(3x-13)}{(x+1)(x-3)^2} = \frac{1}{(x+1)} + \frac{2}{(x-3)} - \frac{3}{(x-3)^2}$$

www.thrivingscholars.com





# Introduction to Logarithms

## What are logarithms?

- A logarithm is the inverse of an exponent
  - If $a^x = b$ then $\log_a(b) = x$ where $a > 0$, $b > 0$, $a \neq 1$
    - This is in the formula booklet
    - The number $a$ is called the **base** of the logarithm
    - Your GDC will be able to use this function to solve equations involving exponents
- Try to get used to 'reading' logarithm statements to yourself
  - $\log_a(b) = x$ would be read as "the power that you raise $a$ to, to get $b$, is $x$"
  - So $\log_5 125 = 3$ would be read as "the power that you raise 5 to, to get 125, is 3"
- Two important cases are:
  - $\ln x = \log_e(x)$
    - Where e is the mathematical constant 2.718…
    - This is called the natural logarithm and will have its own button on your GDC
  - $\log x = \log_{10}(x)$
    - Logarithms of base 10 are used often and so abbreviated to log $x$

## Why use logarithms?

- Logarithms allow us to solve equations where the exponent is the unknown value
  - We can solve some of these by inspection
    - For example, for the equation $2^x = 8$ we know that $x$ must be 3
  - Logarithms allow use to solve more complicated problems
    - For example, the equation $2^x = 10$ does not have a clear answer
    - Instead, we can use our GDCs to find the value of $\log_2 10$



### Exam Tip

- Before going into the exam, make sure you are completely familiar with your GDC and know how to use its logarithm functions

www.thrivingscholars.com

## Worked Example

Solve the following equations:

i)

$x = \log_3 27,$

$x = \log_3 27 \iff 3^x = 27$

We can see from inspection:

$3^3 = 27 \iff x = 3$

$\boxed{x = 3}$

OR: use GDC to find answer directly.

ii)

$2^x = 21.4$, giving your answer to 3 s.f.

$2^x = 21.4$ This cannot be seen from inspection:

$2^x = 21.4 \iff x = \log_2 21.4$

use GDC to find answer directly.

$\log_2 21.4 = 4.4195...$

$\boxed{x = 4.42 \text{ (3 s.f.)}}$

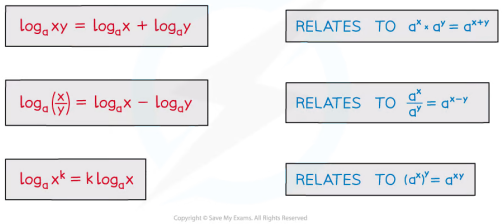
www.thrivingscholars.com



# Laws of Logarithms

## What are the laws of logarithms?

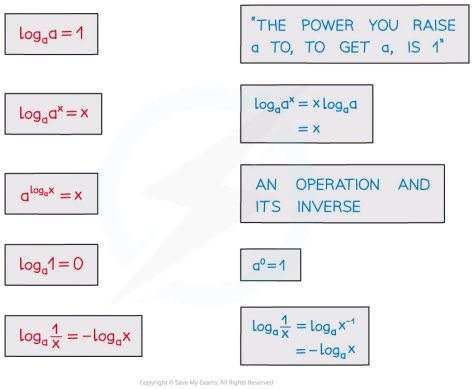
- Laws of logarithms allow you to simplify and manipulate expressions involving logarithms
  - The laws of logarithms are equivalent to the **laws of indices**
- The laws you need to know are, given $a > 0$:
  - $\log_a xy = \log_a x + \log_a y$
    - This relates to $a^x \times a^y = a^{x+y}$
  - $\log_a \frac{x}{y} = \log_a x - \log_a y$
    - This relates to $a^x \div a^y = a^{x-y}$
  - $\log_a x^m = m\log_a x$
    - This relates to $(a^x)^y = a^{xy}$
- These laws are **in the formula booklet** so you do not need to remember them
  - You must make sure you know how to use them




## Useful results from the laws of logarithms

- Given $a > 0$, $a \neq 1$
  - $\log_a 1 = 0$
    - This is equivalent to $a^0 = 1$
- If we substitute *b* for *a* into the given identity in the formula booklet
  - $a^x = b \Leftrightarrow \log_a b = x$ where $a > 0$, $b > 0$, $a \neq 1$
  - $a^x = a \Leftrightarrow \log_a a = x$ gives $a^1 = a \Leftrightarrow \log_a a = 1$
    - This is an important and useful result
- Substituting this into the third law gives the result
  - $\log_a a^k = k$
- Taking the inverse of its operation gives the result
  - $a^{\log_a x} = x$
- From the third law we can also conclude that
  - $\log_a \frac{1}{x} = -\log_a x$

www.thrivingscholars.com




- These useful results are **not in the formula booklet** but can be deduced from the laws that are
- Beware...
  - ...$\log_a(x+y) \neq \log_a x + \log_a y$
- These results apply to $\ln x$ ($\log_e x$) too
  - Two particularly useful results are
    - $\ln e^x = x$
    - $e^{\ln x} = x$
- Laws of logarithms can be used to ...
  - simplify expressions
  - solve logarithmic equations
  - solve exponential equations

## Exam Tip

- Remember to check whether your solutions are valid
  - log (x+k) is only defined if x > -k
  - You will lose marks if you forget to reject invalid solutions

www.thrivingscholars.com

## Worked Example

a)

Write the expression $2 \log 4 - \log 2$ in the form $\log k$, where $k \in \mathbb{Z}$.

Using the law $\log_a x^m = m \log_a x$

$2 \log 4 = \log 4^2 = \log 16$

$2 \log 4 - \log 2 = \log 4^2 - \log 2$

$= \log 16 - \log 2$

Using the law $\log_a \frac{x}{y} = \log_a x - \log_a y$

$\log 16 - \log 2 = \log \frac{16}{2} = \log 8$

$$2 \log 4 - \log 2 = \log 8$$

b) Hence, or otherwise, solve $2 \log 4 - \log 2 = -\log \frac{1}{x}$.

To solve $2 \log 4 - \log 2 = \log \frac{1}{x}$ rewrite as

$\log 8 = -\log \frac{1}{x}$ (from part (a))

Use the index law $\frac{1}{x} = x^{-1}$

$\log 8 = -\log x^{-1}$

$\log 8 = \log x$ ($\log_a x^m = m \log_a x$)

$8 = x$

$$x = 8$$

www.thrivingscholars.com

# Change of Base

## Why change the base of a logarithm?

- The laws of logarithms can only be used if the logs have the same **base**
  - If a problem involves logarithms with different bases, you can change the base of the logarithm and then apply the laws of logarithms
- **Changing the base** of a logarithm can be particularly useful if you need to evaluate a log problem **without a calculator**
  - Choose the base such that you would know how to solve the problem from the equivalent exponent

## How do I change the base of a logarithm?

- The formula for changing the base of a logarithm is

$$\log_a x = \frac{\log_b x}{\log_b a}$$

- This is **in the formula booklet**
- The value you choose for *b* does not matter, however if you do not have a calculator, you can choose *b* such that the problem will be possible to solve

### Exam Tip

- Changing the base is a key skill which can help you with many different types of questions, make sure you are confident with it
  - It is a particularly useful skill for examinations where a GDC is not permitted

www.thrivingscholars.com

## Worked Example

By choosing a suitable value for *b*, use the change of base law to find the value of $\log_8 32$ without using a calculator.

Change of base law: $\log_a x = \frac{\log_b x}{\log_b a}$

$\log_8 32$, $2^5 = 32$, $2^3 = 8$

Choose $b = 2$ to allow for a solution by inspection

$$\log_8 32 = \frac{\log_2 32}{\log_2 8} = \frac{5}{3}$$

$$\log_8 32 = 1\frac{2}{3}$$

www.thrivingscholars.com





# Solving Exponential Equations

## What are exponential equations?

- An exponential equation is an equation where the unknown is a power
  - In simple cases the solution can be spotted without the use of a calculator
  - For example,

$$5^{2x} = 125$$
$$2x = 3$$
$$x = \frac{3}{2}$$

- In more complicated cases the laws of logarithms should be used to solve exponential equations
- The **change of base** law can be used to solve some exponential equations without a calculator
  - For example,

$$27^x = 9$$
$$x = \log_{27} 9$$
$$= \frac{\log_3 9}{\log_3 27}$$
$$= \frac{2}{3}$$

## How do we use logarithms to solve exponential equations?

- An exponential equation can be solved by taking logarithms of both sides
- The **laws of indices** may be needed to rewrite the equation first
- The **laws of logarithms** can then be used to solve the equation
  - **ln ($\log_e$)** is often used
  - The answer is often written in terms of ln
- A question my ask you to give your answer in a particular form
- Follow these steps to solve exponential equations
  - STEP 1: Take logarithms of both sides
  - STEP 2: Use the laws of logarithms to remove the powers
  - STEP 3: Rearrange to isolate $x$
  - STEP 4: Use logarithms to solve for $x$

## What about hidden quadratics?

- Look for hidden squared terms that could be changed to form a quadratic
  - In particular look out for terms such as
    - $4^x = (2^2)^x = 2^{2x} = (2^x)^2$
    - $e^{2x} = (e^2)^x = (e^x)^2$

www.thrivingscholars.com

## Exam Tip

- Always check which form the question asks you to give your answer in, this can help you decide how to solve it
- If the question requires an exact value you may need to leave your answer as a logarithm

## Worked Example

Solve the equation $4^x - 3(2^{x+1}) + 9 = 0$. Give your answer correct to three significant figures.

Spot the hidden quadratic: $4^x = (2^2)^x = (2^x)^2$

By the laws of indices $2^{x+1} = 2^x \times 2^1 = 2 \times 2^x$

$$(2^x)^2 - 3(2^{x+1}) + 9 = 0$$

$$(2^x)^2 - 3 \times 2 \times 2^x + 9 = 0$$

$$(2^x)^2 - 6 \times 2^x + 9 = 0$$

Let $u = 2^x$ $\quad u^2 - 6u + 9 = 0$

$$(u-3)(u-3) = 0$$

$$u = 3 \quad \therefore \quad 2^x = 3$$

Solve the exponential equation $2^x = 3$

Step 1: Take logarithms of both sides: $\ln(2^x) = \ln(3)$

Step 2: Use the law $\log_a x^m = m\log_a x$ $\quad x\ln 2 = \ln 3$

Step 3: Rearrange to isolate $x$ $\quad x = \frac{\ln 3}{\ln 2}$

Step 4: Solve $\quad x = \frac{\ln 3}{\ln 2} = 1.584\ldots$

$$x = 1.58 \text{ (3s.f.)}$$

www.thrivingscholars.com



# 1.3 Sequences & Series

## 1.3.1 Language of Sequences & Series

### Language of Sequences & Series

### What is a sequence?

- A **sequence** is an ordered set of numbers with a rule for finding all of the numbers in the sequence
  - For example 1, 3, 5, 7, 9, … is a sequence with the rule '*start at one and add two to each number*'
- The numbers in a sequence are often called **terms**
- The terms of a sequence are often referred to by letters with a subscript
  - In IB this will be the letter $u$
  - So in the sequence above, $u_1 = 1$, $u_2 = 3$, $u_3 = 5$ and so on
- Each term in a sequence can be found by **substituting** the term number into **formula for the $n^{th}$ term**

### What is a series?

- You get a **series** by summing up the terms in a sequence
  - E.g. For the sequence 1, 3, 5, 7, … the associated series is $1 + 3 + 5 + 7 + \ldots$
- We use the notation $S_n$ to refer to the sum of the first $n$ terms in the series
  - $S_n = u_1 + u_2 + u_3 + \ldots + u_n$
  - So for the series above $S_5 = 1 + 3 + 5 + 7 + 9 = 25$

www.thrivingscholars.com

## Worked Example

Determine the first five terms and the value of $S_5$ in the sequence with terms defined by $u_n = 5 - 2n$.

$u_n = 5 - 2n$

find the term you want by replacing n with it's value.

$n$ = term number

first term

$u_1 = 5 - 2(1) = 3$

$u_2 = 5 - 2(2) = 1$

$u_3 = 5 - 2(3) = -1$

$u_4 = 5 - 2(4) = -3$

$u_5 = 5 - 2(5) = -5$

recognise the pattern.

$-2$

$-2$ ← rule is subtract 2

'start with 3 and subtract 2 from each number'.

$S_5 = 3 + 1 + (-1) + (-3) + (-5) = -5$

the sum of the first 5 terms

3, 1, -1, -3, -5

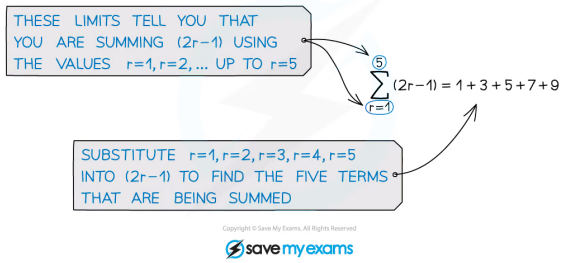
$S_5 = -5$

www.thrivingscholars.com

# Sigma Notation

## What is sigma notation?

- Sigma notation is used to show the sum of a certain number of terms in a sequence
- The symbol $\Sigma$ is the capital Greek letter sigma
- $\Sigma$ stands for 'sum'
  - The expression to the right of the $\Sigma$ tells you what is being summed, and the limits above and below tell you which terms you are summing




- Be careful, the limits don't have to start with 1
  - For example $\sum_{k=0}^{4}(2k+1)$ or $\sum_{k=7}^{14}(2k-13)$
  - $r$ and $k$ are commonly used variables within sigma notation

### Exam Tip

- Your GDC will be able to use sigma notation, familiarise yourself with it and practice using it to check your work

www.thrivingscholars.com

## Worked Example

A sequence can be defined by $u_n = 2 \times 3^{n-1}$ for $n \in \mathbb{Z}^+$.

a)

Write an expression for $u_1 + u_2 + u_3 + \ldots + u_6$ using sigma notation.

$u_n = 2 \times 3^{n-1}$, $n \in \mathbb{Z}^+$ ← n is the set of all positive integers

Using sigma notation

$$u_1 + u_2 + \ldots + u_6 = \sum_{k=1}^{6} u_k$$

$$\sum_{k=1}^{6} (2 \times 3^{k-1})$$

b)

Write an expression for $u_7 + u_8 + u_9 + \ldots + u_{12}$ using sigma notation.

$u_n = 2 \times 3^{n-1}$, $n \in \mathbb{Z}^+$ ← n is the set of all positive integers

Using sigma notation

$$u_7 + u_8 + \ldots + u_{12} = \sum_{k=7}^{12} u_k$$

$$\sum_{k=7}^{12} (2 \times 3^{k-1})$$

www.thrivingscholars.com



# Arithmetic Sequences

## What is an arithmetic sequence?

- In an **arithmetic sequence,** the difference between consecutive terms in the sequence is constant
- This **constant difference** is known as the **common difference, *d,*** of the sequence
  - For example, 1, 4, 7, 10, … is an arithmetic sequence with the rule *'start at one and add three to each number'*
    - The **first term, $u_1$,** is 1
    - The **common difference, *d*,** is 3
  - An arithmetic sequence can be **increasing** (positive common difference) or **decreasing** (negative common difference)
  - Each term of an arithmetic sequence is referred to by the letter *u* with a subscript determining its place in the sequence

## How do I find a term in an arithmetic sequence?

- The $n^{th}$ term formula for an arithmetic sequence is given as

$$u_n = u_1 + (n-1)d$$

  - Where $u_1$ is the first term, and $d$ is the common difference
  - This is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two terms and asked to find both the first term and the common difference
  - Substitute the information into the formula and set up a **system of linear equations**
  - Solve the simultaneous equations
    - You could use your GDC for this

### Exam Tip

- Simultaneous equations are often needed within arithmetic sequence questions, make sure you are confident solving them with and without the GDC

www.thrivingscholars.com

## Worked Example

The fourth term of an arithmetic sequence is 10 and the ninth term is 25, find the first term and the common difference of the sequence.

$u_4 = 10$ , $u_9 = 25$

Formula for $n^{th}$ term of an arithmetic series:

$$u_n = u_1 + (n-1)d$$

Sub in $u_4 = 10$ and $u_9 = 25$

$$u_4 = u_1 + (4-1)d = u_1 + 3d = 10$$

$$u_9 = u_1 + (9-1)d = u_1 + 8d = 25$$

Solve using GDC:

let $u_1 = x$ and $d = y$

$$x + 3y = 10$$
$$x + 8y = 25$$

$x = 1$ , $y = 3$

$$u_1 = 1$$
$$d = 3$$

www.thrivingscholars.com

# Arithmetic Series

## How do I find the sum of an arithmetic series?

- An **arithmetic series** is the sum of the terms in an **arithmetic sequence**
  - For the arithmetic sequence 1, 4, 7, 10, … the arithmetic series is $1+4+7+10+\ldots$
- Use the following formulae to find the sum of the first $n$ terms of the arithmetic series:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) \quad ; \quad S_n = \frac{n}{2}(u_1 + u_n)$$

    - $u_1$ is the first term
    - $d$ is the common difference
    - $u_n$ is the last term
  - Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - If you know the first term and common difference use the first version
  - If you know the first and last term then the second version is easier to use
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this

### Exam Tip

- The formulae you need for arithmetic series are in the formula book, you do not need to remember them
  - Practice finding the formulae so that you can quickly locate them in the exam

www.thrivingscholars.com

## Worked Example

The sum of the first 10 terms of an arithmetic sequence is 630.

a)

Find the common difference, $d$, of the sequence if the first term is 18.

$S_{10} = 630$

Formula for the sum of an arithmetic series:

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

Sub in $S_{10} = 630$, $u_1 = 18$

$S_{10} = \frac{10}{2}(2(18) + (10-1)d) = 630$

$5(36 + 9d) = 630$

Solve: $36 + 9d = 126$

$9d = 90$

$d = 10$

$\boxed{d = 10}$

b)

Find the first term of the sequence if the common difference, $d$, is 11.

www.thrivingscholars.com



Sub in $S_{10} = 630$, $d = 11$

$$S_{10} = \frac{10}{2}(2u_1 + (10-1)(11)) = 630$$

$$5(2u_1 + 99) = 630$$

Solve:

$$2u_1 + 99 = 126$$

$$2u_1 = 27$$

$$\boxed{u_1 = 13.5}$$

www.thrivingscholars.com





# Geometric Sequences

## What is a geometric sequence?

- In a **geometric sequence**, there is a **common ratio, *r*,** between consecutive terms in the sequence
  - For example, 2, 6, 18, 54, 162, … is a sequence with the rule *'start at two and multiply each number by three'*
    - The **first term, $u_1$,** is 2
    - The **common ratio, *r*,** is 3
- A geometric sequence can be **increasing** ($r > 1$) or **decreasing** ($0 < r < 1$)
- If the common ratio is a **negative number** the terms will alternate between positive and negative values
  - For example, 1, -4, 16, -64, 256, … is a sequence with the rule *'start at one and multiply each number by negative four'*
    - The **first term, $u_1$,** is 1
    - The **common ratio, *r*,** is -4
- Each term of a geometric sequence is referred to by the letter *u* with a subscript determining its place in the sequence

## How do I find a term in a geometric sequence?

- The $n^{th}$ term formula for a geometric sequence is given as

$$u_n = u_1 r^{n-1}$$

  - Where $u_1$ is the first term, and $r$ is the common ratio
  - This formula allows you to find **any term** in the geometric sequence
  - It is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common ratio
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two or more consecutive terms and asked to find both the first term and the common ratio
  - Find the common ratio by dividing a term by the one before it
  - Substitute this and one of the terms into the formula to find the first term
- Sometimes you may be given a term and the formula for the $n^{th}$ term and asked to find the value of *n*
  - You can solve these using **logarithms** on your GDC

www.thrivingscholars.com



### Exam Tip

- You will sometimes need to use logarithms to answer geometric sequences questions
  - Make sure you are confident doing this
  - Practice using your GDC for different types of questions

www.thrivingscholars.com

## Worked Example

The sixth term, $u_6$, of a geometric sequence is 486 and the seventh term, $u_7$, is 1458.

Find,

i)
the common ratio, $r$, of the sequence,

$u_6 = 486$ , $u_7 = 1458$

The common ratio, r, is given by

$$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \ldots = \frac{u_{n+1}}{u_n}$$

Sub in $u_6 = 486$ , $u_7 = 1458$

$$r = \frac{u_7}{u_6} = \frac{1458}{486} = 3$$

$$\boxed{r = 3}$$

ii)
the first term of the sequence, $u_1$.

Formula for $n^{th}$ term of a geometric series:

$$u_n = u_1 r^{n-1}$$

Sub in $r = 3$ and either $u_6 = 486$ or $u_7 = 1458$

$$u_6 = u_1(3)^{6-1} = 486$$

Solve: $243\, u_1 = 486$

$u_1 = 2$

$$\boxed{u_1 = 2}$$

www.thrivingscholars.com

# Geometric Series

## How do I find the sum of a geometric series?

- A **geometric series** is the sum of a certain number of terms in a **geometric sequence**
  - For the geometric sequence 2, 6, 18, 54, … the geometric series is $2 + 6 + 18 + 54 + \ldots$
- The following formulae will let you find the sum of the first *n* terms of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

    - $u_1$ is the first term
    - $r$ is the common ratio
  - Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - The first version of the formula is more convenient if $r > 1$ and the second is more convenient if $r < 1$
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term, the common ratio, or the number of terms within the sequence
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this

### Exam Tip

- The geometric series formulae are in the formulae booklet, you don't need to memorise them
  - Make sure you can locate them quickly in the formula booklet

www.thrivingscholars.com

## Worked Example

A geometric sequence has $u_1 = 25$ and $r = 0.8$. Find the value of $u_5$ and $S_5$.

$u_1 = 25$, $r = 0.8$

Formula for $n^{th}$ term of a geometric series:

$$u_n = u_1 r^{n-1}$$

Sub in $u_1 = 25$, $r = 0.8$

$$u_5 = 25(0.8)^4 = 10.24$$

Formula for the sum of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

$r < 1$ so this version is easier to use.

Sub in $u_1 = 25$, $r = 0.8$

$$S_5 = \frac{u_1(1 - r^5)}{1 - r} = \frac{25(1 - 0.8^5)}{1 - 0.8} = 84.04$$

$u_5 = 10.24$

$S_5 = 84.04$

www.thrivingscholars.com

# Sum to Infinity

## What is the sum to infinity of a geometric series?

- A geometric sequence will either increase or decrease away from zero or the terms will get progressively closer to zero
  - Terms will get closer to zero if the common ratio, *r*, is between 1 and -1
- If the terms are getting closer to zero then the series is said to **converge**
  - This means that the sum of the series will approach a limiting value
  - As the number of terms increase, the sum of the terms will get closer to the limiting value

## How do we calculate the sum to infinity?

- If asked to find out if a geometric sequence converges find the value of *r*
  - If $|r| < 1$ then the sequence converges
  - If $|r| \geq 1$ then the sequence does not converge and the sum to infinity cannot be calculated
  - $|r| < 1$ means $-1 < r < 1$
- If $|r| < 1$, then the geometric series **converges** to a finite value given by the formula

$$S_\infty = \frac{u_1}{1-r}, \quad |r| < 1$$

  - $u_1$ is the first term
  - $r$ is the common ratio
  - This is **in the formula book**, you do not need to remember it

### Exam Tip

- Learn and remember the conditions for when a sum to infinity can be calculated

www.thrivingscholars.com

## Worked Example

The first three terms of a geometric sequence are $6$ , $2$ , $\frac{2}{3}$. Explain why the series converges and find the sum to infinity.

$u_1 = 6, \quad u_2 = 2, \quad u_3 = \frac{2}{3}$

Find the value of r: $r = \frac{u_2}{u_1}$

$$r = \frac{u_2}{u_1} = \frac{2}{6} = \frac{1}{3}$$

$|r| < 1$ so the series converges

Find the sum to infinity: $S_\infty = \frac{u_1}{1-r}$

$$S_\infty = \frac{u_1}{1-r} = \frac{6}{1-\frac{1}{3}} = \frac{6}{2/3} = 9$$

$S_\infty = 9$

www.thrivingscholars.com





# Applications of Arithmetic Sequences & Series

Many real-life situations can be modelled using sequences and series, including but not limited to: patterns made when tiling floors; seating people around a table; the rate of change of a population; the spread of a virus and many more.

## What do I need to know about applications of arithmetic sequences and series?

- If a quantity is changing repeatedly by having a fixed amount **added to** or **subtracted from** it then the use of **arithmetic sequences** and **arithmetic series** is appropriate to **model** the situation
  - If a sequence seems to fit the pattern of an arithmetic sequence it can be said to be **modelled** by an arithmetic sequence
  - The scenario can be **modelled** using the given information and the formulae from the formula booklet
- A common application of arithmetic sequences and series is **simple interest**
  - Simple interest is when an initial investment is made and then a percentage of the initial investment is added to this amount on a regular basis (usually per year)
- Arithmetic sequences can be used to make estimations about how something will change in the future

### Exam Tip

- Exam questions won't always tell you to use sequences and series methods, practice spotting them by looking for clues in the question
- If a given amount is repeated periodically then it is likely the question is on arithmetic sequences or series

www.thrivingscholars.com

## Worked Example

Jasper is saving for a new car. He puts USD $100 into his savings account and then each month he puts in USD $10 more than the month before. Jasper needs USD $1200 for the car. Assuming no interest is added, find,

i)
the amount Jasper has saved after four months,

Identify the arithmetic sequence :

$u_1 = 100$ , $d = 10$

After 4 months Jasper will have saved:

$u_1 + u_2 + u_3 + u_4 = S_4$

Formula for the sum of an arithmetic series :

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$S_4 = \frac{4}{2}(2u_1 + (4-1)d)$

Sub in $u_1 = 100$ and $d = 10$

$S_4 = \frac{4}{2}(2(100) + (4-1)(10))$

$= 2(200 + 30)$

$= 2(230)$

$S_4 = \$460$

ii)
the month in which Jasper reaches his goal of USD $1200.

www.thrivingscholars.com

Sub $S_n = 1200$, $u_1 = 100$, $d = 10$ into formula:

$$1200 = \frac{n}{2}(2(100) + (n-1)(10))$$

Solve using algebraic solver on GDC:

$n = 8.67...$ or $n = -27.67...$

↑ disregard as $n$ cannot be negative.

$\therefore$ $S_8 < 1200$

$S_9 > 1200$ reaches total in 9th month

Jasper will reach USD $1200 in the 9th month.

www.thrivingscholars.com

# Applications of Geometric Sequences & Series

## What do I need to know about applications of geometric sequences and series?

- If a quantity is changing repeatedly by a fixed **percentage,** or by being **multiplied** repeatedly by a fixed amount, then the use of **geometric sequences** and **geometric series** is appropriate to **model** the situation
  - If a sequence seems to fit the pattern of a geometric sequence it can be said to be **modelled** by a geometric sequence
  - The scenario can be **modelled** using the given information and the formulae from the formula booklet
- A common application of geometric sequences and series is **compound interest**
  - Compound interest is when an initial investment is made and then interest is paid on the initial amount **and on the interest already earned** on a regular basis (usually every year)
- Geometric sequences can be used to make estimations about how something will change in the future
- The questions won't always tell you to use sequences and series methods, so be prepared to spot 'hidden' sequences and series questions
  - Look out for questions on savings accounts, salaries, sales commissions, profits, population growth and decay, spread of bacteria etc

### Exam Tip

- Exam questions won't always tell you to use sequences and series methods, practice spotting them by looking for clues in the question
- If a given amount is changing by a percentage or multiple then it is likely the question is on geometric sequences or series

www.thrivingscholars.com

## Worked Example

A new virus is circulating on a remote island. On day one there were 10 people infected, with the number of new infections increasing at a rate of 40% per day.

a)

Find the expected number of people newly infected on the $7^{th}$ day.

Identify the geometric sequence:

$u_1 = 10$ , $r = 1.4$ ← 40% increase so 140% of the day before

New infections : $u_7$

Formula for $n^{th}$ term of a geometric series:

$$u_n = u_1 r^{n-1}$$

Sub in $u_1 = 10$ , $r = 1.4$

$$u_7 = 10(1.4)^6 = 75.29...$$

**Expected number of new infections = 75**

b)

Find the expected number of infected people after one week (7 days), assuming no one has recovered yet.

Total infections: $S_7$

Formula for the sum of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$ ← $r > 1$ so this version is easier to use.

Sub in $u_1 = 10$ , $r = 1.4$

$$S_7 = \frac{10(1.4^7 - 1)}{1.4 - 1} = 238.53...$$

**Expected number of total infections = 239**

www.thrivingscholars.com





# Compound Interest

## What is compound interest?

- Interest is a small percentage paid by a bank or company that is added on to an initial investment
  - Interest can also refer to an amount paid on a loan or debt, however IB compound interest questions will always refer to interest on **investments**
- **Compound interest** is where interest is paid on **both the initial investment** and any interest that has **already been paid**
  - Make sure you know the difference between compound interest and simple interest
    - Simple interest pays interest only on the initial investment
- The interest paid each time will increase as it is a percentage of a higher number
- Compound interest will be paid in instalments in a given timeframe
  - The interest rate, *r*, will be per annum (per year)
    - This could be written *r* % p.a.
  - Look out for phrases such as **compounding annually** (interest paid yearly) **or compounding monthly** (interest paid monthly)
    - If $\alpha\%$ p.a. (per annum) is paid compounding monthly, then $\frac{\alpha}{12}\%$ will be paid each month
    - The formula for compound interest allows for this so you do not have to compensate separately

## How is compound interest calculated?

- The formula for calculating compound interest is:

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

  - Where
    - *FV* is the future value
    - *PV* is the present value
    - *n* is the number of years
    - *k* is the number of compounding periods per year
    - *r* % is the nominal annual rate of interest
  - This formula is **given in the formula booklet**, you do not have to remember it
- Be careful with the *k* value
  - Compounding annually means *k* = 1
  - Compounding half-yearly means *k* = 2
  - Compounding quarterly means *k* = 4
  - Compounding monthly means *k* = 12
- Your GDC will have a finance solver app on it which you can use to find the future value
  - This may also be called the TVM (time value of money) solver
  - You will have to enter the information from the question into your calculator
- Be aware that many questions will be set up such that you will have to use the formula

www.thrivingscholars.com

- So for compound interest questions it is better to use the formula from your formula booklet than your GDC

### Exam Tip

- Your GDC will be able to solve some compound interest problems so it is a good idea to make sure you are confident using it, however you must also familiarise yourself with the formula and make sure you can find it in the formula booklet

### Worked Example

Kim invests MYR 2000 (Malaysian Ringgit) in an account that pays a nominal annual interest rate of 2.5% **compounded monthly**. Calculate the amount that Kim will have in her account after 5 years.

Compound interest formula:

$$FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$$

FV: future value; PV: present value; k: compounding periods; r: interest rate; n: number of years

Substitute values in:

PV = 2000 (initial investment)
k = 12 (compounding monthly)
r = 2.5%
n = 5 (number of years)

$$FV = 2000\left(1 + \frac{2.5}{(100)(12)}\right)^{(12\times 5)}$$

$$= 2266.002\ldots$$

**FV ≈ MYR 2270 (3sf)**

www.thrivingscholars.com

# Depreciation

## What is depreciation?

- Depreciation is when something looses value over time
  - The most common examples of depreciation are the value of cars and technology or the temperature of a cooling cup of coffee

## How is compound depreciation calculated?

- The formula for calculating compound depreciation is:

$$FV = PV \times \left(1 - \frac{r}{100}\right)^n$$

  - Where
    - $FV$ is the future value
    - $PV$ is the present value
    - $n$ is the number of years
    - $r\%$ is the rate of depreciation
  - This formula is **not** given in the formula booklet, however it is almost the same as the formula for compound interest but
    - with a **subtraction** instead of an addition
    - the value of $k$ will always be 1
- Your GDC **could** again be used to solve some compound depreciation questions, but watch out for those which are set up such that you will have to use the formula

### Exam Tip

- Although the formula is not the same as the one given in the formula booklet for compound interest, it is very similar
  - Practice finding this formula and recognising the differences

www.thrivingscholars.com

## Worked Example

Kyle buys a new car for AUD $14 999. The value of the car depreciates by 15% each year.

a)

Find the value of the car after 5 years.

Depreciation formula:

$$FV = PV\left(1 - \frac{r}{100}\right)^n$$

$r$: rate of depreciation; $n$: number of years

$FV$: future value; $PV$: present value

Substitute values in:

$PV = 14\,999$ (initial cost)

$r = 15\%$

$n = 5$ (number of years)

$$FV = 14\,999\left(1 - \frac{15}{100}\right)^5$$

$$= 6655.13\ldots$$

$FV \approx$ AUD $6660 (3sf)

b)

Find the number of years and months it will take for the value of the car to be approximately AUD $9999.

www.thrivingscholars.com

$$FV = PV\left(1 - \frac{r}{100}\right)^n$$

$FV \approx 9999$

$PV = 14999$

$r = 15\%$

Substitute values in:

$$9999 \approx 14999\left(1 - \frac{15}{100}\right)^n$$

Use GDC to solve:

$n = 2.495\ldots$

$2$ years; $0.495^{th}$ of a year

Convert to years and months:

$2$ years $+ 0.495\ldots \times 12$ months

$\approx$ 2 years and 6 months

www.thrivingscholars.com



# 1.4 Simple Proof & Reasoning

## 1.4.1 Proof

### Language of Proof

### What is proof?

- Proof is a series of logical steps which show a result is **true** for all specified numbers
  - 'Seeing' that a result works for a few numbers is not enough to **show** that it will work for all numbers
  - Proof allows us to show (usually algebraically) that the result will work for **all values**
- You must be familiar with the notation and language of proof
- LHS and RHS are standard abbreviations for left-hand side and right-hand side
- **Integers** are used frequently in the language of proof
  - The set of **integers** is denoted by $\mathbb{Z}$
  - The set of **positive integers** is denoted by $\mathbb{Z}^+$

### How do we prove a statement is true for all values?

- Most of the time you will need to use algebra to show that the left-hand side (LHS) is the same as the right-hand side (RHS)
  - You **must not** move terms from one side to the other
  - Start with one side (usually the LHS) and manipulate it to show that it is the same as the other
- A **mathematical identity** is a statement that is true for all values of $x$ (or $\theta$ in trigonometry)
  - The symbol $\equiv$ is used to identify an identity
  - If you see this symbol then you can use proof methods to show it is true
- You can complete your proof by stating that RHS = LHS or writing QED



**Exam Tip**

- You will need to show each step of your proof clearly and set out your method in a logical manner in the exam
  - Be careful not to skip steps

www.thrivingscholars.com

## Worked Example

Prove that $(2x-2)(x-3)+2(x-1)=2(x-2)(x-1)$.

Work with LHS first:

Expand brackets:

LHS: $(2x-2)(x-3)+2(x-1)$ (FOIL)

$2x^2-6x-2x+6+2x-2$

Simplify, take care with negatives:

$2x^2-6x+4$

Factorise the 2:

$2(x^2-3x+2)$

Factorise remaining quadratic:

$2(x-2)(x-1)$ = RHS as required.

$$(2x-2)(x-3)+2(x-1)=2(x-2)(x-1)$$

www.thrivingscholars.com

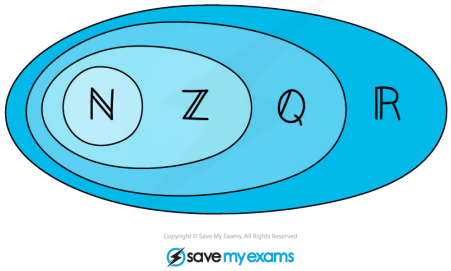
# Proof by Deduction

## What is proof by deduction?

- A mathematical and logical argument that shows that a result is true

## How do we do proof by deduction?

- A proof by deduction question will often involve showing that a result is true for all integers, consecutive integers or even or odd numbers
  - You can begin by letting an integer be $n$
    - Use conventions for even ($2n$) and odd ($2n - 1$) numbers
- You will need to be familiar with sets of numbers ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)
  - $\mathbb{N}$ – the set of **natural numbers**
  - $\mathbb{Z}$ – the set of **integers**
  - $\mathbb{Q}$ – the set of **quotients (rational numbers)**
  - $\mathbb{R}$ – the set of **real numbers**



Copyright © Save My Exams. All Rights Reserved
save my exams

## What is proof by exhaustion?

- Proof by exhaustion is a way to show that the desired result works for every allowed value
  - This is a good method when there are only a limited number of cases to check
- Using proof by exhaustion means testing every allowed value not just showing a few examples
  - The allowed values could be specific values
  - They could also be split into cases such as even and odd

### Exam Tip

- Try the result you are proving with a few different values
  - Use a sequence of them (eg 1, 2, 3)
  - Try different types of numbers (positive, negative, zero)
- This may help you see a pattern and spot what is going on

www.thrivingscholars.com

## Worked Example

Prove that the sum of any two consecutive odd numbers is always even.

Let $2n-1$ be an odd number

($2n-1$ must be even)

Let two consecutive odd integers be:

$2n-1$ , $2n+1$ (next odd number)

Then their sum is:

$$2n-1+2n+1 \equiv 4n$$

$$= 2(2n)$$

Any multiple of 2 must be even.

www.thrivingscholars.com

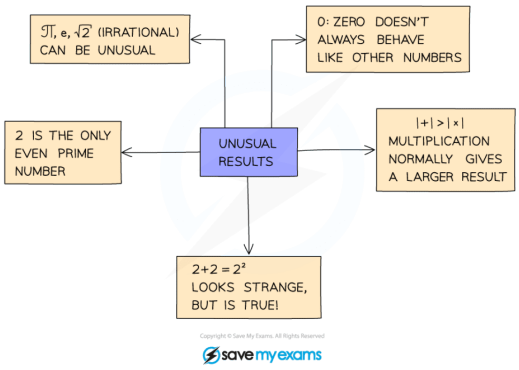
# Disproof by Counter Example

## What is disproof by counter-example?

- Disproving a result involves finding a value that does **not** work in the result
- That value is called a **counter-example**

## How do I disprove a result?

- You only need to find **one** value that does not work
- Look out for the set of numbers for which the statement is made, it will often be just integers or natural numbers
- Numbers that have unusual results are often involved
  - It is often a good idea to try the values 0 and 1 first as they often behave in different ways to other numbers
  - The number 2 also behaves differently to other even numbers
    - It is the only even prime number
    - It is the only number that satisfies $n + n = n^n$
  - If it is the set of real numbers consider how rational and irrational numbers behave differently
  - Think about how positive and negative numbers behave differently
    - Particularly when working with inequalities




### Exam Tip

- Read the question carefully, looking out for the set of numbers for which you need to prove the result

www.thrivingscholars.com

## Worked Example

For each of the following statements, show that they are false by giving a counterexample:

a)

Given $n \in \mathbb{Z}^+$, if $n^2$ is a multiple of 4, then $n$ is also a multiple of 4.

$n \in \mathbb{Z}^+$ ← set of positive integers only

We are only interested in positive integers so start by trying 1, 2 etc.

$1^2 = 1$ (not a multiple of 4)

$2^2 = 4$ $\quad n^2 = 4$ (multiple of 4)

$n = 2$ (not a multiple of 4)

Let $n = 2$: $n^2 = 4$ (multiple of 4)

$n = 2$ (not a multiple of 4)

b)

Given $x \in \mathbb{Z}$ then $3x$ is always greater than $2x$.

www.thrivingscholars.com

$x \in \mathbb{Z}$ ← Set of integers only

We are interested in both positive and negative integers and zero so consider how each of these groups behave:

Positive integers, e.g. let $x = 1$:

$2x = 2$

$3x = 3 \quad \therefore 3x > 2x$

Zero:

Let $x = 0$

$2x = 0$

$3x = 0 \quad \therefore \quad 3x = 2x$ (this is enough to disprove the result)

Negative integers, e.g. let $x = -1$:

$2x = -2$

$3x = -3 \quad \therefore 2x > 3x$ (Any negative integer can disprove the result)

Let $x = 0$

$2x = 0$

$3x = 0 \quad \therefore \quad 3x \not> 2x$

www.thrivingscholars.com

# 1.5 Further Proof & Reasoning

## 1.5.1 Proof by Induction

### Proof by Induction

### What is proof by induction?

- **Proof by induction** is a way of proving a **result is true for a set of integers** by showing that if it is **true for one integer then it is true for the next integer**
- It can be thought of as dominoes:
  - All dominoes will fall down if:
    - The first domino falls down
    - Each domino falling down causes the next domino to fall down

### What are the steps for proof by induction?

- **STEP 1: The basic step**
  - **Show** the result is true for the **base case**
  - This is **normally $n = 1$ or 0** but it could be any integer
    - For example: To prove $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ is true for all integers $n \geq 1$ you would first need to show it is true for $n = 1$:
    - $\sum_{r=1}^{1} r^2 = \frac{1}{6}(1)((1)+1)(2(1)+1)$
- **STEP 2: The assumption step**
  - **Assume** the result is true for **$n = k$** for some integer $k$
    - For example: Assume $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$ is true
  - There is nothing to do for this step apart from writing down the assumption
- **STEP 3: The inductive step**
  - **Using the assumption show** the result is true for **$n = k + 1$**
  - It can be helpful to simplify LHS & RHS separately and show they are identical
  - The assumption from STEP 2 will be needed at some point
    - For example: $LHS = \sum_{r=1}^{k+1} r^2$ and $RHS = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$
- **STEP 4: The conclusion step**
  - **State** the result is true
  - **Explain in words** why the result is true
  - It must include:
    - If true for $n = k$ then it is true for $n = k + 1$
    - Since true for $n = 1$ the statement is true for all $n \in \mathbb{Z}, n \geq 1$ by mathematical induction
  - The sentence will be the same for each proof just change the base case from $n = 1$ if necessary

### What type of statements might I be asked to prove by induction?

www.thrivingscholars.com



- **Sums of sequences**
  - If the terms involve factorials then $(k+1)! = (k+1) \times (k!)$ is useful
  - These can be written in the form $\sum_{r=1}^{n} f(r) = g(n)$
  - A useful trick for the inductive step is using $\sum_{r=1}^{k+1} f(r) = f(k+1) + \sum_{r=1}^{k} f(r)$
- **Divisibility** of an expression by an integer
  - These can be written in the form $f(n) = m \times q_n$ where $m$ & $q_n$ are integers
  - A useful trick for the inductive step is using $a^{k+1} = a \times a^k$
- **Complex numbers**
  - You can use proof by induction to prove de Moivre's theorem
- **Derivatives**
  - Such as chain rule, product rule & quotient rule
  - These can be written in the form $f^{(n)}(x) = g(x)$
  - A useful trick for the inductive step is using $f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f^{(k)}(x))$
  - You will have to use the differentiation rules



### Exam Tip

- Learn the steps for proof by induction and make sure you can use the method for a number of different types of questions before going into the exam
- The trick to answering these questions well is practicing the pattern of using each step regularly

www.thrivingscholars.com

## Worked Example

Prove by induction that $\sum_{r=1}^{n} r(r-3) = \frac{1}{3}n(n-4)(n+1)$ for $n \in \mathbb{Z}^+$.

Want to prove $\sum_{r=1}^{n} r(r-3) = \frac{1}{3}n(n-4)(n+1)$

**Basic step**
Show true for n=1

LHS $= \sum_{r=1}^{1} r(r-3) = (1)(1-3) = -2$

RHS $= \frac{1}{3}(1)(1-4)(1+1) = -2$ $\therefore$ LHS = RHS so true for n=1

**Assumption step**
Assume true for n=k

Assume $\sum_{r=1}^{k} r(r-3) = \frac{1}{3}k(k-4)(k+1)$

**Inductive step**
Show true for n=k+1

RHS $= \frac{1}{3}(k+1)((k+1)-4)((k+1)+1) = \frac{1}{3}(k+1)(k-3)(k+2)$

LHS $= \sum_{r=1}^{k+1} r(r-3) = (k+1)((k+1)-3) + \sum_{r=1}^{k} r(r-3)$

$= (k+1)(k-2) + \frac{1}{3}k(k-4)(k+1)$ ← Using assumption

$= \frac{1}{3}(k+1)\left[3(k-2) + k(k-4)\right]$ ← Factorise $\frac{1}{3}(k+1)$

$= \frac{1}{3}(k+1)\left[k^2 - k - 6\right]$

$= \frac{1}{3}(k+1)(k-3)(k+2)$

$\therefore$ LHS = RHS so true for n=k+1

**Conclusion step**
Explain

If true for n=k then true for n=k+1.
Since it is true for n=1, the statement is true for all $n \in \mathbb{Z}^+$

$\sum_{r=1}^{n} r(r-3) = \frac{1}{3}n(n-4)(n+1)$

www.thrivingscholars.com





# Proof by Contradiction

## What is proof by contradiction?

- **Proof by contradiction** is a way of proving a **result is true** by showing that **the negation can not be true**
- It is done by:
  - Assuming the negation (opposite) of the result is true
  - Showing that this then leads to a contradiction

## How do I determine the negation of a statement?

- The **negation** of a statement is the **opposite**
  - It is the statement that makes the original statement false
- To negate statements that mention "all", "every", "and" "both":
  - Replace these phrases with "there is at least one", "or" or "there exists" and include the opposite
- To negate statements that mention "there is at least one", "or" or "there exists":
  - Replace these phrases with "all", "every", "and" or "both" and include the opposite
- To negate a statement with "if A occurs then B occurs":
  - Replace with "A occurs and the negation of B occurs"
- Examples include:

| Statement | Negation |
|---|---|
| $a$ is rational | $a$ is irrational |
| every even number bigger than 2 can be written as the sum of two primes | there exists an even number bigger than 2 which cannot be written as a sum of two primes |
| $n$ is even and prime | $n$ is not even or $n$ is not prime |
| there is at least one odd perfect number | all perfect numbers are even |
| $n$ is a multiple of 5 or a multiple of 3 | $n$ is not a multiple of 5 and $n$ is not a multiple of 3 |
| if $n^2$ is even then $n$ is even | $n^2$ is even and $n$ is odd |

## What are the steps for proof by contradiction?

- **STEP 1: Assume the negation** of the statement is **true**
  - You assume it is true but then try to prove your assumption is wrong
    - For example: To prove that there is no smallest positive number you start by assuming there is a smallest positive number called $a$
- **STEP 2:** Find **two results which contradict** each other
  - Use algebra to help with this

www.thrivingscholars.com



    - Consider how a contradiction might arise
        - For example: ½*a* is positive and it is smaller than *a* which contradicts that *a* was the smallest positive number
- **STEP 3**: **Explain** why the **original statement is true**
    - In your explanation mention:
        - The **negation can't be true** as it led to a contradiction
        - Therefore the **original statement must be true**

## What type of statements might I be asked to prove by contradiction?

- **Irrational numbers**
    - To show $\sqrt[n]{p}$ is irrational where *p* is a prime
        - Assume $\sqrt[n]{p} = \frac{a}{b}$ where *a* & *b* are integers with no common factors and $b \neq 0$
        - Use algebra to show that *p* is a factor of both *a* & *b*
    - To show that $\log_p(q)$ is irrational where *p* & *q* are different primes
        - Assume $\log_p(q) = \frac{a}{b}$ where *a* & *b* are integers with no common factors and $b \neq 0$
        - Use algebra to show $q^b = p^a$
    - To show that *a* or *b* must be irrational if their sum or product is irrational
        - Assume *a* & *b* are rational and write as fractions
        - Show that $a + b$ or $ab$ is rational
- **Prime numbers**
    - To show a polynomial is never prime
        - Assume that it is prime
        - Show there is at least one factor that cannot equal 1
    - To show that there is an infinite number of prime numbers
        - Assume there are *n* primes $p_1, p_2, ..., p_n$
        - Show that $p = 1 + p_1 \times p_2 \times ... \times p_n$ is a prime that is bigger than the *n* primes
- **Odds and evens**
    - To show that *n* is even if $n^2$ is even
        - Assume $n^2$ is even and *n* is odd
        - Show that $n^2$ is odd
- **Maximum and minimum values**
    - To show that there is no maximum multiple of 3
        - Assume there is a maximum multiple of 3 called *a*
        - Multiply *a* by 3

### Exam Tip

- A question won't always state that you should use proof by contradiction, you will need to recognise that it is the correct method to use
    - There will only be two options (e.g. a number is rational or irrational)
    - Contradiction is often used when no other proof seems reasonable

www.thrivingscholars.com

## Worked Example

Prove the following statements by contradiction.

a)

For any integer $n$, if $n^2$ is a multiple of 3 then $n$ is a multiple of 3.

Assume the negation is true for a contradiction.

Assume $n^2$ is a multiple of 3 and $n$ is not a multiple of 3.

Every integer can be written as one of $3k-1, 3k, 3k+1$ for some $k \in \mathbb{Z}$

As $n$ is not a multiple of 3 then $n = 3k+1$ or $n = 3k-1$ for some $k \in \mathbb{Z}$

If $n = 3k+1$ : $n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ so not a multiple of 3

If $n = 3k-1$ : $n^2 = (3k-1)^2 = 9k^2 - 6k + 1 = 3(3k^2 - 2k) + 1$ so not a multiple of 3

$\therefore n^2$ is not a multiple of 3

This contradicts the statement "$n^2$ is a multiple of 3".

Therefore the assumption is incorrect.

Therefore if $n^2$ is a multiple of 3 then $n$ is a multiple of 3.

b)

$\sqrt{3}$ is an irrational number.

Assume the negation is true for a contradiction.

Assume $\sqrt{3}$ is rational so can be written $\sqrt{3} = \frac{a}{b}$ where $a$ and $b$ are integers with no common factors and $b \neq 0$.

Square both sides and rearrange

$3 = \frac{a^2}{b^2} \Rightarrow 3b^2 = a^2 \Rightarrow a^2$ is a multiple of 3 $\Rightarrow a$ is a multiple of 3

Let $a = 3k$ for some $k \in \mathbb{Z}$

$3b^2 = a^2 \Rightarrow 3b^2 = 9k^2 \Rightarrow b^2 = 3k^2 \Rightarrow b^2$ is a multiple of 3

$\therefore b$ and $a$ are multiples of 3

This contradicts the statement "$a$ and $b$ have no common factors".

Therefore the assumption is incorrect.

Therefore $\sqrt{3}$ is irrational.

www.thrivingscholars.com

# 1.6 Binomial Theorem

## 1.6.1 Binomial Theorem

### Binomial Theorem

### What is the Binomial Theorem?

- The **binomial theorem** (sometimes known as the binomial expansion) gives a method for expanding a **two-term** expression in a bracket raised to a power
  - A **binomial expression** is in fact any two terms inside the bracket, however in IB the expression will usually be linear
- To expand a bracket with a two-term expression in:
  - First choose the most appropriate parts of the expression to assign to *a* and *b*
  - Then use the formula for the binomial theorem:

$$(a+b)^n = a^n + {}^nC_1\, a^{n-1}\, b + \ldots + {}^nC_r\, a^{n-r}\, b^r + \ldots + b^n$$

  - where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
    - See below for more information on ${}^nC_r$
    - You may also see ${}^nC_r$ written as $\binom{n}{r}$ or ${}_nC_r$

- You will usually be asked to find the first three or four terms of an expansion
- Look out for whether you should give your answer in **ascending** or **descending** powers of *x*
  - For **ascending** powers start with the constant term, $a^n$
  - For **descending** powers start with the term with *x* in
    - You may wish to swap *a* and *b* over so that you can follow the general formula given in the formula book
- If you are not writing the full expansion you can either
  - show that the sequence continues by putting an ellipsis (…) after your final term
  - or show that the terms you have found are an approximation of the full sequence by using the sign for approximately equals to (≈)

### How do I find the coefficient of a single term?

- Most of the time you will be asked to find the coefficient of a term, rather than carry out the whole expansion
- Use the formula for the general term

$${}^nC_r\, a^{n-r}\, b^r$$

- The question will give you the power of *x* of the term you are looking for
  - Use this to choose which value of *r* you will need to use in the formula
  - This will depend on where the *x* is in the bracket
  - The laws of indices can help you decide which value of *r* to use:
    - For $(a + bx)^n$ to find the coefficient of $x^r$ use $a^{n-r}(bx)^r$

www.thrivingscholars.com



- For $(a + bx^2)^n$ to find the coefficient of $x^r$ use $a^{\frac{n-r}{2}}(bx^2)^{\frac{r}{2}}$
- For $(a + \frac{b}{x})^n$ look at how the powers will cancel out to decide which value of $r$ to use
- So for $\left(3x + \frac{2}{x}\right)^8$ to find the coefficient of $x^2$ use the term with $r = 3$ and to find the constant term use the term with $r = 4$
- There are a lot of variations of this so it is usually easier to see this by inspection of the exponents

- You may also be given the coefficient of a particular term and asked to find an unknown in the brackets
  - Use the laws of indices to choose the correct term and then use the binomial theorem formula to form and solve and equation

## Exam Tip

- Binomial expansion questions can get messy, use separate lines to keep your working clear and always put terms in brackets

## Worked Example

Find the first three terms, in ascending powers of $x$, in the expansion of $(3 - 2x)^5$.

$a = 3 \quad b = -2x \quad n = 5$

Substitute values into the formula for $(a+b)^n$

$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n$

Question asks for ascending powers of $x$, so start with the constant term, $a^n$.

$(3 - 2x)^5 = 3^5 + {}^5C_1 (3)^{5-1}(-2x) + {}^5C_2 (3)^{5-2}(-2x)^2 + \ldots$

Watch out for the negative

$\approx 243 + 5 \times 81 \times -2x + 10 \times 27 \times 4x^2$

$\approx 243 - 810x + 1080x^2$

$(3 - 2x)^5 \approx 243 - 810x + 1080x^2$

www.thrivingscholars.com

# The Binomial Coefficient nCr

## What is ${}^{n}C_{r}$?

- If we want to find the number of ways to **choose** *r* items out of *n* different objects we can use the formula for ${}^{n}C_{r}$
  - The formula for *r* **combinations** of *n* items is ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$
  - This formula is given in the formula booklet along with the formula for the binomial theorem
  - The function ${}^{n}C_{r}$ can be written $\binom{n}{r}$ or ${}_{n}C_{r}$ and is often read as '*n choose r*'
    - Make sure you can find and use the button on your GDC

## How does ${}^{n}C_{r}$ relate to the binomial theorem?

- The formula ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$ is also known as a **binomial** coefficient
- For a binomial expansion $(a + b)^{n}$ the coefficients of each term will be ${}^{n}C_{0}$, ${}^{n}C_{1}$ and so on up to ${}^{n}C_{n}$
  - The coefficient of the $r^{th}$ term will be ${}^{n}C_{r}$
- ${}^{n}C_{n} = {}^{n}C_{0} = 1$
- The binomial coefficients are symmetrical, so ${}^{n}C_{r} = {}^{n}C_{n-r}$
  - This can be seen by considering the formula for ${}^{n}C_{r}$
  - ${}^{n}C_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{r!(n-r)!} = nC_{r}$

### Exam Tip

- You will most likely need to use the formula for nCr at some point in your exam
  - Practice using it and don't always rely on your GDC
  - Make sure you can find it easily in the formula booklet

www.thrivingscholars.com

## Worked Example

Without using a calculator, find the coefficient of the term in $x^3$ in the expansion of $(1 + x)^9$.

$n = 9$, $a = 1$, $b = x$

Substitute values into the formula for the binomial theorem:

$$(a+b)^n = a^n + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + b^n$$

where $${}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(1+x)^9 = \sum_{r=0}^{9} {}^9C_r\, (1)^{9-r}\, (x)^r$$

← Coefficient of $x^3$ occurs when $r = 3$.

$r = 3$ gives ${}^9C_3 \times (1)^{9-3} (x)^3$

Non-calculator, so work out ${}^nC_r$ separately:

$${}^9C_3 = \frac{9!}{3!(9-3)!} = \frac{9 \times 8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}}{(3 \times 2)(\cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2})}$$

$$= \frac{9 \times 8 \times 7}{6} = 84$$

so the term when $r = 3$ is $84 \times (1)^6 \times x^3$

$$= 84x^3$$

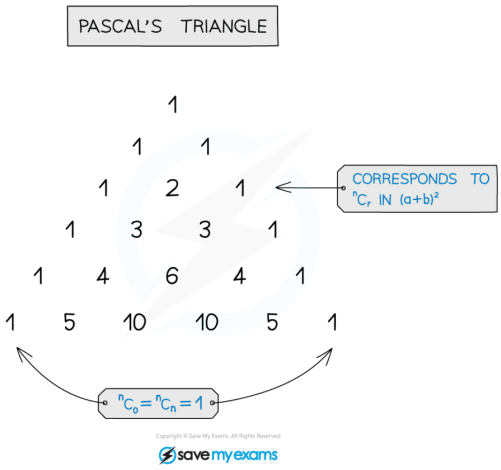
**Coefficient of $x^3$ = 84**

www.thrivingscholars.com

# Pascal's Triangle

## What is Pascal's Triangle?

- Pascal's triangle is a way of arranging the binomial coefficients and neatly shows how they are formed
  - Each term is formed by adding the two terms above it
  - The first row has just the number 1
  - Each row begins and ends with a number 1
  - From the third row the terms in between the 1s are the sum of the two terms above it




## How does Pascal's Triangle relate to the binomial theorem?

- Pascal's triangle is an alternative way of finding the binomial coefficients, ${}^{n}\mathrm{C}_{r}$
  - It can be useful for finding for smaller values of $n$ without a calculator
  - However for larger values of $n$ it is slow and prone to arithmetic errors
- Taking the first row as zero, $({}^{0}\mathrm{C}_{0} = 1)$, each row corresponds to the $n^{th}$ row and the term within that row corresponds to the $r^{th}$ term

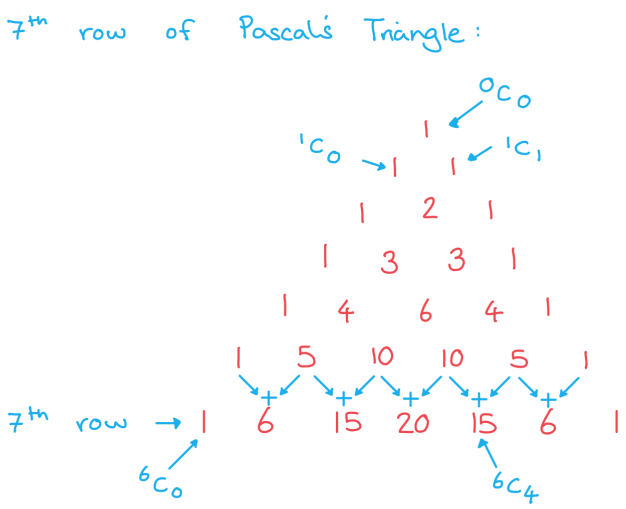
### Exam Tip

- In the non-calculator exam Pascal's triangle can be helpful if you need to get the coefficients of an expansion quickly, provided the value of *n* is not too big

www.thrivingscholars.com

## Worked Example

Write out the 7th row of Pascal's triangle and use it to find the value of ${}^{6}C_{4}$.




7th row of Pascal's Triangle: 1, 6, 15, 20, 15, 6, 1

${}^{6}C_{4} = 15$

www.thrivingscholars.com



# Binomial Theorem: Fractional & Negative Indices

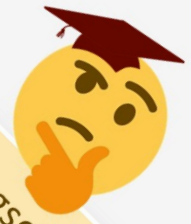
## How do I use the binomial theorem for fractional and negative indices?

- The formula given in the formula booklet for the binomial theorem applies to positive integers only
  - $(a+b)^n = a^n + {}^nC_1 a^{n-1}b + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n$
  - where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
- For **negative** or **fractional powers** the expression in the brackets must first be changed such that the value for *a* is 1
  - $(a+b)^n = a^n\left(1+\dfrac{b}{a}\right)^n$
  - $(a+b)^n = a^n\left(1+n\left(\dfrac{b}{a}\right)+\dfrac{n(n-1)}{2!}\left(\dfrac{b}{a}\right)^2+\ldots\right), n\in\mathbb{Q}$
  - This **is given in the formula booklet**
- If *a* = 1 and *b* = *x* the binomial theorem is simplified to
  - $(1+x)^n = 1+nx+\dfrac{n(n-1)}{2!}x^2+\dfrac{n(n-1)(n-2)}{3!}x^3+\ldots,\ n\in\mathbb{Q},\ |x|<1$
  - This is **not** in the formula booklet, you must remember it or be able to derive it from the formula given
- You need to be able to recognise a negative or fractional power
  - The expression may be on the denominator of a fraction
    - $\dfrac{1}{(a+b)^n} = (a+b)^{-n}$
  - Or written as a surd
    - $\sqrt[n]{(a+b)^m} = (a+b)^{\frac{m}{n}}$
- For $n \notin \mathbb{N}$ the expansion is infinitely long
  - You will usually be asked to find the first three terms
- The expansion is only valid for $|x|<1$
  - This means $-1<x<1$
  - This is known as the **interval of convergence**
  - For an expansion $(a+bx)^n$ the interval of convergence would be $-\dfrac{a}{b}<x<\dfrac{a}{b}$

## How do we use the binomial theorem to estimate a value?

- The binomial expansion can be used to form an approximation for a value raised to a power
- Since $|x|<1$ higher powers of *x* will be very small
  - Usually only the first three or four terms are needed to form an approximation
  - The more terms used the closer the approximation is to the true value
- The following steps may help you use the binomial expansion to approximate a value
  - STEP 1: Compare the value you are approximating to the expression being expanded

www.thrivingscholars.com



    - e.g. $(1 - x)^{\frac{1}{2}} = 0.96^{\frac{1}{2}}$
  - STEP 2: Find the value of *x* by solving the appropriate equation
    - e.g. $1 - x = 0.96$
      $x = 0.04$
  - STEP 3: Substitute this value of *x* into the expansion to find the approximation
    - e.g. $1 - \frac{1}{2}(0.04) - \frac{1}{8}(0.04)^2 = 0.9798$
- Check that the value of *x* is within the **interval of convergence** for the expression
  - If *x* is outside the interval of convergence then the approximation may not be valid

### Exam Tip

- Students often struggle with the extension of the binomial theorem questions in the exam, however the formula is given in the formula booklet
  - Make sure you can locate the formula easily and practice substituting values in
  - Mistakes are often made with negative numbers or by forgetting to use brackets properly
    - Writing one term per line can help with both of these

www.thrivingscholars.com

## Worked Example

Consider the binomial expansion of $\frac{1}{\sqrt{9-3x}}$.

a)
Write down the first three terms.

Rewrite $\frac{1}{\sqrt{9-3x}}$ in the form $k(1+\frac{x}{a})^n$

$$\frac{1}{\sqrt{9-3x}} = (9-3x)^{-\frac{1}{2}} = 9^{-\frac{1}{2}}\left(1-\frac{3x}{9}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{3}\left(1-\frac{x}{3}\right)^{-\frac{1}{2}}$$

Substitute values into the formula for $(1+x)^n$

$$\frac{1}{3}\left(1-\frac{x}{3}\right)^{-\frac{1}{2}} = \frac{1}{3}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{3}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{3}\right)^2+\ldots\right]$$

$$= \frac{1}{3}\left[1+\frac{x}{6}+\frac{x^2}{24}+\ldots\right]$$

$$= \frac{1}{3}+\frac{x}{18}+\frac{x^2}{72}+\ldots$$

$$\frac{1}{\sqrt{9-3x}} \approx \frac{1}{3}+\frac{x}{18}+\frac{x^2}{72}$$

b)
State the interval of convergence for the complete expansion.

$n \geq 0$ and $n \notin \mathbb{N}$, so the series converges when $|x| < 1$

$\frac{1}{3}\left(1-\frac{x}{3}\right)^{-\frac{1}{2}}$ ← $x$-term

$\left|-\frac{x}{3}\right| < 1$

$|x| < 3 \Rightarrow -3 < x < 3$

Converges for $-3 < x < 3$

c)
Use the terms found in part (a) to estimate $\frac{1}{\sqrt{10}}$. Give your answer as a fraction.

www.thrivingscholars.com

Find the value of $x$ for which $\frac{1}{\sqrt{9-3x}} = \frac{1}{\sqrt{10}}$

$9 - 3x = 10$

$x = -\frac{1}{3}$ ← $-3 < x < 3$ so can use the expansion

Substitute $x = -\frac{1}{3}$ into the expansion for $\frac{1}{\sqrt{9-3x}}$

$$\frac{1}{\sqrt{9-3\left(-\frac{1}{3}\right)}} \approx \frac{1}{3} + \frac{\left(-\frac{1}{3}\right)}{18} + \frac{\left(-\frac{1}{3}\right)^2}{72}$$

$$\frac{1}{\sqrt{10}} \approx \frac{205}{648}$$

www.thrivingscholars.com



# 1.7 Permutations & Combinations

## 1.7.1 Counting Principles

### Counting Principles

### What is meant by counting principles?

- The fundamental counting principle states that if there are $m$ ways to do one thing and $n$ ways to do another there are $m \times n$ ways to do **both** things
- Applying counting principles allows us to…
  - … analyse patterns and make generalisations about real work situations
  - … find the number of **permutations** of $n$ items
  - … find the number of ways of choosing an item from a list of $n$ items
  - … find the number of ways of choosing $r$ items from $n$ items
  - … find the number of ways of permutating $r$ items from $n$ items
- The topic of counting principles is a particularly interesting part of mathematics that can lead to the development of working with very large numbers
- It is always vital to consider whether objects taken from each list can be repeated or not
  - For example a four digit PIN from ten numbers where each number can be used repeatedly would be $10 \times 10 \times 10 \times 10$
    - There are 10 options for the first and ten options for the second number and so on
  - If the numbers could only be used once then the number of options for each digit would **reduce** with each digit
    - There are 10 options for the first, nine options for the second, eight for the third and so on
    - This concept will be explored further in the **permutations** revision note

### How do I choose an item from a list of $m$ items AND another item from a list of $n$ items?

- If a question requires you to choose an item from one list **AND** an item from another list you should **multiply** the number of options in each list
  - In general if you see the word 'AND' you will most likely need to 'MULTIPLY'
- For example if you are choosing a pen and a pencil from 4 pens and 5 pencils:
  - You can choose 1 item from 4 pens AND 1 item from 5 pencils
  - You will have $4 \times 5$ different options to choose from

### How do I choose an item from a list of $m$ items OR another item from a list of $n$ items?

- If a question requires you to choose an item from one list **OR** an item from another list you should **add** the number of options in each list
  - In general if you see the word 'OR' you will most likely need to 'ADD'
- For example if you are choosing a pen or a pencil from 4 pens and 5 pencils:
  - You can choose 1 item from 4 pens OR 1 item from 5 pencils
  - You will have $4 + 5$ different options to choose from

www.thrivingscholars.com

## Exam Tip

- Counting principles and factorials are tightly interlinked with permutations and combinations
- Make sure you fully understand the concepts in this revision note as they will be fundamental to answering perms and combs exam questions

## Worked Example

Harry is going to a formal event and is choosing what accessories to add to his outfit. He has seven different ties, four different bow ties and five different pairs of cufflinks. How many different ways can Harry get ready if he chooses:

a)
Either a tie, a bow tie or a pair of cufflinks?

Harry has $7 + 4 + 5$ different items to choose from
He wants a tie OR a bow tie OR a pair of cufflinks
OR means ADD $7 + 4 + 5 = 16$

**16 different ways**

b)
A pair of cufflinks and either a tie or a bow tie?

He wants a tie AND a pair of cufflinks
OR a bow tie AND a pair of cufflinks
AND means MULTIPLY
A tie AND cufflinks $= 7 \times 5 = 35$ ways
A bowtie AND cufflinks $= 4 \times 5 = 20$ ways
OR means ADD $35 + 20 = 55$ ways

**55 different ways**

www.thrivingscholars.com





# Permutations

## What are Permutations?

- A **permutation** is the number of possible arrangements of a set of objects when the **order** of the arrangements matters
- A permutation can either be finding the number of ways to arrange *n* items or finding the number of ways to arrange *r* out of *n* items

## How many ways can *n* different objects be arranged?

- When considering how many ways you can arrange a number of **different** objects in a row consider how many of the objects can go in the first position, how many can go in the second and so on
- For *n* = 2 there are two options for the first position and then there will only be one option to go in the second position so:
  - The first object has **two** places it could go and the second object has **one** place
  - By the fundamental counting principle **both** objects have 2 × 1 places to go
  - For example to arrange the letters A and B we have
    - AB and BA
- For *n* = 3 there are three options for the first position and then there will be two options for the second position and one for the third position so
  - The first object has **three** places it could go, the second object has **two** places and the third object has **one** place
  - By the fundamental counting principle **the three** objects have 3 × 2 × 1 places to go
  - For example to arrange the letters A, B and C we have
    - ABC, ACB, BAC, BCA, CAB and CBA
- For *n* objects there are *n* options for the first position, *n* - 1 options for the second position and so on until there is only one object left to go in final position
- The number of **permutations** of *n* different objects is *n* factorial (*n*!)
  - Where $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$
  - For 5 **different** items there are $5! = 5 \times 4 \times 3 \times 2 = 120$ permutations
  - For 6 **different** items there are $6! = 6 \times 5 \times 4 \times 3 \times 2 = 720$ permutations
  - It is easy to see how quickly the number of possible permutations of different items can increase
  - For 10 different items there are 10! = 3 628 800 possible permutations

## What are factorials?

- Factorials are a type of mathematical operation (just like +, -, ×, ÷)
- The symbol for factorial is !
  - So to take a factorial of any non-negative integer, *n*, it will be written *n*! and pronounced '*n factorial*'
- The factorial function for any positive integer, *n*, is $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$
  - For example, 5 factorial is $5! = 5 \times 4 \times 3 \times 2 \times 1$
- The factorial of a negative number is not defined
  - You cannot arrange a negative number of items
- 0! = 1

www.thrivingscholars.com



- There are no positive integers less than zero, so zero items can only be arranged once
- Your GDC will have a mode for calculating factorials, make sure you can put yours into the correct mode
- Most normal calculators cannot handle numbers greater than about 70!, experiment with yours to see the greatest value of *x* such that your calculator can handle *x*!

## What are the key properties of using factorials?

- Some important relationships to be aware of are:
  - $n! = n \times (n-1)!$
    - Therefore $\frac{n!}{(n-1)!} = n$
  - $n! = n \times (n-1) \times (n-2)!$
    - Therefore $\frac{n!}{(n-2)!} = n \times (n-1)$
- Expressions with factorials in can be simplified by considering which values cancel out in the fraction
  - Dividing a large factorial by a smaller one allows many values to cancel out
    - $\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = 8 \times 7 \times 6$

## How do we find *r* permutations of *n* items?

- If we only want to find the number of ways to arrange a few out of $n$ different objects, we should consider how many of the objects can go in the first position, how many can go in the second and so on
- If we wanted to arrange 3 out of 5 different objects, then we would have 3 positions to place the objects in, but we would have 5 options for the first position, 4 for the second and 3 for the third
  - This would be 5 × 4 × 3 ways of permutating 3 out of 5 different objects
  - This is equivalent to $\frac{5!}{2!} = \frac{5!}{(5-3)!}$
- If we wanted to arrange 4 out of 10 different objects, then we would have 4 positions to place the objects in, but we would have 10 options for the first position, 9 for the second, 8 for the third and 7 for the fourth
  - This would be 10 × 9 × 8 × 7 ways of permutating 4 out of 10 different objects
  - This is equivalent to $\frac{10!}{6!} = \frac{10!}{(10-4)!}$
- If we wanted to arrange *r* out of *n* different objects, then we would have *r* positions to place the objects in, but we would have *n* options for the first position, $(n-1)$ for the second, $(n-2)$ for the third and so on until we reach $(n-(r-1))$
  - This would be $n \times (n-1) \times \ldots \times (n-r+1)$ ways of permutating r out of n different objects
  - This is equivalent to $\frac{n!}{(n-r)!}$
- The function $\frac{n!}{(n-r)!}$ can be written as ${}^{n}\mathrm{P}_{r}$
  - Make sure you can find and use this button on your calculator

www.thrivingscholars.com



- The same function works if we have *n* spaces into which we want to arrange *r* objects, consider
  - for example arranging five people into a row of ten empty chairs

## Permutations when two or more items must be together

- If two or more items must stay together within an arrangement, it is easiest to think of these items as 'stuck' together
- These items will become one within the arrangement
- Arrange this 'one' item with the others as normal
- Arrange the items within this 'one' item separately
- Multiply these two arrangements together

## Permutations when two or more items cannot be all together

- If **two** items must be **separated …**
  - consider the number of ways these two items would be together
  - subtract this from the total number of arrangements without restrictions
- If **more than two** items must be separated…
  - consider whether all of them must be **completely separate** (none can be next to each other) or whether they **cannot all be together** (but two could still be next to each other)
  - If they **cannot all be together** then we can treat it the same way as separating two items and subtract the number of ways they would all be together from the total number of permutations of the items, the final answer will include all permutations where two items are still together
  - If the items must all be **completely separate** then
    - lay out the rest of the items in a line with a space in between each of them where one of the items which cannot be together could go
    - remember that this could also include the space before the first and after the last item
    - You would then be able to fit the items which cannot be together into any of these spaces, using the *r* permutations of *n* items rule $\left({}^{n}\mathrm{P}_{r}\right)$
    - You do not need to fill every space

## Permutations when two or more items must be in specific places

- Most commonly this would be arranging a word where specific letters would go in the first and last place
- Or arranging objects where specific items have to be at the ends/in the middle
  - Imagine these specific items are stuck in place, then you can find the number of ways to arrange the rest of the items around these 'stuck' items
- Sometimes the items must be grouped
  - for example all vowels must be before the consonants
  - Or all the red objects must be on one side and the blue objects must be on the other
  - Find the number of permutations within each group separately and multiply them together
  - Be careful to check whether the groups could be in either place
    - e.g. the vowels on one side and consonants on the other
    - or if they must be in specific places (the vowels **before** the consonants)
  - If the groups could be in either place than your answer would be multiplied by two

www.thrivingscholars.com

- If there were $n$ groups that could be in any order then you're answer would be multiplied by $n!$

## Exam Tip

- The wording is very important in permutations questions, just one word can change how you answer the question
- Look out for specific details such as whether three items must all be separated or just cannot be all together (there is a difference)
- Pay attention to whether items must be in alternating order (e.g. red and blue items must alternate, either RBRB… or BRBR…) or whether a particular item must come first (red **then** blue and so on)
- If items should be at the ends, look out for whether they can be at either end or whether one must be at the beginning and the other at the end

## Worked Example

Find the number of ways nine different tasks can be carried out given that two particular tasks must not be carried out consecutively.

Start by considering the tasks that can be carried out consecutively:

$T_1\ T_2\ T_3\ T_4\ T_5\ T_6\ T_7$

There are $7!$ ways of carrying out the other seven tasks.

Then consider the tasks with restrictions:

$\times\ T_1 \times T_2 \times T_3 \times T_4 \times T_5 \times T_6 \times T_7 \times$ positions where $T_8$ or $T_9$ could go

There are 8 positions in which the two tasks could go but only 2 tasks to fill the spaces $= 8 \times 7$ options

$= {}^8P_2$

(8 options for $T_8$ and 7 options for $T_9$)

**Total $= 7! \times 8 \times 7 = 282240$ ways**

Alternative method:

Put the two tasks together and then subtract from the total.

$T_1\ T_2\ T_3\ T_4\ T_5\ T_6\ T_7\ (T_8\ T_9)$

The two tasks are 'stuck' together.

number of ways without restrictions

$9! - (8! \times 2!)$

7 'free' tasks plus the two tasks 'stuck' together.

There are $2!$ ways of carrying out the two 'stuck' tasks

www.thrivingscholars.com

# Combinations

## What is the difference between permutations and combinations?

- A **combination** is the number of possible arrangements of a set of objects when the **order** of the arrangements **does not matter**
  - On the other hand a **permutation** is when the order of arrangement **does matter**
- A combination will be finding the number of ways to **choose** *r* out of *n* items
  - The order in which the *r* items are chosen is not important
  - For example if we are choosing two letters from the word CAB, AB and BA would be considered the same combination but different permutations

## How do we find *r* combinations of *n* items?

- If we want to find the number of ways to **choose** 2 out of 3 different objects, but we don't mind the order in which they are chosen, then we could find the number of **permutations of 2 items from 3** and then divide by the number of ways of arranging each combination
  - For example if we want to choose 2 letters from *A*, *B* and *C*
    - There are 6 permutations of 2 letters:
    - *AB*, *BA*, *AC*, *CA*, *BC*, *CB*
    - For each combination of 2 letters there are 2 (2 × 1) ways of arranging them
    - (for example, *AB* and *BA*)
    - So divide the total number of permutations (6) by the number of ways of arranging each combination (2) to get 3 combinations

- If we want to find the number of ways to **choose** 3 out of 5 different objects, but we don't mind the order in which they are chosen, then we could find the number of **permutations of 3 items from 5** and then divide by the number of ways of arranging each combination
  - For example if we want to choose 3 letters from *A*, *B*, *C*, *D* and *E*
    - There are 60 permutations of 3 letters:
    - *ABC*, *ACB*, *BAC*, *BCA*, *CAB*, *CBA*, *ABD*, *ADB*, etc
    - For each combination of 3 letters there are 6 (3 × 2 ×1) ways of arranging them (for example, *ABC*, *ACB*, *BAC*, *BCA*, *CAB* and *CBA*)
    - So divide the total number of permutations (60) by the number of ways of arranging each combination (3! = 6) to get 10 combinations

- If we want to find the number of ways to **choose** *r* items out of *n* different objects, but we don't mind the order in which they are chosen, then we could find the number of **permutations of *r* items from *n*** and then divide by the number of ways of arranging each combination
- Recall that the formula for *r* permutations of *n* items is
  - $^{n}\mathrm{P}_{r} = \dfrac{n!}{(n-r)!}$
- This would include *r*! ways of repeating each combination
- The formula for *r* **combinations** of *n* items is
  - $\dfrac{^{n}\mathrm{P}_{r}}{r!} = \dfrac{n!}{(n-r)!\,r!}$
- The function $\dfrac{n!}{(n-r)!\,r!}$ can be written as $^{n}\mathrm{C}_{r}$ or $\binom{n}{r}$ and is often read as '*n choose r*'

www.thrivingscholars.com



- Make sure you can find and use this button on your calculator
- The formulae for permutations and combinations satisfy the following relationship:
  - $^{n}\mathrm{C}_{r} = \frac{^{n}\mathrm{P}_{r}}{r!}$
- The formula $^{n}\mathrm{C}_{r} = \frac{n!}{r!(n-r)!}$ can be found in the formula booklet

## What do I need to know about combinations?

- The formula $^{n}C_{r} = \frac{n!}{(n-r)!r!}$ is also known as a **binomial** coefficient
- $^{n}C_{n} = {}^{n}C_{0} = 1$
  - It is easy to see that there is only one way of arranging *n* objects out of *n* and also there can only be one way of arranging 0 objects out of *n*
  - By considering the formula for this, it reinforces the fact that 0! Must equal 1
- The binomial coefficients are symmetrical, so $^{n}C_{r} = {}^{n}C_{n-r}$
  - This can be seen by considering the formula for $^{n}C_{r}$
  - $^{n}\mathrm{C}_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{r!(n-r)!} = {}^{n}\mathrm{C}_{r}$

## How do I know when to multiply or add?

- Many questions will ask you to find combinations of a group of different items from a bigger group of a specified number of those different items
  - For example, find the number of ways five questions could be chosen from a bank of twenty different pure and ten different statistics questions
  - The hint in this example is the word 'chosen', this tells you that the order in which the questions are chosen doesn't matter
- Sometimes questions will have restrictions,
  - For example there should be three pure and two statistics chosen from the bank of questions,
  - Or there must be at least two pure questions within the group
- If unsure about whether to add or multiply your options, ask yourself if A **and** B are both needed, or if A **or** B is needed
  - Always **multiply** if the answer is **and,** and **add** if the answer is **or**
  - For example if we needed exactly three pure **and** two statistics questions we would find the amount of each and multiply them
  - If we could have either five statistics **or** five pure questions we would find them separately and add the answers

www.thrivingscholars.com

### Exam Tip

- It is really important that you can tell whether a question is about permutations or combinations
  - Look out for key words such as **arrange** (for permutations) or **choose** or **select** (for combinations)
- Don't be confused if a question asks for the number of **ways**, this could be for either a permutations or a combinations question
  - Look out for other clues

www.thrivingscholars.com

## Worked Example

Oscar has to choose four books from a reading list to take home over the summer. There are four fantasy books, five historical fiction books and two classics available for him to choose from. In how many ways can Oscar choose four books if he decides to have:

i)
Two fantasy books and two historical fictions?

Choosing two fantasy from four: $4C2$

Choosing two historical fiction from five: $5C2$

Total = $4C2 \times 5C2 = 6 \times 10$

**60 options**

ii)
At least one of each type of book?

To choose two of one and one of each of the others:
Let F represent a fantasy book, H represent historical fiction and C represent a classic:

| | F | H | C | |
|---|---|---|---|---|
| | 4 | 5 | 2 | these are the numbers of options Oscar can choose from |
| these are the choices he can make | 1 | 1 | 2 | $4C1 \times 5C1 \times 2C2 = 4 \times 5 \times 1$ |
| | 1 | 2 | 1 | $4C1 \times 5C2 \times 2C1 = 4 \times 10 \times 2$ |
| | 2 | 1 | 1 | $4C2 \times 5C1 \times 2C1 = 6 \times 5 \times 2$ |

'and' so multiply.

Oscar can choose from one of these options so we add them:

$20 + 80 + 60$ ('or' so add)

**160 options**

iii)
At least two fantasy books?

www.thrivingscholars.com

Oscar can choose two, three or four fantasy books. It does not matter whether the other books are classics or historical fiction.

This is just 'Not Fantasy' (5H + 2C)

| F | NF | |
|---|---|---|
| 2 | 2 | $4C2 \times 7C2 = 6 \times 21 = 126$ |
| 3 | 1 | $4C3 \times 7C1 = 4 \times 7 = 28$ |
| 4 | 0 | $4C4 \times 7C0 = 1 \times 1 = 1$ |

↑ Use a system to make sure you don't forget any. Check that all the options add up to 4.

Total number of options = 126 + 28 + 1

**155 Options**

www.thrivingscholars.com

# 1.8 Complex Numbers

## 1.8.1 Intro to Complex Numbers

### Cartesian Form

### What is an imaginary number?

- Up until now, when we have encountered an equation such as $x^2 = -1$ we would have stated that there are "no real solutions"
  - The solutions are $x = \pm\sqrt{-1}$ which are not real numbers
- To solve this issue, mathematicians have defined one of the square roots of negative one as $\mathrm{i}$; an imaginary number
  - $\sqrt{-1} = \mathrm{i}$
  - $\mathrm{i}^2 = -1$
- The square roots of other negative numbers can be found by rewriting them as a multiple of $\sqrt{-1}$
  - using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

### What is a complex number?

- Complex numbers have both a real part and an imaginary part
  - For example: $3 + 4\mathrm{i}$
  - The real part is 3 and the imaginary part is 4i
- Complex numbers are often denoted by $z$
  - We refer to the real and imaginary parts respectively using $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$
- Two complex numbers are equal if, and only if, both the real and imaginary parts are identical.
  - For example, $3 + 2\mathrm{i}$ and $3 + 3\mathrm{i}$ are **not equal**
- The set of all complex numbers is given the symbol $\mathbb{C}$

### What is Cartesian Form?

- There are a number of different forms that complex numbers can be written in
- The form $z = a + b\mathrm{i}$ is known as **Cartesian Form**
  - $a, b \in \mathbb{R}$
  - This is the first form given in the formula booklet
- In general, for $z = a + b\mathrm{i}$
  - Re(z) = $a$
  - Im(z) = $b$
- A complex number can be easily represented geometrically when it is in Cartesian Form
- Your GDC may call this **rectangular form**
  - When your GDC is set in rectangular settings it will give answers in Cartesian Form
  - If your GDC is **not** set in a complex mode it will not give any output in complex number form
  - Make sure you can find the settings for using complex numbers in Cartesian Form and practice inputting problems
- Cartesian form is the easiest form for adding and subtracting complex numbers

www.thrivingscholars.com

## Exam Tip

- Remember that complex numbers have both a real part and an imaginary part
  - 1 is purely real (its imaginary part is zero)
  - i is purely imaginary (its real part is zero)
  - 1 + i is a complex number (both the real and imaginary parts are equal to 1)

## Worked Example

a)
Solve the equation $x^2 = -9$

$x^2 = -9$
$x = \pm\sqrt{-9}$
Using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ $x = \pm\sqrt{9}\sqrt{-1}$
$x = \pm 3i$

b)
Solve the equation $(x+7)^2 = -16$, giving your answers in Cartesian form.

$(x+7)^2 = -16$
$x + 7 = \pm\sqrt{-16}$
$x + 7 = \pm\sqrt{16}\sqrt{-1}$
$x + 7 = \pm 4i$ ← Using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
Rearrange answer into Cartesian form:
$x = -7 \pm 4i$

www.thrivingscholars.com



# Complex Addition, Subtraction & Multiplication

## How do I add and subtract complex numbers in Cartesian Form?

- Adding and subtracting complex numbers should be done when they are in **Cartesian form**
- When adding and subtracting complex numbers, simplify the real and imaginary parts separately
  - Just like you would when collecting like terms in algebra and surds, or dealing with different components in vectors
  - $(a+b\mathrm{i})+(c+d\mathrm{i})=(a+c)+(b+d)\mathrm{i}$
  - $(a+b\mathrm{i})-(c+d\mathrm{i})=(a-c)+(b-d)\mathrm{i}$

## How do I multiply complex numbers in Cartesian Form?

- Complex numbers can be multiplied by a constant in the same way as algebraic expressions:
  - $k(a+b\mathrm{i})=ka+kb\mathrm{i}$
- Multiplying two complex numbers in Cartesian form is done in the same way as multiplying two linear expressions:
  - $(a+b\mathrm{i})(c+d\mathrm{i})=ac+(ad+bc)\mathrm{i}+bd\mathrm{i}^2=ac+(ad+bc)\mathrm{i}-bd$
  - This is a complex number with real part $ac-bd$ and imaginary part $ad+bc$
  - The most important thing when multiplying complex numbers is that
    - $\mathrm{i}^2=-1$
- Your GDC will be able to multiply complex numbers in Cartesian form
  - Practise doing this and use it to check your answers
- It is easy to see that multiplying more than two complex numbers together in Cartesian form becomes a lengthy process prone to errors
  - It is easier to multiply complex numbers when they are in different forms and usually it makes sense to convert them from Cartesian form to either Polar form or Euler's form first
- Sometimes when a question describes multiple complex numbers, the notation $z_1$, $z_2$, ... is used to represent each complex number

## How do I deal with higher powers of i?

- Because $\mathrm{i}^2=-1$ this can lead to some interesting results for higher powers of i
  - $\mathbf{i}^3=\mathbf{i}^2\times\mathbf{i}=-\mathbf{i}$
  - $\mathbf{i}^4=(\mathbf{i}^2)^2=(-1)^2=1$
  - $\mathbf{i}^5=(\mathbf{i}^2)^2\times\mathbf{i}=\mathbf{i}$
  - $\mathbf{i}^6=(\mathbf{i}^2)^3=(-1)^3=-1$
- We can use this same approach of using $\mathrm{i}^2$ to deal with much higher powers
  - $\mathbf{i}^{23}=(\mathbf{i}^2)^{11}\times\mathbf{i}=(-1)^{11}\times\mathbf{i}=-\mathbf{i}$
  - Just remember that -1 raised to an even power is 1 and raised to an odd power is -1

www.thrivingscholars.com

## Exam Tip

- When revising for your exams, practice using your GDC to check any calculations you do with complex numbers by hand
  - This will speed up using your GDC in rectangular form whilst also giving you lots of practice of carrying out calculations by hand

## Worked Example

a)

Simplify the expression $2(8-6i)-5(3+4i)$.

Expand the brackets

$2(8-6i)-5(3+4i) = 16-12i-15-20i$

Collect the real and imaginary parts

$16-15-12i-20i$

Simplify

$1-32i$

b)

Given two complex numbers $z_1 = 3+4i$ and $z_2 = 6+7i$, find $z_1 \times z_2$.

Expand the brackets

$(3+4i)(6+7i) = 18+21i+24i+28i^2$

$= 18+21i+24i+(28)(-1)$ ← Using $i^2 = -1$

Collect the real and imaginary parts

$18+21i+24i-28 = 18-28+(21+24)i$

Simplify

$-10+45i$

www.thrivingscholars.com

# Complex Conjugation & Division

When **dividing** complex numbers, the **complex conjugate** is used to change the denominator to a real number.

## What is a complex conjugate?

- For a given complex number $z = a + b\text{i}$, **the complex conjugate of $z$ is denoted as $z^*$**, where $z^* = a - b\text{i}$
- If $z = a - b\text{i}$ then $z^* = a + b\text{i}$
- You will find that:
  - $z + z^*$ is always real because $(a + b\text{i}) + (a - b\text{i}) = 2a$
    - For example: $(6 + 5\text{i}) + (6 - 5\text{i}) = 6 + 6 + 5\text{i} - 5\text{i} = 12$
  - $z - z^*$ is always imaginary because $(a + b\text{i}) - (a - b\text{i}) = 2b\text{i}$
    - For example: $(6 + 5\text{i}) - (6 - 5\text{i}) = 6 - 6 + 5\text{i} - (-5\text{i}) = 10\text{i}$
  - $z \times z^*$ is always real because $(a + b\text{i})(a - b\text{i}) = a^2 + ab\text{i} - ab\text{i} - b^2\text{i}^2 = a^2 + b^2$ (as $\text{i}^2 = -1$)
    - For example: $(6 + 5\text{i})(6 - 5\text{i}) = 36 + 30\text{i} - 30\text{i} - 25\text{i}^2 = 36 - 25(-1) = 61$

## How do I divide complex numbers?

- To divide two complex numbers:
  - STEP 1: Express the calculation in the form of a fraction
  - STEP 2: Multiply **the top and bottom by the conjugate of the denominator**:
    - $\dfrac{a + b\text{i}}{c + d\text{i}} = \dfrac{a + b\text{i}}{c + d\text{i}} \times \dfrac{c - d\text{i}}{c - d\text{i}}$
    - This ensures we are multiplying by 1; so not affecting the overall value
  - STEP 3: Multiply out and simplify your answer
    - This should have a real number as the denominator
  - STEP 4: Write your answer in Cartesian form as two terms, simplifying each term if needed
    - OR convert into the required form if needed
- Your GDC will be able to divide two complex numbers in Cartesian form
  - Practise doing this and use it to check your answers if you can

### Exam Tip

- We can speed up the process for finding $zz^*$ by using the basic pattern of $(x + a)(x - a) = x^2 - a^2$
- We can apply this to complex numbers: $(a + b\text{i})(a - b\text{i}) = a^2 - b^2\text{i}^2 = a^2 + b^2$ (using the fact that $\text{i}^2 = -1$)
  - So $3 + 4\text{i}$ multiplied by its conjugate would be $3^2 + 4^2 = 25$

www.thrivingscholars.com

## Worked Example

Find the value of $(1+7i) \div (3-i)$.

Rewrite as a fraction: $\frac{1+7i}{3-i}$ ← complex conjugate of $3-i$ is $3+i$

Multiply top and bottom of the fraction by the complex conjugate of the denominator.

$$\frac{1+7i}{3-i} \times \frac{3+i}{3+i} = \frac{(1+7i)(3+i)}{(3-i)(3+i)}$$

$$= \frac{3+i+21i+7i^2}{9+3i-3i-i^2} \quad (i^2 = -1)$$

← the imaginary parts eliminate each other

$$= \frac{3+22i+(-7)}{9-(-1)}$$

Simplify $= \frac{-4+22i}{10}$

Write in Cartesian form $= -\frac{4}{10} + \frac{22}{10}i$

$\boxed{-\frac{2}{5} + \frac{11}{5}i}$ Simplify final answer.

www.thrivingscholars.com



# Modulus & Argument

## How do I find the modulus of a complex number?

- The modulus of a complex number is its **distance** from the origin when plotted on an Argand diagram
- The modulus of $z$ is written $|z|$
- If $z = x + \mathrm{i}y$, then we can use **Pythagoras** to show...
  - $|z| = \sqrt{x^2 + y^2}$
- A modulus is **never negative**

## What features should I know about the modulus of a complex number?

- the modulus is related to the complex **conjugate** by...
  - $zz^* = z^*z = |z|^2$
  - This is because $zz^* = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2$
- In general, $|z_1 + z_2| \neq |z_1| + |z_2|$
  - e.g. both $z_1 = 3 + 4\mathrm{i}$ and $z_2 = -3 + 4\mathrm{i}$ have a modulus of 5, but $z_1 + z_2$ simplifies to $8\mathrm{i}$ which has a modulus of 8

## How do I find the argument of a complex number?

- The argument of a complex number is the **angle** that it makes on an **Argand diagram**
  - The angle must be taken from the **positive real axis**
  - The angle must be in a **counter-clockwise** direction
- Arguments are measured in **radians**
  - They can be given exact in terms of π
- The argument of $z$ is written $\arg z$
- Arguments can be calculated using right-angled **trigonometry**
  - This involves using the tan ratio plus a sketch to decide whether it is positive/negative and acute/obtuse

## What features should I know about the argument of a complex number?

- Arguments are usually given in the range $-\pi < \arg z \leq \pi$
  - Negative arguments are for complex numbers in the third and fourth quadrants
  - Occasionally you could be asked to give arguments in the range $0 < \arg z \leq 2\pi$
    - The question will make it clear which range to use
- The argument of zero, $\arg 0$ is undefined (no angle can be drawn)

## What are the rules for moduli and arguments under multiplication and division?

- When two complex numbers, $z_1$ and $z_2$, are **multiplied** to give $z_1 z_2$, their **moduli** are also **multiplied**
  - $|z_1 z_2| = |z_1||z_2|$

www.thrivingscholars.com



- When two complex numbers, $z_1$ and $z_2$, are **divided** to give $\frac{z_1}{z_2}$, their **moduli** are also **divided**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- When two complex numbers, $z_1$ and $z_2$, are **multiplied** to give $z_1 z_2$, their **arguments** are **added**
  - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- When two complex numbers, $z_1$ and $z_2$, are **divided** to give $\frac{z_1}{z_2}$, their **arguments** are **subtracted**



## Exam Tip

- Always draw a quick sketch to help you see what quadrant the complex number lies in when working out an argument
- Look for the range of values within which you should give your argument
  - If it is $-\pi < \arg z \leq \pi$ then you may need to measure it in the negative direction
  - If it is $0 < \arg z \leq 2\pi$ then you will always measure in the positive direction (counter - clockwise)

www.thrivingscholars.com

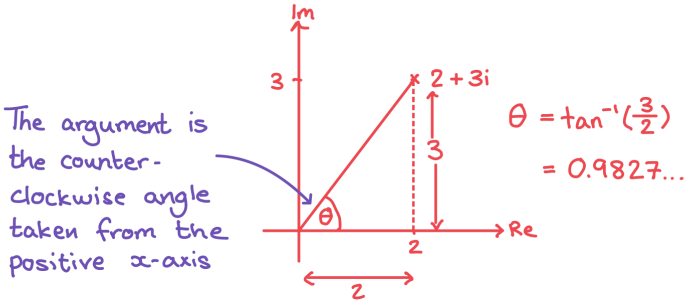
## Worked Example

a)

Find the modulus and argument of $z = 2 + 3i$

$|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$

Draw a sketch to help find the argument:



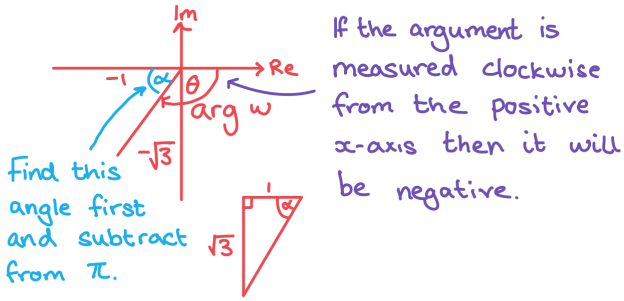

**Mod z = |z| = √13**

**arg z = θ = 0.983 (3sf)**

b)

Find the modulus and argument of $w = -1 - \sqrt{3}i$

$|w| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{4}$




$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

**Mod z = |z| = 2**

**arg z = −θ = $-\frac{2\pi}{3}$**

www.thrivingscholars.com





# Argand Diagrams

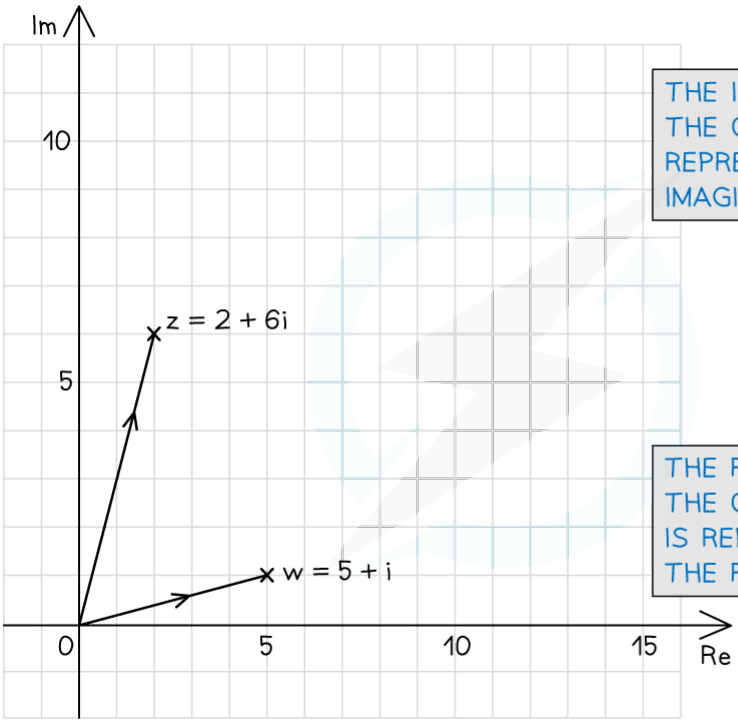
## What is the complex plane?

- The complex plane, sometimes also known as the Argand plane, is a two-dimensional plane on which complex numbers can be represented geometrically
- It is similar to a two-dimensional Cartesian coordinate grid
  - The $x$-axis is known as the **real** axis (Re)
  - The $y$-axis is known as the **imaginary** axis (Im)
- The complex plane emphasises the fact that a complex number is two dimensional
  - i.e it has two parts, a real and imaginary part
  - Whereas a real number only has one dimension represented on a number line (the $x$-axis only)

## What is an Argand diagram?

- An Argand diagram is a geometrical representation of complex numbers on a **complex plane**
  - A complex number can be represented as either a point or a vector
- The complex number $x + yi$ is represented by the point with cartesian coordinate $(x, y)$
  - The **real** part is represented by the point on the **real** ($x$-) axis
  - The **imaginary** part is represented by the point on the **imaginary** ($y$-) axis
- Complex numbers are often represented as **vectors**
  - A line segment is drawn from the origin to the cartesian coordinate point
  - An arrow is added in the direction away from the origin
  - This allows for geometrical representations of complex numbers

www.thrivingscholars.com




Copyright © Save My Exams. All Rights Reserved

## Exam Tip

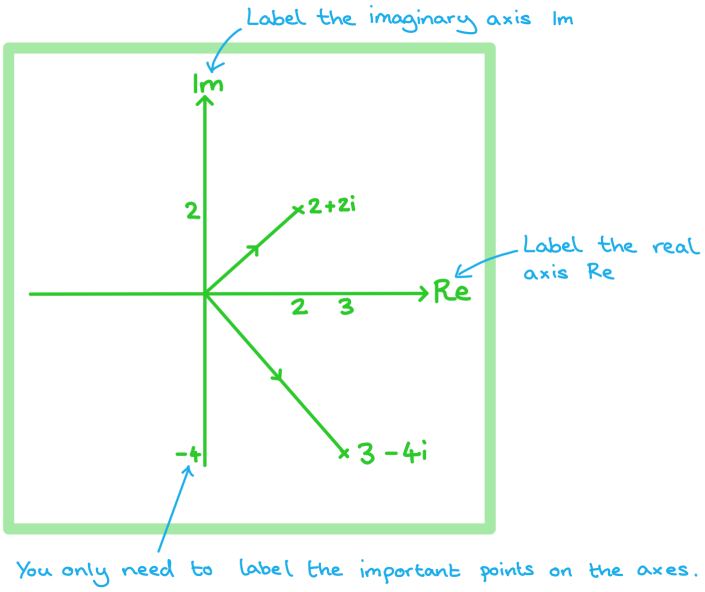
- When setting up an Argand diagram you do not need to draw fully scaled axes, you only need the essential information for the points you want to show, this will save a lot of time

www.thrivingscholars.com

## Worked Example

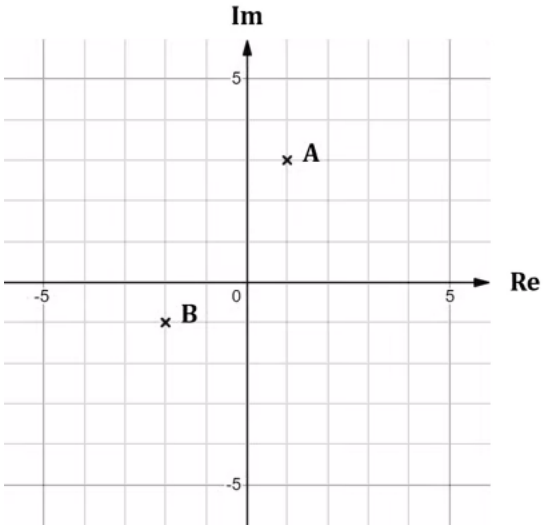
a)

Plot the complex numbers $z_1 = 2 + 2i$ and $z_2 = 3 - 4i$ as points on an Argand diagram.




b)

Write down the complex numbers represented by the points A and B on the Argand diagram below.




A: $1 + 3i$
B: $-2 - i$

www.thrivingscholars.com

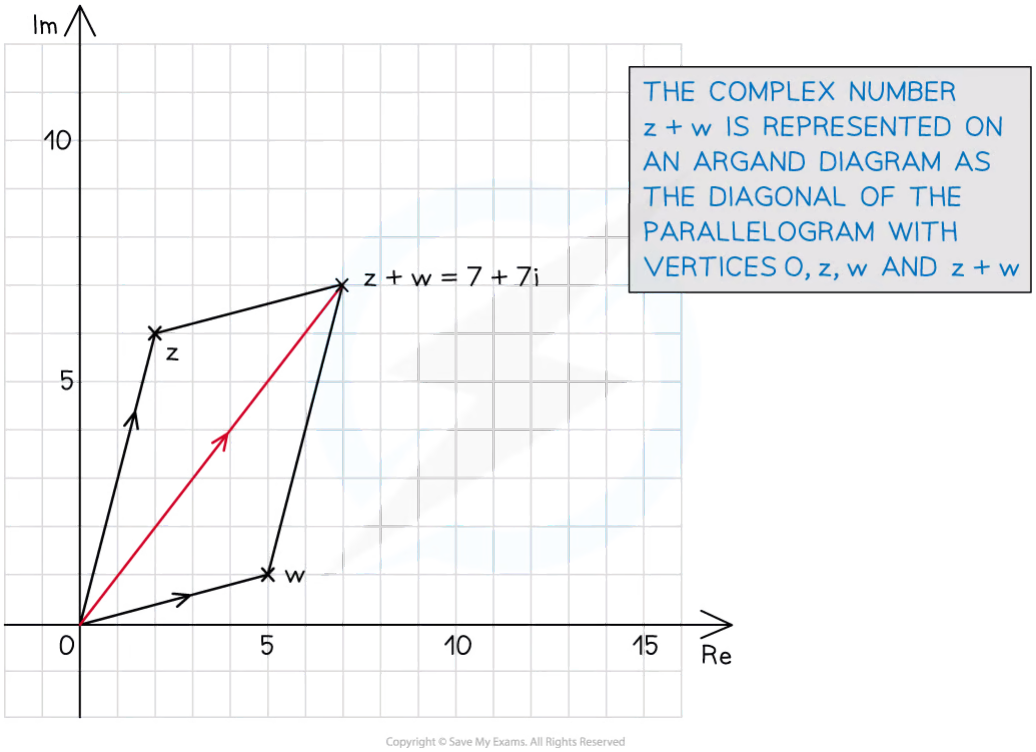
# 1.9 Further Complex Numbers

## 1.9.1 Geometry of Complex Numbers

### Geometry of Complex Addition & Subtraction

### What does addition look like on an Argand diagram?

- In Cartesian form two complex numbers are added by adding the real and imaginary parts
- When plotted on an Argand diagram the complex number $z_1 + z_2$ is the longer diagonal of the parallelogram with vertices at the origin, $z_1$, $z_2$ and $z_1 + z_2$

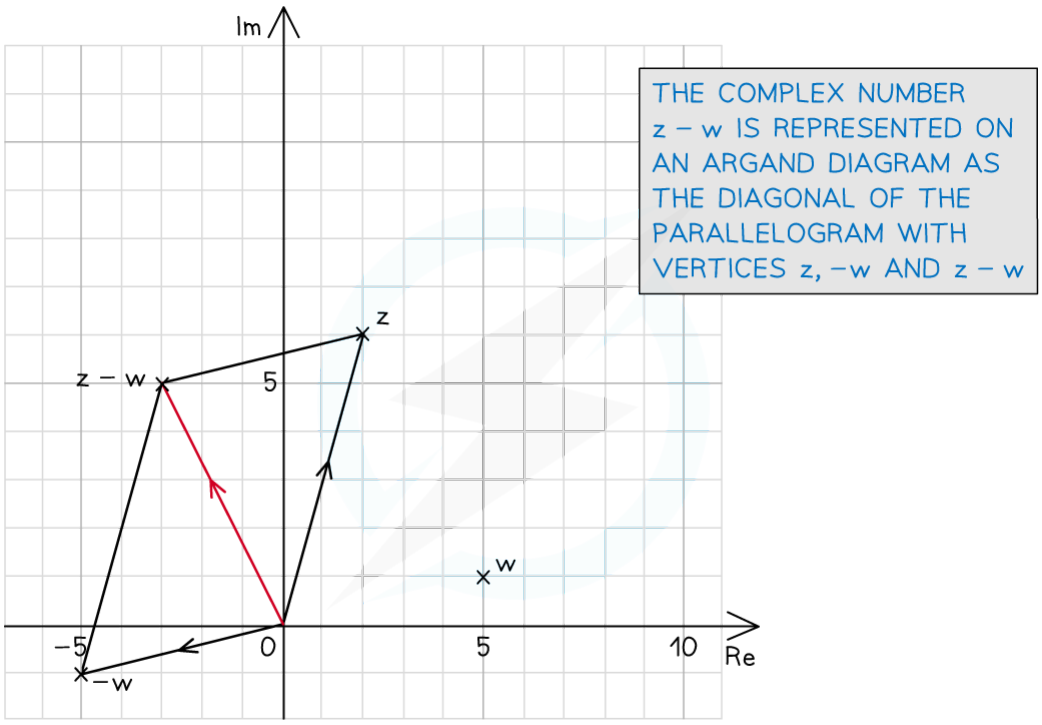



Copyright © Save My Exams. All Rights Reserved

### What does subtraction look like on an Argand diagram?

- In Cartesian form the difference of two complex numbers is found by subtracting the real and imaginary parts
- When plotted on an Argand diagram the complex number $z_1 - z_2$ is the shorter diagonal of the parallelogram with vertices at the origin, $z_1$, $-z_2$ and $z_1 - z_2$

www.thrivingscholars.com




REMEMBER TO PLOT THE POINT −w BEFORE DRAWING THE PARALLELOGRAM

Copyright © Save My Exams. All Rights Reserved

## What are the geometrical representations of complex addition and subtraction?

- Let $w$ be a given complex number with **real part** $a$ and **imaginary part** $b$
  - $w = a + b\mathrm{i}$
- Let $z$ be any complex number represented on an Argand diagram
- **Adding $w$ to $z$** results in $z$ being:
  - **Translated** by vector $\begin{pmatrix} a \\ b \end{pmatrix}$
- **Subtracting $w$ from $z$** results in $z$ being:
  - **Translated** by vector $\begin{pmatrix} -a \\ -b \end{pmatrix}$

### Exam Tip

- Take extra care when representing a subtraction of a complex number geometrically
  - Remember that your answer will be a translation of the shorter diagonal of the parallelogram made up by the two complex numbers

www.thrivingscholars.com

## Worked Example

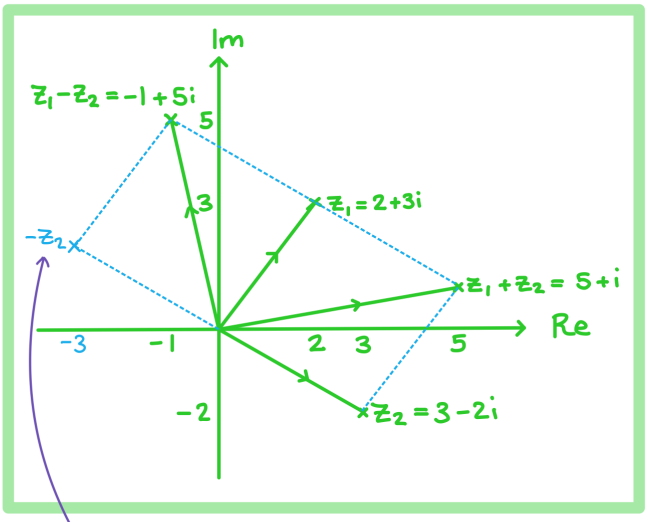
Consider the complex numbers $z_1 = 2 + 3i$ and $z_2 = 3 - 2i$.

On an Argand diagram represent the complex numbers $z_1, z_2, z_1 + z_2$ and $z_1 - z_2$.

First find $z_1+z_2$ and $z_1 - z_2$ :

$z_1 + z_2 = (2+3i) + (3-2i) = 5 + i$

$z_1 - z_2 = (2+3i) - (3-2i) = -1 + 5i$




The geometrical properties can be seen by adding in $-z_2 = -3 + 2i$

www.thrivingscholars.com

# Geometry of Complex Multiplication & Division

## What do multiplication and division look like on an Argand diagram?

- The geometrical effect of multiplying a complex number by a real number, $a$, will be an enlargement of the vector by scale factor $a$
  - For positive values of $a$ the direction of the vector will not change but the distance of the point from the origin will increase by scale factor $a$
  - For negative values of $a$ the direction of the vector will change and the distance of the point from the origin will increase by scale factor $a$
- The geometrical effect of dividing a complex number by a real number, $a$, will be an enlargement of the vector by scale factor $1/a$
  - For positive values of $a$ the direction of the vector will not change but the distance of the point from the origin will increase by scale factor $1/a$
  - For negative values of $a$ the direction of the vector will change and the distance of the point from the origin will increase by scale factor $1/a$
- The geometrical effect of multiplying a complex number by i will be a rotation of the vector 90° counter-clockwise
  - $i(x + yi) = -y + xi$
- The geometrical effect of multiplying a complex number by an imaginary number, $ai$, will be a rotation 90° counter-clockwise and an enlargement by scale factor $a$
  - $ai(x + yi) = -ay + axi$
- The geometrical effect of multiplying or dividing a complex number by a complex number will be an enlargement and a rotation
  - The direction of the vector will change
    - The angle of rotation is the **argument**
  - The distance of the point from the origin will change
    - The scale factor is the **modulus**

## What does complex conjugation look like on an Argand diagram?

- The geometrical effect of plotting a **complex conjugate** on an Argand diagram is a reflection in the real axis
  - The **real** part of the complex number will stay the same and the **imaginary** part will change sign

### Exam Tip

- Make sure you remember the transformations that different operations have on complex numbers, this could help you check your calculations in an exam

www.thrivingscholars.com

## Worked Example

Consider the complex number $z = 2 - i$.

On an Argand diagram represent the complex numbers z, 3z, iz, $z^*$ and $zz^*$.

First find 3z, iz and $z^*$

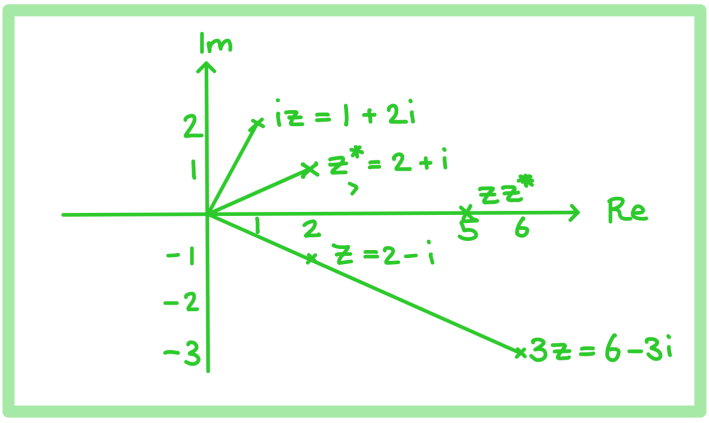
$z = 2 - i$

$3z = 3(2-i) = 6 - 3i$

$iz = i(2-i) = 2i - i^2 = 2i - (-1) = 1 + 2i$

$z^* = 2 + i$

$zz^* = (2-i)(2+i) = 4 - i^2 = 4 - (-1) = 5$



www.thrivingscholars.com



# Modulus-Argument (Polar) Form

## How do I write a complex number in modulus-argument (polar) form?

- The **Cartesian form** of a complex number, $z = x + \mathrm{i}y$, is written in terms of its real part, $x$, and its imaginary part, $y$
- If we let $r = |z|$ and $\theta = \arg z$, then it is possible to write a complex number in terms of its modulus, $r$, and its argument, $\theta$, called the **modulus-argument (polar) form**, given by...
  - $z = r(\cos\theta + \mathrm{i}\sin\theta)$
  - This is often written as $z = r\,\mathrm{cis}\,\theta$
  - This is given in the formula book under Modulus-argument (polar) form and exponential (Euler) form
- It is usual to give arguments in the range $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$
  - Negative arguments should be shown clearly
  - e.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right) = 2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$
    - without simplifying $\cos\left(-\frac{\pi}{3}\right)$ to either $\cos\left(\frac{\pi}{3}\right)$ or $\frac{1}{2}$
- The **complex conjugate** of $r\,\mathrm{cis}\,\theta$ is $r\,\mathrm{cis}\,(-\theta)$
- If a complex number is given in the form $z = r(\cos\theta - \mathrm{i}\sin\theta)$, then it is not in modulus-argument (polar) form due to the minus sign
  - It can be converted by considering transformations of trigonometric functions
    - $-\sin\theta = \sin(-\theta)$ and $\cos\theta = \cos(-\theta)$
  - So $z = r(\cos\theta - \mathrm{i}\sin\theta) = z = r(\cos(-\theta) + \mathrm{i}\sin(-\theta)) = r\,\mathrm{cis}\,(-\theta)$
- To convert from modulus-argument (polar) form back to Cartesian form, evaluate the real and imaginary parts
  - E.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ becomes $z = 2\left(\frac{1}{2} + \mathrm{i}\left(-\frac{\sqrt{3}}{2}\right)\right) = 1 - \sqrt{3}\,\mathrm{i}$

## How do I multiply complex numbers in modulus-argument (polar) form?

- The main benefit of writing complex numbers in modulus-argument (polar) form is that they multiply and divide very easily
- To **multiply** two complex numbers in modulus-argument (polar) form we **multiply their moduli** and **add their arguments**
  - $|z_1 z_2| = |z_1||z_2|$
  - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- So if $z_1 = r_1\,\mathrm{cis}\,(\theta_1)$ and $z_2 = r_2\,\mathrm{cis}\,(\theta_2)$
  - $z_1 z_2 = r_1 r_2\,\mathrm{cis}\,(\theta_1 + \theta_2)$
- Sometimes the new argument, $\theta_1 + \theta_2$, does not lie in the range $-\pi < \theta \leq \pi$ (or $0 \leq \theta < 2\pi$ if this is being used)
  - An out-of-range argument can be adjusted by either **adding or subtracting** $2\pi$
  - E.g. If $\theta_1 = \frac{2\pi}{3}$ and $\theta_2 = \frac{\pi}{2}$ then $\theta_1 + \theta_2 = \frac{7\pi}{6}$
  - This is currently not in the range $-\pi < \theta \leq \pi$

www.thrivingscholars.com



- Subtracting $2\pi$ from $\frac{7\pi}{6}$ to give $-\frac{5\pi}{6}$, a new argument is formed
  - This lies in the correct range and represents the same angle on an Argand diagram
- The rules of **multiplying the moduli** and **adding the arguments** can also be applied when…
  - …multiplying three complex numbers together, $z_1 z_2 z_3$, or more
  - …finding powers of a complex number (e.g. $z^2$ can be written as $zz$)
- The rules for multiplication can be proved algebraically by multiplying $z_1 = r_1 \text{ cis } (\theta_1)$ by $z_2 = r_2 \text{ cis } (\theta_2)$, expanding the brackets and using compound angle formulae

## How do I divide complex numbers in modulus-argument (polar) form?

- To **divide** two complex numbers in modulus-argument (polar) form, we **divide their moduli** and **subtract their arguments**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
  - $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
- So if $z_1 = r_1 \text{ cis } (\theta_1)$ and $z_2 = r_2 \text{ cis } (\theta_2)$ then
  - $\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}\left(\theta_1 - \theta_2\right)$
- Sometimes the new argument, $\theta_1 - \theta_2$, can lie out of the range $-\pi < \theta \leq \pi$ (or the range $0 < \theta \leq 2\pi$ if this is being used)
  - You can **add or subtract** $2\pi$ to bring out-of-range arguments back in range
- The rules for division can be proved algebraically by dividing $z_1 = r_1 \text{ cis } (\theta_1)$ by $z_2 = r_2 \text{ cis } (\theta_2)$ using **complex division** and the compound angle formulae

### Exam Tip

- Remember that $r$ cis $\theta$ only refers to $r(\cos\theta + i\sin\theta)$
  - If you see a complex number written in the form $z = r(\cos\theta - i\sin\theta)$ then you will need to convert it to the correct form first
  - Make sure you are confident with basic trig identities to help you do this

www.thrivingscholars.com

## Worked Example

Let $z_1 = 4\sqrt{2}\ \text{cis}\ \frac{3\pi}{4}$ and $z_2 = \sqrt{8}\left(\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\right)$

a)

Find $z_1 z_2$, giving your answer in the form $r(\cos\theta + i\sin\theta)$ where $0 \le \theta < 2\pi$

$z_1 = 4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)$, $z_2 = \sqrt{8}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = 2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)$

For $z_1 z_2$, multiply the moduli and add the arguments.

$z_1 z_2 = \left(4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)\right)\left(2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)\right)$

$= (4\sqrt{2})(2\sqrt{2})\ \text{cis}\left(\frac{3\pi}{4} + \left(-\frac{\pi}{2}\right)\right)$

$= 16\ \text{cis}\left(\frac{\pi}{4}\right)$

$$z_1 z_2 = 16\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

b)

Find $\frac{z_1}{z_2}$, giving your answer in the form $r(\cos\theta + i\sin\theta)$ where $-\pi \le \theta < \pi$

For $\frac{z_1}{z_2}$, divide the moduli and subtract the arguments

$\frac{z_1}{z_2} = \frac{4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)}{2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)} = \frac{4\sqrt{2}}{2\sqrt{2}}\ \text{cis}\left(\frac{3\pi}{4} - \left(-\frac{\pi}{2}\right)\right)$

$= 2\ \text{cis}\left(\frac{5\pi}{4}\right)$

$= 2\ \text{cis}\left(\frac{5\pi}{4} - 2\pi\right)$

$\frac{5\pi}{4}$ is not in the range $-\pi \le \theta \le \pi$ so subtract $2\pi$ to bring it into range

$$\frac{z_1}{z_2} = 2\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$$

www.thrivingscholars.com

# Exponential (Euler's) Form

## How do we write a complex number in Euler's (exponential) form?

- A complex number can be written in Euler's form as $z = re^{i\theta}$
  - This relates to the modulus-argument (polar) form as $z = re^{i\theta} = r\,\text{cis}\,\theta$
  - This shows a clear link between exponential functions and trigonometric functions
  - This is given in the formula booklet under 'Modulus-argument (polar) form and exponential (Euler) form'
- The argument is normally given in the range $0 \leq \theta < 2\pi$
  - However in exponential form other arguments can be used and the same convention of adding or subtracting $2\pi$ can be applied

## How do we multiply and divide complex numbers in Euler's form?

- Euler's form allows for quick and easy multiplication and division of complex numbers
- If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then
  - $z_1 \times z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
    - Multiply the moduli and add the arguments
  - $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$
    - Divide the moduli and subtract the arguments
- Using these rules makes multiplying and dividing more than two complex numbers much easier than in Cartesian form
- When a complex number is written in Euler's form it is easy to raise that complex number to a power
  - If $z = re^{i\theta}$, $z^2 = r^2 e^{2i\theta}$ and $z^n = r^n e^{ni\theta}$

## What are some common numbers in exponential form?

- As $\cos(2\pi) = 1$ and $\sin(2\pi) = 0$ you can write:
  - $1 = e^{2\pi i}$
- Using the same idea you can write:
  - $1 = e^0 = e^{2\pi i} = e^{4\pi i} = e^{6\pi i} = e^{2k\pi i}$
  - where $k$ is any integer
- As $\cos(\pi) = -1$ and $\sin(\pi) = 0$ you can write:
  - $e^{\pi i} = -1$
  - Or more commonly written as $e^{i\pi} + 1 = 0$
    - This is known as Euler's identity and is considered by some mathematicians as the most beautiful equation
- As $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$ you can write:
  - $i = e^{\frac{\pi}{2} i}$

www.thrivingscholars.com

## Exam Tip

- Euler's form allows for easy manipulation of complex numbers, in an exam it is often worth the time converting a complex number into Euler's form if further calculations need to be carried out
  - Familiarise yourself with which calculations are easier in which form, for example multiplication and division are easiest in Euler's form but adding and subtracting are easiest in Cartesian form

## Worked Example

Consider the complex number $z = 2e^{\frac{\pi}{3}i}$. Calculate $z^2$ giving your answer in the form $re^{i\theta}$.

$$z^2 = \left(2e^{\frac{\pi}{3}i}\right)^2 = \left(2e^{\frac{\pi}{3}i}\right)\left(2e^{\frac{\pi}{3}i}\right) = 4e^{2\left(\frac{\pi}{3}i\right)}$$

multiply the moduli
add the arguments

$$z^2 = 4e^{\frac{2\pi}{3}i}$$

www.thrivingscholars.com

## Conversion of Forms

### Converting from Cartesian form to modulus-argument (polar) form or exponential (Euler's) form.

- To convert from Cartesian form to modulus-argument (polar) form or exponential (Euler) form use
  - $r = |z| = \sqrt{x^2 + y^2}$
- and
  - $\theta = \arg z$

### Converting from modulus-argument (polar) form or exponential (Euler's) form to Cartesian form.

- To convert from modulus-argument (polar) form to Cartesian form
  - Write $z = r(\cos\theta + i\sin\theta)$ as $z = r\cos\theta + (r\sin\theta)i$
  - Find the values of the trigonometric ratios $r\sin\theta$ and $r\cos\theta$
    - You may need to use your knowledge of trig exact values
  - Rewrite as $z = a + bi$ where
    - $a = r\cos\theta$ and $b = r\sin\theta$
- To convert from exponential (Euler's) form to Cartesian form first rewrite $z = re^{i\theta}$ in the form $z = r\cos\theta + (r\sin\theta)i$ and then follow the steps above

### Exam Tip

- When converting from Cartesian form into Polar or Euler's form, always leave your modulus and argument as an exact value
  - Rounding values too early may result in inaccuracies later on

www.thrivingscholars.com

## Worked Example

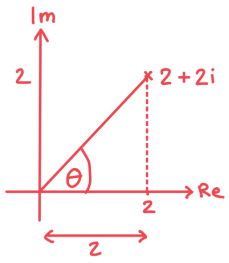
Two complex numbers are given by $z_1 = 2 + 2\mathrm{i}$ and $z_2 = 3\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$.

a)
Write $z_1$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

$z_1 = 2 + 2i$

Find the modulus: $|z_1| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Draw a sketch to help find the argument:




$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

$r = 2\sqrt{2}$, $\theta = \frac{\pi}{4}$

$$z_1 = 2\sqrt{2}\,\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}$$

b)
Write $z_2$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$ and then convert it to Cartesian form.

$z_2 = 3\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}} = 3\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$

$= 3\left(-\frac{1}{2} + \mathrm{i}\left(\frac{\sqrt{3}}{2}\right)\right)$

$$z_2 = \frac{3}{2}(-1 + \sqrt{3}\,\mathrm{i})$$

www.thrivingscholars.com



# Complex Roots of Quadratics

## What are complex roots?

- A quadratic equation can either have two real roots (zeros), a repeated real root or no real roots
  - This depends on the location of the graph of the quadratic with respect to the $x$-axis
- If a quadratic equation has no real roots we would previously have stated that it has **no real solutions**
  - The quadratic equation will have a **negative discriminant**
  - This means taking the square root of a negative number
- Complex numbers provide solutions for quadratic equations that have **no real roots**

## How do we solve a quadratic equation when it has complex roots?

- If a quadratic equation takes the form $ax^2 + bx + c = 0$ it can be solved by either using the quadratic formula or completing the square
- If a quadratic equation takes the form $ax^2 + b = 0$ it can be solved by rearranging
- The property $i = \sqrt{-1}$ is used
  - $\sqrt{-a} = \sqrt{a \times -1} = \sqrt{a} \times \sqrt{-1}$
- If the coefficients of the quadratic are real then the complex roots will occur in complex conjugate pairs
  - If $\boldsymbol{z = m + ni\ (n \neq 0)}$ is a root of a quadratic with real coefficients then $\boldsymbol{z^* = m - ni}$ is also a root
- The **real part** of the solutions will have the same value as the $x$ coordinate of the turning point on the graph of the quadratic
- When the coefficients of the quadratic equation are **non-real**, the solutions will **not** be complex conjugates
  - To solve these you can use the quadratic formula

## How do we factorise a quadratic equation if it has complex roots?

- If we are given a quadratic equation in the form $az^2 + bz + c = 0$, where $a$, $b$, and $c \in \mathbb{R}$, $a \neq 0$ we can use its complex roots to write it in **factorised form**
  - Use the quadratic formula to find the two roots, $z = p + qi$ and $z^* = p - qi$
  - This means that $z - (p + qi)$ and $z - (p - qi)$ must both be factors of the quadratic equation
  - Therefore we can write $az^2 + bz + c = (z - (p + qi))(z - (p - qi))$
  - This can be rearranged into the form $(z - p - qi)(z - p + qi)$

## How do we find a quadratic equation when given a complex root?

- If we are given a complex root in the form $z = p + qi$ we can find the quadratic equation in the form $az^2 + bz + c = 0$, where $a$, $b$, and $c \in \mathbb{R}$, $a \neq 0$
  - We know that the second root must be $z^* = p - qi$
  - This means that $z - (p + qi)$ and $z - (p - qi)$ must both be factors of the quadratic equation
  - Therefore we can write $az^2 + bz + c = (z - (p + qi))(z - (p - qi))$
  - Rewriting this as $((z - p) - qi))((z - p) + qi))$ makes expanding easier

www.thrivingscholars.com



- Expanding this gives the quadratic equation $z^2 - 2pz + (p^2 + q^2)$
  - $a = 1$
  - $b = -2p$
  - $c = p^2 + q^2$
- This demonstrates the important property $(x - z)(x - z^*) = x^2 - 2\text{Re}(z)x + |z|^2$



### Exam Tip

- Once you have your final answers you can check your roots are correct by substituting your solutions back into the original equation
  - You should get 0 if correct! [Note: 0 is equivalent to $0 + 0i$]

www.thrivingscholars.com

## Worked Example

a)

Solve the quadratic equation $z^2 - 2z + 5 = 0$ and hence, factorise $z^2 - 2z + 5$.

Use the quadratic formula or completing the square to find the solutions.

| | |
|---|---|
| Solutions of a quadratic equation | $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \ a \neq 0$ |

$a = 1$
$b = -2$
$c = 5$

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm \sqrt{16}\sqrt{-1}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$z_1 = 1 + 2i \qquad z_2 = 1 - 2i$$

If the solutions are $z_1 = 1 + 2i$ and $z_2 = 1 - 2i$

then the factors must be $z - (1+2i)$ and $z - (1-2i)$

$$z^2 - 2z + 5 = (z - (1 + 2i))(z - (1 - 2i))$$

$$(z - 1 - 2i)(z - 1 + 2i)$$

b)

Given that one root of a quadratic equation is $z = 2 - 3i$, find the quadratic equation in the form $az^2 + bz + c = 0$, where $a$, $b$, and $c \in \mathbb{R}$, $a \neq 0$.

If $2 - 3i$ is one root then $2 + 3i$ must be the other root and the two factors must be $z - (2 - 3i)$ and $z - (2 + 3i)$.

$$(z - (2 - 3i))(z - (2 + 3i)) = 0$$

$$((z - 2) + 3i)((z - 2) - 3i) = 0$$

$$(z - 2)^2 - (3i)^2 = 0$$

$$z^2 - 4z + 4 - 9i^2 = 0$$

$i^2 = -1 \therefore -9i^2 = 9$

$$z^2 - 4z + 13 = 0$$

www.thrivingscholars.com

www.thrivingscholars.com



# Complex Roots of Polynomials

## How many roots should a polynomial have?

- We know that every **quadratic** equation has **two roots** (not necessarily distinct or real)
- This is a particular case of a more general rule:
  - Every polynomial equation, with real coefficients, of degree *n* has *n* roots
  - The *n* roots are not necessarily all **distinct** and therefore we need to count any **repeated** roots that may occur individually
- From the above rule we can state the following:
  - A **cubic** equation of the form $ax^3 + bx^2 + cx + d = 0$ can have either:
    - 3 **real** roots
    - Or 1 **real** root and a complex **conjugate** pair
- A **quartic** equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ will have one of the following cases for roots:
  - 4 **real** roots
  - 2 **real** and 2 **nonreal** (a complex conjugate pair)
  - 4 **nonreal** (two complex conjugate pairs)
- When a real polynomial of any degree has one complex root it will always also have the complex conjugate as a root

## How do we solve a cubic equation with complex roots?

- Steps to solve a cubic equation with complex roots
  - If we are told that $p + qi$ is a root, then we know $p - qi$ is also a root
  - This means that $z - (p + qi)$ and $z - (p - qi)$ must both be factors of the cubic equation
  - Multiplying the above factors together gives us a quadratic factor of the form $(Az^2 + Bz + C)$
  - We need to find the third factor $(z - \alpha)$
  - **Multiply** the factors and **equate** to our original equation to get
    - $(Az^2 + Bz + C)(z - \alpha) = ax^3 + bx^2 + cx + d$
  - From there either
    - **Expand** and **compare coefficients** to find
    - Or use **polynomial division** to find the factor $(z - \alpha)$
  - Finally, write your **three roots** clearly

## How do we solve a polynomial of any degree with complex roots?

- When asked to find the roots of any polynomial when we are given one, we use almost the same method as for a cubic equation
  - State the initial root and its conjugate and write their factors as a quadratic factor (as above) we will have two unknown roots to find, write these as factors $(z - \alpha)$ and $(z - \beta)$
  - The unknown factors also form a quadratic factor $(z - \alpha)(z - \beta)$
  - Then continue with the steps from above, either **comparing coefficients** or using **polynomial division**
    - If using polynomial division, then solve the quadratic factor you get to find the roots $\alpha$ and $\beta$

## How do we solve polynomial equations with unknown coefficients?

www.thrivingscholars.com

- Steps to find unknown variables in a given equation when given a root:
  - **Substitute** the given root $p + qi$ into the equation $f(z) = 0$
  - **Expand** and **group** together the **real** and **imaginary** parts (these expressions will contain our unknown values)
  - **Solve** as simultaneous equations to find the unknowns
  - **Substitute** the values into the **original** equation
  - From here continue using the previously described methods for finding other roots for the polynomial

## How do we factorise a polynomial when given a complex root?

- If we are given a root of a polynomial of any degree in the form $z = p + qi$
  - We know that the complex conjugate, $z^* = p - qi$ is another root
  - We can write $(z - (p + qi))$ and $(z - (p - qi))$ as two linear factors
    - Or rearrange into one quadratic factor
  - This can be multiplied out with another factor to find further factors of the polynomial
- For higher order polynomials more than one root may be given
  - If the further given root is complex then its complex conjugate will also be a root
  - This will allow you to find further factors

### Exam Tip

- You can speed up multiplying two complex conjugate factors together by
  - rewrite $(z - (p + qi))(z - (p - qi))$ as $((z - p) - qi))((z - p) + qi))$
  - Then $((z - p) - qi))((z - p) + qi)) = (z - p)^2 - (qi)^2 = (z - p)^2 + q^2$
- If you are working on a calculator paper read the question carefully to see how much of the working needs to be shown but always remember to use your GDC to check your working where you can

www.thrivingscholars.com

## Worked Example

Given that one root of a polynomial $p(x) = z^3 + z^2 - 7z + 65$ is $2 - 3i$, find the other roots.

If $2-3i$ is one root then $2+3i$ must be the other root and two of the factors must be $z-(2-3i)$ and $z-(2+3i)$

Therefore a quadratic factor is $z^2 - 4z + 13$

There must exist a linear factor $(az+b)$

$\therefore (az+b)(z^2-4z+13) = z^3 + z^2 - 7z + 65$

Compare coefficients: $az^3 = z^3$ coefficient of $z^3$

$a = 1$

$13b = 65$ constant coefficient

$b = 5$

Therefore the factors are $z-(2-3i)$, $z-(2+3i)$ and $(z+5)$

$(z-(2-3i))(z-(2+3i))(z+5) = 0$

$z = (2 \pm 3i)$ and $z = -5$

www.thrivingscholars.com





# De Moivre's Theorem

## What is De Moivre's Theorem?

- De Moivre's theorem can be used to find powers of complex numbers
- It states that for $z = r\,\text{cis}\,\theta$, $z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$
  - Where
    - $z \neq 0$
    - $r$ is the modulus, $|z|$, $r \in \mathbb{R}^+$
    - $\theta$ is the argument, $\arg z$, $\theta \in \mathbb{R}$
    - $n \in \mathbb{R}$
- In Euler's form this is simply:
  - $(re^{i\theta})^n = r^n e^{in\theta}$
- In words de Moivre's theorem tells us to raise the modulus by the power of $n$ and multiply the argument by $n$
- In the formula booklet de Moivre's theorem is given in both polar and Euler's form:
  - $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta} = r^n\ \text{cis}\ n\theta$

## How do I use de Moivre's Theorem to raise a complex number to a power?

- If a complex number is in Cartesian form you will need to convert it to either modulus-argument (polar) form or exponential (Euler's) form first
  - This allows de Moivre's theorem to be used on the complex number
- You may need to convert it back to Cartesian form afterwards
- If a complex number is in the form $z = r(\cos(\theta) - i\sin(\theta))$ then you will need to rewrite it as $z = r(\cos(-\theta) + i\sin(-\theta))$ before applying de Moivre's theorem
- A useful case of de Moivre's theorem allows us to easily find the reciprocal of a complex number:
  - $\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) = \frac{1}{r}e^{-i\theta}$
  - Using the trig identities $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$ gives
  - $\frac{1}{z} = z^{-1} = r^{-1}[\cos(\theta) - i\sin(\theta)] = \frac{1}{r}[\cos(\theta) - i\sin(\theta)]$
- In general
  - $z^{-n} = r^{-n}[\cos(-n\theta) + i\sin(-n\theta)] = r^{-n}[\cos(n\theta) - i\sin(n\theta)]$

### Exam Tip

- You may be asked to find all the powers of a complex number, this means there will be a repeating pattern
  - This can happen if the modulus of the complex number is 1
  - Keep an eye on your answers and look for the point at which they begin to repeat themselves

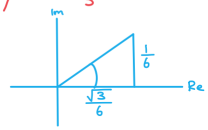
www.thrivingscholars.com

## Worked Example

Find the value of $\left(\frac{\sqrt{3}}{6}+\frac{1}{6}i\right)^{-3}$, giving your answer in the form $a+bi$.

Write in Polar form: $r=\sqrt{\left(\frac{\sqrt{3}}{6}\right)^2+\left(\frac{1}{6}\right)^2}=\frac{1}{3}$

$\theta=\tan^{-1}\left(\frac{\frac{1}{6}}{\frac{\sqrt{3}}{6}}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$




$\left(\frac{\sqrt{3}}{6}+\frac{1}{6}i\right)^{-3}=\left(\frac{1}{3}\,\text{cis}\left(\frac{\pi}{6}\right)\right)^{-3}$

Apply DeMoivre's Theorem: $\left(\frac{1}{3}\,\text{cis}\left(\frac{\pi}{6}\right)\right)^{-3}=\left(\frac{1}{3}\right)^{-3}\text{cis}\left(\frac{-3\pi}{6}\right)$

Convert back to Cartesian form:

$27\,\text{cis}\left(-\frac{\pi}{2}\right)=27\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$

$=27(0-i)$

$-27i$

www.thrivingscholars.com

# Proof of De Moivre's Theorem

## How is de Moivre's Theorem proved?

- When written in Euler's form the proof of de Moivre's theorem is easy to see:
  - Using the index law of brackets: $(re^{i\theta})^n = r^n e^{in\theta}$
- However Euler's form cannot be used to prove de Moivre's Theorem when it is in modulus-argument (polar) form
- **Proof by induction** can be used to prove de Moivre's Theorem for positive integers:
  - To prove de Moivre's Theorem for all positive integers, $n$
  - $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$
- STEP 1: Prove it is true for $n = 1$
  - $[r(\cos\theta + i\sin\theta)]^1 = r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta)$
  - So de Moivre's Theorem is true for $n = 1$
- STEP 2: Assume it is true for $n = k$
  - $[r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$
- STEP 3: Show it is true for $n = k + 1$
  - $[r(\cos\theta + i\sin\theta)]^{k+1} = ([r(\cos\theta + i\sin\theta)]^k)([r(\cos\theta + i\sin\theta)]^1)$
  - According to the assumption this is equal to
    - $(r^k(\cos k\theta + i\sin k\theta))\ (r(\cos\theta + i\sin\theta))$
  - Using laws of indices and multiplying out the brackets:
    - $= r^{k+1}[\cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta]$
  - Letting $i^2 = -1$ and collecting the real and imaginary parts gives:
    - $= r^{k+1}[\cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)]$
  - Recognising that the real part is equivalent to $\cos(k\theta + \theta)$ and the imaginary part is equivalent to $\sin(k\theta + \theta)$ gives
    - $(r\,\mathrm{cis}\,\theta)^{k+1} = r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]$
  - So de Moivre's Theorem is true for $n = k + 1$
- STEP 4: Write a conclusion to complete the proof
  - The statement is true for $n = 1$, and if it is true for $n = k$ it is also true for $n = k + 1$
  - Therefore, by the principle of mathematical induction, the result is true for all positive integers, $n$
- De Moivre's Theorem works for all real values of $n$
  - However you could only be asked to prove it is true for positive integers

### Exam Tip

- Learning the standard proof for de Moivre's theorem will also help you to memorise the steps for proof by induction, another important topic for your AA HL exam

www.thrivingscholars.com

## Worked Example

Show, using proof by mathematical induction, that for a complex number $z = r\,\text{cis}\,\theta$ and for all positive integers, $n$,

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

**Step 1:** Prove it is true for $n=1$

$z^1 = [r(\cos\theta + i\sin\theta)]^1 = r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta)$

**Step 2:** Assume it is true for $n=k$

$z^k = [r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$

**Step 3:** Show it is true for $n=k+1$

$z^{k+1} = [r(\cos\theta + i\sin\theta)]^{k+1}$ Addition Law of indices: $a^k a^1 = a^{k+1}$

$= ([r(\cos\theta + i\sin\theta)]^k)([r(\cos\theta + i\sin\theta)]^1)$ using the assumption

$r^k \times r^1 = r^{k+1}$

$= [r^k(\cos k\theta + i\sin k\theta)][r(\cos\theta + i\sin\theta)]$

$= r^{k+1}(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= r^{k+1}[\cos k\theta\cos\theta + \cos k\theta(i\sin\theta) + \cos\theta(i\sin k\theta) + i^2\sin k\theta\sin\theta]$ $i^2 = -1$

$= r^{k+1}[\cos k\theta\cos\theta + i(\cos k\theta\sin\theta + \cos\theta\sin k\theta) - \sin k\theta\sin\theta]$

$= r^{k+1}[\underbrace{\cos k\theta\cos\theta - \sin k\theta\sin\theta}_{=\cos(k\theta+\theta)} + i\underbrace{(\cos k\theta\sin\theta + \sin k\theta\cos\theta)}_{=\sin(k\theta+\theta)}]$ collect Re and Im parts

$= r^{k+1}[\cos(k\theta+\theta) + i\sin(k\theta+\theta)]$

$= r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]$ So true for $n=k+1$

**Step 4:** Write a conclusion:

De Moivre's theorem is true for $n=1$, and if it is true for $n=k$ it is also true for $n=k+1$.
Therefore it is true for all $n \in \mathbb{Z}^+$

www.thrivingscholars.com



# Roots of Complex Numbers

## How do I find the square root of a complex number?

- The square roots of a complex number will themselves be complex:
  - i.e. if $z^2 = a + b\text{i}$ then $z = c + d\text{i}$
- We can then square $(c + d\text{i})$ and equate it to the original complex number $(a + b\text{i})$, as they both describe $z^2$:
  - $a + b\text{i} = (c + d\text{i})^2$
- Then expand and simplify:
  - $a + b\text{i} = c^2 + 2cd\text{i} + d^2\text{i}^2$
  - $a + b\text{i} = c^2 + 2cd\text{i} - d^2$
- As both sides are equal we are able to equate real and imaginary parts:
  - Equating the real components: $a = c^2 - d^2$ (1)
  - Equating the imaginary components: $b = 2cd$ (2)
- These equations can then be solved simultaneously to find the real and imaginary components of the square root
  - In general, we can rearrange (2) to make $\frac{b}{2d} = c$ and then substitute into (1)
  - This will lead to a quartic equation in terms of d; which can be solved by making a substitution to turn it into a quadratic
- The values of $d$ can then be used to find the corresponding values of $c$, so we now have both components of both square roots $(c + d\text{i})$
- Note that one root will be the negative of the other root
  - g. $c + d\text{i}$ and $-c - d\text{i}$

## How do I use de Moivre's Theorem to find roots of a complex number?

- De Moivre's Theorem states that a complex number in modulus-argument form can be raised to the power of *n* by
  - Raising the modulus to the power of *n* and multiplying the argument by *n*
- When in modulus-argument (polar) form de Moivre's Theorem can then be used to find the roots of a complex number by
  - Taking the *n*th root of the modulus and dividing the argument by *n*
  - If $z = r(\cos\theta + \text{i}\sin\theta)$ then $\sqrt[n]{z} = [r(\cos(\theta + 2\pi k) + \text{i}\sin(\theta + 2\pi k))]^{\frac{1}{n}}$
    - $k = 0,\ 1,\ 2,\ \ldots,\ n-1$
    - Recall that adding $2\pi$ to the argument of a complex number does not change the complex number
    - Therefore we must consider how different arguments will give the same result
  - This can be rewritten as $\sqrt[n]{z} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + \text{i}\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$
- This can be written in exponential (Euler's) form as
  - For $z^n = re^{\text{i}\theta}$, $z = \sqrt[n]{r}e^{\frac{\theta + 2\pi k}{n}\text{i}}$
- The *n*th root of complex number will have *n* roots with the properties:

www.thrivingscholars.com



    - The modulus is $\sqrt[n]{r}$ for all roots
    - There will be *n* different arguments spaced at equal intervals on the unit circle
    - This creates some geometrically beautiful results:
        - The five roots of a complex number raised to the power 5 will create a regular pentagon on an Argand diagram
        - The eight roots of a complex number raised to the power 8 will create a regular octagon on an Argand diagram
        - The *n* roots of a complex number raised to the power *n* will create a regular *n*-sided polygon on an Argand diagram
- Sometimes you may need to use your GDC to find the roots of a complex number
    - Using your GDC's store function will help when entering complicated modulus and arguments
    - Make sure you choose the correct form to enter your complex number in
    - Your GDC should be able to give you the answer in your preferred form



### Exam Tip

- De Moivre's theorem makes finding roots of complex numbers very easy, but you must be confident converting from Cartesian form into Polar and Euler's form first
    - If you are in a calculator exam your GDC will be able to do this for you but you must clearly show how you got to your answer
    - You must also be prepared to do this by hand in a non-calculator paper

www.thrivingscholars.com

## Worked Example

a)

Find the square roots of $5+12i$, giving your answers in the form $a+bi$.

Let $z^2 = 5+12i$, then $z = a+bi$

$z^2 = a^2 + 2abi + b^2i^2$ ($i^2 = -1$)

$= a^2 + 2abi - b^2$

Therefore $5 + 12i = (a^2 - b^2) + 2abi$

Equate the real components: $a^2 - b^2 = 5$ ①

Equate the imaginary components: $2ab = 12$ ②

Solve the simultaneous equations: $a = \frac{6}{b} \Rightarrow \left(\frac{6}{b}\right)^2 - b^2 = 5$

$b^4 + 5b^2 - 36 = 0$

$(b^2 + 9)(b^2 - 4) = 0$

$b^2 = -9$ or $b^2 = 4$

No real solutions

$b = \pm 2$

$a = \pm 3$

$z_1 = 3+2i$ , $z_2 = -3-2i$

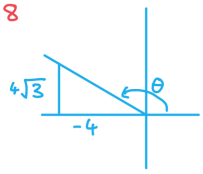
b)

Solve the equation $z^3 = -4 + 4\sqrt{3}i$ giving your answers in the form $r\,\text{cis}\,\theta$.

Convert $-4 + 4\sqrt{3}i$ to Polar form:

$r = \sqrt{(-4)^2 + (4\sqrt{3})^2} = \sqrt{64} = 8$

$\theta = \pi - \left(\tan^{-1}\left(\frac{4\sqrt{3}}{4}\right)\right) = \frac{2\pi}{3}$




$\therefore \quad -4 + 4\sqrt{3}i = 8\,\text{cis}\left(\frac{2\pi}{3}\right)$

$z^3 = -4 + 4\sqrt{3}i$

$z = \sqrt[3]{-4 + 4\sqrt{3}i} = \left(8\,\text{cis}\left(\frac{2\pi}{3}\right)\right)^{\frac{1}{3}}$

$= \left(8^{\frac{1}{3}}\right)\text{cis}\left(\frac{\frac{2\pi}{3} + 2\pi k}{3}\right)$

Order 3 so there are 3 roots, use $k = 0, 1, 2$:

$z = 2\,\text{cis}\left(\frac{2\pi}{9}\right)$, $2\,\text{cis}\left(\frac{8\pi}{9}\right)$, $2\,\text{cis}\left(\frac{14\pi}{9}\right)$

www.thrivingscholars.com

# 1.10 Systems of Linear Equations

## 1.10.1 Systems of Linear Equations

### Introduction to Systems of Linear Equations

### What are systems of linear equations?

- A linear equation is an equation of the first order (**degree 1**)
  - This means that the **maximum degree** of each term is 1
  - These are examples of linear equations:
    - $2x + 3y = 5$ & $5x - y = 10 + 5z$
  - These are examples of non-linear equations:
    - $x^2 + 5x + 3 = 0$ & $3x + 2xy - 5y = 0$
    - The terms $x^2$ and $xy$ have degree 2
- A system of linear equations is where **two or more linear equations** work together
  - These are also called **simultaneous equations**
- If there are ***n* variables** then you will need **at least *n* equations** in order to solve it
  - For your exam $n$ will be 2 or 3
- A **2×2 system** of linear equations can be written as
  - $a_1x + b_1y = c_1$
    $a_2x + b_2y = c_2$
- A **3×3 system** of linear equations can be written as
  - $a_1x + b_1y + c_1z = d_1$
    $a_2x + b_2y + c_2z = d_2$
    $a_3x + b_3y + c_3z = d_3$

### What do systems of linear equations represent?

- The most common application of systems of linear equations is in **geometry**
- For a **2×2** system
  - Each equation will represent a **straight line in 2D**
  - The solution (if it exists and is unique) will correspond to the **coordinates** of the point where the **two lines intersect**
- For a **3×3** system
  - Each equation will represent a **plane in 3D**
  - The solution (if it exists and is unique) will correspond to the **coordinates** of the point where the **three planes intersect**

www.thrivingscholars.com

# Systems of Linear Equations

## How do I set up a system of linear equations?

- Not all questions will have the equations written out for you
- There will be **bits of information** given about the variables
  - **Two bits** of information for a **2×2 system**
  - **Three bits** of information for a **3×3 system**
  - Look out for clues such as 'assuming a linear relationship'
- Choose to assign ***x*, *y* & *z*** to the given variables
  - This will be helpful if using a GDC to solve
- Or you can choose to use more meaningful variables if you prefer
  - Such as *c* for the number of cats and *d* for the number of dogs

## How do I use my GDC to solve a system of linear equations?

- You can use your **GDC to solve** the system on the **calculator papers (paper 2 & paper 3)**
- Your GDC will have a function within the algebra menu to solve a system of linear equations
- You will need to choose the number of equations
  - For two equations the variables will be $x$ and $y$
  - For three equations the variables will be $x$, $y$ and $z$
- If required, write the equations in the given form
  - $ax + by = c$
  - $ax + by + cz = d$
- Your GDC will display the values of $x$ and $y$ (or $x$, $y$, and $z$)

### Exam Tip

- Make sure that you are familiar with how to use your GDC to solve a system of linear equations because even if you are asked to use an algebraic method and show your working, you can use your GDC to check your final answer
- If a systems of linear equations question is asked on a non-calculator paper, make sure you check your final answer by inputting the values into all original equations to ensure that they satisfy the equations

www.thrivingscholars.com

## Worked Example

On a mobile phone game, a player can purchase one of three power-ups (fire, ice, electricity) using their points.

- Adam buys 5 fire, 3 ice and 2 electricity power-ups costing a total of 1275 points.
- Alice buys 2 fire, 1 ice and 7 electricity power-ups costing a total of 1795 points.
- Alex buys 1 fire and 1 ice power-ups which in total costs 5 points less than a single electricity power up.

Find the cost of each power-up.

Let $x$ be the cost of a fire power-up
Let $y$ be the cost of an ice power-up
Let $z$ be the cost of an electricity power-up

Form 3 equations

$5x + 3y + 2z = 1275$

$2x + y + 7z = 1795$

$x + y = z - 5 \quad \rightarrow \quad x + y - z = -5$

Write in form $ax + by + cz = d$

Type the 3 equations into the GDC and solve

$x = 120$, $y = 85$, $z = 210$

Fire costs 120 points
Ice costs 85 points
Electricity costs 210 points

www.thrivingscholars.com



# Row Reduction

## How can I write a system of linear equations?

- To save space we can just write the **coefficients without the variables**
- For 2 variables: $\begin{aligned} a_1x+b_1y&=c_1 \\ a_2x+b_2y&=c_2 \end{aligned}$ can be written shorthand as $\left[\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right]$
- For 3 variables: $\begin{aligned} a_1x+b_1y+c_1z&=d_1 \\ a_2x+b_2y+c_2z&=d_2 \\ a_3x+b_3y+c_3z&=d_3 \end{aligned}$ can be written shorthand as $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$

## What is a row reduced system of linear equations?

- A system of linear equations is in row reduced form if it is written as:
  - $\left[\begin{array}{ccc|c} A_1 & B_1 & C_1 & D_1 \\ 0 & B_2 & C_2 & D_2 \\ 0 & 0 & C_3 & D_3 \end{array}\right]$ which corresponds to $\begin{aligned} A_1x+B_1y+C_1z&=D_1 \\ B_2y+C_2z&=D_2 \\ C_3z&=D_3 \end{aligned}$
    - It is very helpful if the values of $A_1$, $B_2$, $C_3$ are **equal to 1**

## What are row operations?

- **Row operations** are used to make the linear equations **simpler to solve**
  - They **do not affect the solution**
- You can **switch any two rows**
  - $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$ can be written as $\left[\begin{array}{ccc|c} a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \end{array}\right]$ using $r_1 \leftrightarrow r_3$
    - This is useful for getting zeros to the bottom
    - Or getting a one to the top
- You can **multiply any row by a (non-zero) constant**
  - $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$ can be written as $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ ka_2 & kb_2 & kc_2 & kd_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$ using $k \times r_2 \rightarrow r_2$
    - This is useful for getting a 1 as the first non-zero value in a row
- You can **add multiples of a row to another row**
  - $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$ can be written as $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2+5a_3 & b_2+5b_3 & c_2+5c_3 & d_2+5d_3 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$ using $r_2+5r_3 \rightarrow r_2$
    - This is useful for creating zeros under a 1

## How can I row reduce a system of linear equations?

www.thrivingscholars.com



- **STEP 1**: Get a **1 in the top left corner**
  - You can do this by **dividing the row** by the current value in its place
  - If the current value is 0 or an awkward number then you can **swap rows first**
    - $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ * & * & * & * \\ * & * & * & * \end{array}\right]$
- **STEP 2**: Get **0's in the entries below the 1**
  - You can do this by **adding/subtracting a multiple of the first row** to each row
    - $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right]$
- **STEP 3**: **Ignore the first row and column** as they are now complete
  - **Repeat STEPS 1 – 2** to the remaining section
    - Get a 1: $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & * & * & * \end{array}\right]$
    - Then 0 underneath: $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & 0 & * & * \end{array}\right]$
- **STEP 4**: Get a **1 in the third row**
  - Using the same idea as **STEP 1**
    - $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & 0 & 1 & D_3 \end{array}\right]$

## How do I solve a system of linear equations once it is in row reduced form?

- Once you row reduced the equations you can then **convert back to the variables**
  - $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & 0 & 1 & D_3 \end{array}\right]$ corresponds to $\begin{aligned} x + B_1 y + C_1 z &= D_1 \\ y + C_2 z &= D_2 \\ z &= D_3 \end{aligned}$
- Solve the equations **starting at the bottom**
  - You have the value for $z$ from the third equation
  - Substitute $z$ into the second equation and solve for $y$
  - Substitute $z$ and $y$ into the first each and solve for $x$

www.thrivingscholars.com

### Exam Tip

- To reduce the number of operations you do whilst solving a system of operations, you can do a couple of things:
  - You can set up your original matrix with the equations in any order, so if one of the equations already has a 1 in the top left corner, put that one first
  - You do not need to make every equation so that it only has a single variable with a value of 1, you just need to do that for 1 of the equations and use that result to work out the others

www.thrivingscholars.com

## Worked Example

Solve the following system of linear equations using algebra.

$$2x-3y+4z=14$$
$$x+2y-2z=-2$$
$$3x-y-2z=10$$

Write without the variables $\left[\begin{array}{ccc|c} 2 & -3 & 4 & 14 \\ 1 & 2 & -2 & -2 \\ 3 & -1 & -2 & 10 \end{array}\right]$

Swap rows to get 1 in top left corner $\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 2 & -3 & 4 & 14 \\ 3 & -1 & -2 & 10 \end{array}\right]$ $R_1 \leftrightarrow R_2$

Add multiples of $R_1$ to $R_2$ and $R_3$ to get zeros under the 1 $\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & -7 & 8 & 18 \\ 0 & -7 & 4 & 16 \end{array}\right]$ $R_2 - 2R_1 \to R_2$, $R_3 - 3R_1 \to R_3$

Multiple the second row to get a 1 $\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -\frac{8}{7} & -\frac{18}{7} \\ 0 & -7 & 4 & 16 \end{array}\right]$ $R_2 \times -\frac{1}{7} \to R_2$

Repeat the steps $\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -\frac{8}{7} & -\frac{18}{7} \\ 0 & 0 & -4 & -2 \end{array}\right]$ $R_3 + 7R_2 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -\frac{8}{7} & -\frac{18}{7} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$ $R_3 \times -\frac{1}{4} \to R_3$

Write out the equations starting at the bottom

$z = \frac{1}{2}$

$y - \frac{8}{7}z = -\frac{18}{7} \Rightarrow y - \frac{4}{7} = -\frac{18}{7} \Rightarrow y = -\frac{14}{7} = -2$

$x + 2y - 2z = -2 \Rightarrow x - 4 - 1 = -2 \Rightarrow x = 3$

$x = 3, \quad y = -2, \quad z = \frac{1}{2}$

www.thrivingscholars.com

# Number of Solutions to a System

## How many solutions can a system of linear equations have?

- There could be
  - 1 **unique solution**
  - **No solutions**
  - An **infinite number** of solutions
- You can determine the case by looking at the row reduced form

## How do I know if the system of linear equations has no solutions?

- Systems with **no solutions** are called **inconsistent**
- When trying to solve the system after using the row reduction method you will end up with a **mathematical statement which is never true**:
  - Such as: 0 = 1
- The **row reduced system will contain**:
  - **At least one row** where the entries to the **left of the line are zero** and the entry on the **right of the line is non-zero**
    - Such a row is called **inconsistent**
  - For example:
    - Row 2 is inconsistent $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 0 & 0 & D_2 \\ 0 & 0 & 1 & D_3 \end{array}\right]$ if $D_2$ is non-zero

## How do I know if the system of linear equations has infinite solutions?

- Systems with **at least one solution** are called **consistent**
  - The solution could be unique or there could be an infinite number of solutions
- When trying to solve the system after using the row reduction method you will end up with a **mathematical statement which is always true**
  - Such as: 0 = 0
- The **row reduced system will contain**:
  - **At least one row** where **all the entries are zero**
  - **No inconsistent rows**
  - For example:
    - $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

## How do I find the general solution of a dependent system?

- A **dependent system** of linear equations is one where there are **infinite number of solutions**
- The general solution will depend on **one or two parameters**
- In the case where **two rows are zero**
  - Let the **variables corresponding** to the **zero rows** be equal to the **parameters** $\lambda$ & $\mu$
    - For example: If the first and second rows are zero rows then let $x = \lambda$ & $y = \mu$

www.thrivingscholars.com



    - Find the **third** variable in terms of the two parameters using the equation from the third row
        - For example: $z = 4\lambda - 5\mu + 6$
- In the case where **only one row is zero**
    - Let the **variable corresponding** to the **zero row** be equal to the **parameter** $\lambda$
        - For example: If the first row is a zero row then let $x = \lambda$
    - Find the **remaining two variables in terms of the parameter** using the equations formed by the other two rows
        - For example: $y = 3\lambda - 5$ & $z = 7 - 2\lambda$



### Exam Tip

- Common questions that pop up in an IB exam include questions with equations of lines
- Being able to recognise whether there are no solutions, 1 solution or infinite solutions is really useful for identifying if lines are coincident, skew or intersect!

www.thrivingscholars.com

## Worked Example

$$x+2y-z=3$$
$$3x+7y+z=4$$
$$x-9z=k$$

a)
Given that the system of linear equations has an infinite number of equations, find the value of $k$.

Write without the variables $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & 7 & 1 & 4 \\ 1 & 0 & -9 & k \end{array}\right]$

Use the row reduction method

$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & k-3 \end{array}\right]$ $r_2-3r_1\to r_2$, $r_3-r_1\to r_3$ $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & k-13 \end{array}\right]$ $r_3+2r_2\to r_3$

There are an infinite number of solutions if a row is zero

$k-13=0$

$\boxed{k=13}$

b)
Find a general solution to the system.

The third row is zero so let the third variable ($z$) equal a parameter

$z=\lambda$

Use equations to find expressions for the other variables

$y+4z=-5 \Rightarrow y+4\lambda=-5 \Rightarrow y=-4\lambda-5$

$x+2y-1=3 \Rightarrow x-8\lambda-10-\lambda=3 \Rightarrow x=9\lambda+13$

$\boxed{x=9\lambda+13,\ y=-4\lambda-5,\ z=\lambda \text{ for } \lambda\in\mathbb{R}}$

www.thrivingscholars.com

www.thrivingscholars.com

# IB Maths DP

## 2. Functions

**CONTENTS**


2.1 Linear Functions & Graphs
- 2.1.1 Equations of a Straight Line

2.2 Quadratic Functions & Graphs
- 2.2.1 Quadratic Functions
- 2.2.2 Factorising & Completing the Square
- 2.2.3 Solving Quadratics
- 2.2.4 Quadratic Inequalities
- 2.2.5 Discriminants

2.3 Functions Toolkit
- 2.3.1 Language of Functions
- 2.3.2 Composite & Inverse Functions
- 2.3.3 Symmetry of Functions
- 2.3.4 Graphing Functions

2.4 Other Functions & Graphs
- 2.4.1 Exponential & Logarithmic Functions
- 2.4.2 Solving Equations
- 2.4.3 Modelling with Functions

2.5 Reciprocal & Rational Functions
- 2.5.1 Reciprocal & Rational Functions

2.6 Transformations of Graphs
- 2.6.1 Translations of Graphs
- 2.6.2 Reflections of Graphs
- 2.6.3 Stretches Graphs
- 2.6.4 Composite Transformations of Graphs

2.7 Polynomial Functions
- 2.7.1 Factor & Remainder Theorem
- 2.7.2 Polynomial Division
- 2.7.3 Polynomial Functions
- 2.7.4 Roots of Polynomials

2.8 Inequalities
- 2.8.1 Solving Inequalities Graphically
- 2.8.2 Polynomial Inequalities

www.thrivingscholars.com

2.9 Further Functions & Graphs
2.9.1 Modulus Functions
2.9.2 Modulus Transformations
2.9.3 Modulus Equations & Inequalities
2.9.4 Reciprocal & Square Transformations

www.thrivingscholars.com



## 2.1 Linear Functions & Graphs

### 2.1.1 Equations of a Straight Line

# Equations of a Straight Line

## How do I find the gradient of a straight line?

- Find two points that the line passes through with coordinates $(x_1, y_1)$ and $(x_2, y_2)$
- The gradient between these two points is calculated by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

  - This is given in the **formula booklet**
- The gradient of a straight line measures its **slope**
  - A line with gradient 1 will go up 1 unit for every unit it goes to the right
  - A line with gradient -2 will go down two units for every unit it goes to the right

## What are the equations of a straight line?

- $y = mx + c$
  - This is the **gradient-intercept form**
  - It clearly shows the gradient $m$ and the $y$-intercept $(0, c)$
- $y - y_1 = m(x - x_1)$
  - This is the **point-gradient form**
  - It clearly shows the gradient $m$ and a point on the line $(x_1, y_1)$
- $ax + by + d = 0$
  - This is the **general form**
  - You can quickly get the $x$-intercept $\left(-\frac{d}{a}, 0\right)$ and $y$-intercept $\left(0, -\frac{d}{b}\right)$

## How do I find an equation of a straight line?

- You will need the gradient
  - If you are given two points then first find the gradient
- It is easiest to start with the **point-gradient form**
  - then rearrange into whatever form is required
    - multiplying both sides by any denominators will get rid of fractions
- You can check your answer by using your GDC
  - Graph your answer and check it goes through the point(s)
  - If you have two points then you can enter these in the **statistics mode** and find the regression line $y = ax + b$

www.thrivingscholars.com

## Exam Tip

- A sketch of the graph of the straight line(s) can be helpful, even if not demanded by the question
  - Use your GDC to plot them
- Ensure you state equations of straight lines in the format required
  - Usually $y = mx + c$ or $ax + by + d = 0$
  - Check whether coefficients need to be integers (they usually are for $ax + by + d = 0$)

## Worked Example

The line $l$ passes through the points $(-2, 5)$ and $(6, -7)$.

Find the equation of $l$, giving your answer in the form $ax + by + d = 0$ where $a$, $b$ and $c$ are integers to be found.

Find the gradient between (-2,5) and (6,-7)

Formula booklet

$m = \frac{-7-5}{6-(-2)} = -\frac{3}{2}$

| Gradient formula | $m = \frac{y_2 - y_1}{x_2 - x_1}$ |
|---|---|

Use the point-gradient formula

Formula booklet

| Equations of a straight line | $y - y_1 = m(x - x_1)$ |
|---|---|

$(x_1, y_1) = (-2, 5)$ $\quad m = -\frac{3}{2}$

$y - 5 = -\frac{3}{2}(x - (-2))$ Simplify

$y - 5 = -\frac{3}{2}(x + 2)$

$2(y-5) = -3(x+2)$ Multiply by denominator

$2y - 10 = -3x - 6$ Expand

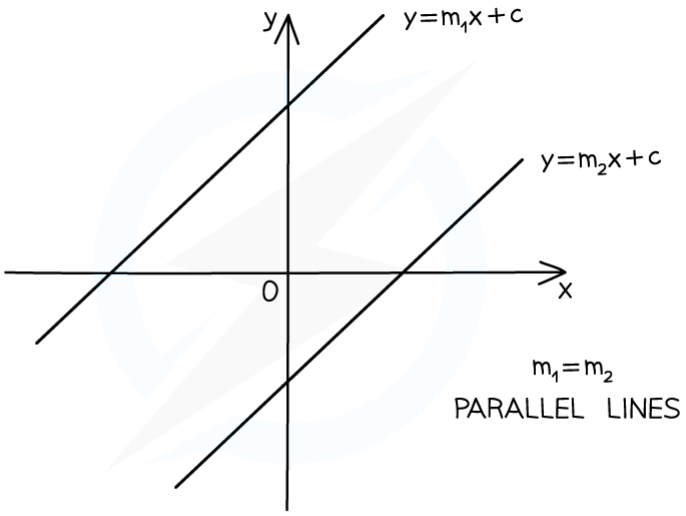
$3x + 2y - 4 = 0$ Rearrange

www.thrivingscholars.com

# Parallel Lines

## How are the equations of parallel lines connected?

- **Parallel lines** are always equidistant meaning they never intersect
- Parallel lines have the same gradient
  - If the gradient of line $l_1$ is $m_1$ and gradient of line $l_2$ is $m_2$ then...
    - $m_1 = m_2 \Rightarrow l_1 \ \& \ l_2$ are parallel
    - $l_1 \ \& \ l_2$ are parallel $\Rightarrow m_1 = m_2$
- To determine if two lines are parallel:
  - Rearrange into the gradient-intercept form $y = mx + c$
  - Compare the coefficients of $x$
  - If they are equal then the lines are parallel




Copyright © Save My Exams. All Rights Reserved
save my exams

www.thrivingscholars.com

## Worked Example

The line $l$ passes through the point $(4, -1)$ and is parallel to the line with equation $2x - 5y = 3$.

Find the equation of $l$, giving your answer in the form $y = mx + c$.

Rearrange into $y = mx + c$ to find the gradient

$5y = 2x - 3 \Rightarrow y = \frac{2}{5}x - \frac{3}{5}$ $\therefore$ gradient $= \frac{2}{5}$

Parallel lines $\Rightarrow m_1 = m_2$

$m = \frac{2}{5}$

Use the point-gradient formula

Formula booklet

| | |
|---|---|
| Equations of a straight line | $y - y_1 = m(x - x_1)$ |

$(x_1, y_1) = (4, -1)$ $m = \frac{2}{5}$

$y + 1 = \frac{2}{5}(x - 4)$

$y + 1 = \frac{2}{5}x - \frac{8}{5}$

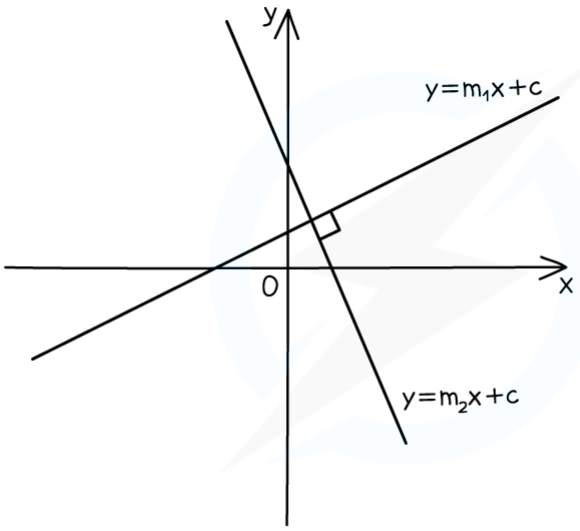
$\boxed{y = \frac{2}{5}x - \frac{13}{5}}$

www.thrivingscholars.com

# Perpendicular Lines

## How are the equations of perpendicular lines connected?

- **Perpendicular lines** intersect at right angles
- The gradients of two perpendicular lines are negative reciprocals
  - If the gradient of line $l_1$ is $m_1$ and gradient of line $l_2$ is $m_2$ then...
    - $m_1 \times m_2 = -1 \Rightarrow l_1 \ \& \ l_2$ are perpendicular
    - $l_1 \ \& \ l_2$ are perpendicular $\Rightarrow m_1 \times m_2 = -1$
- To determine if two lines are perpendicular:
  - Rearrange into the gradient-intercept form $y = mx + c$
  - Compare the coefficients of $x$
  - If their product is -1 then they are perpendicular
- Be careful with horizontal and vertical lines
  - $x = p$ and $y = q$ are perpendicular where $p$ and $q$ are constants




$m_1 \times m_2 = -1$

PERPENDICULAR LINES

Copyright © Save My Exams. All Rights Reserved
savemyexams

www.thrivingscholars.com

## Worked Example

The line $l_1$ is given by the equation $3x - 5y = 7$.

The line $l_2$ is given by the equation $y = \frac{1}{4} - \frac{5}{3}x$.

Determine whether $l_1$ and $l_2$ are perpendicular. Give a reason for your answer.

Rearrange $l_1$ into $y = mx + c$ form

$5y = 3x - 7 \Rightarrow y = \frac{3}{5}x - \frac{7}{5}$

Identity gradients

$m_1 = \frac{3}{5}$ $m_2 = -\frac{5}{3}$

$m_1 \times m_2 = -1 \Rightarrow$ Perpendicular lines

$\frac{3}{5} \times -\frac{5}{3} = -1$

$l_1$ and $l_2$ are perpendicular as $m_1 \times m_2 = -1$

www.thrivingscholars.com

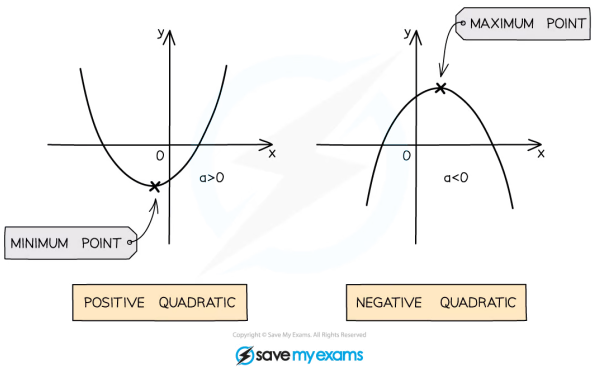
# 2.2 Quadratic Functions & Graphs

## 2.2.1 Quadratic Functions

### Quadratic Functions & Graphs

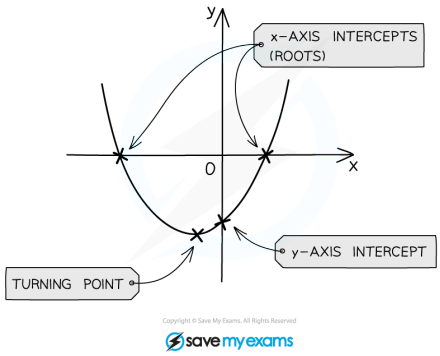
### What are the key features of quadratic graphs?

- A **quadratic** graph can be written in the form $y = ax^2 + bx + c$ where $a \neq 0$
- The value of *a* affects the shape of the curve
  - If *a* is **positive** the shape is **concave up** ∪
  - If *a* is **negative** the shape is **concave down** ∩
- The ***y*-intercept** is at the point (0, c)
- The **zeros or roots** are the solutions to $ax^2 + bx + c = 0$
  - These can be found by
    - Factorising
    - Quadratic formula
    - Using your GDC
  - These are also called the *x*-intercepts
  - There can be 0, 1 or 2 *x*-intercepts
    - This is determined by the value of the **discriminant**
- There is an **axis of symmetry** at $x = -\frac{b}{2a}$
  - This is given in your **formula booklet**
  - If there are two *x*-intercepts then the axis of symmetry goes through the midpoint of them
- The **vertex** lies on the axis of symmetry
  - It can be found by **completing the square**
  - The *x*-coordinate is $x = -\frac{b}{2a}$
  - The *y*-coordinate can be found using the GDC or by calculating *y* when $x = -\frac{b}{2a}$
  - If *a* is **positive** then the vertex is the **minimum point**
  - If *a* is **negative** then the vertex is the **maximum point**




Copyright © Save My Exams. All Rights Reserved
savemyexams

www.thrivingscholars.com



Copyright © Save My Exams. All Rights Reserved
save my exams

## What are the equations of a quadratic function?

- $f(x) = ax^2 + bx + c$
  - This is the **general form**
  - It clearly shows the $y$-intercept $(0, c)$
  - You can find the axis of symmetry by $x = -\frac{b}{2a}$
    - This is given in the formula booklet
- $f(x) = a(x - p)(x - q)$
  - This is the **factorised form**
  - It clearly shows the roots $(p, 0)$ & $(q, 0)$
  - You can find the axis of symmetry by $x = \frac{p + q}{2}$
- $f(x) = a(x - h)^2 + k$
  - This is the **vertex form**
  - It clearly shows the vertex $(h, k)$
  - The axis of symmetry is therefore $x = h$
  - It clearly shows how the function can be transformed from the graph $y = x^2$
    - Vertical stretch by scale factor $a$
    - Translation by vector $\begin{pmatrix} h \\ k \end{pmatrix}$

## How do I find an equation of a quadratic?

- If you have the **roots** $x = p$ and $x = q$...
  - Write in **factorised form** $y = a(x - p)(x - q)$
  - You will need a third point to find the value of $a$
- If you have the **vertex** $(h, k)$ then...
  - Write in **vertex form** $y = a(x - h)^2 + k$
  - You will need a second point to find the value of $a$
- If you have **three random points** $(x_1, y_1)$, $(x_2, y_2)$ & $(x_3, y_3)$ then...
  - Write in the **general form** $y = ax^2 + bx + c$
  - Substitute the three points into the equation
  - Form and solve a system of three linear equations to find the values of $a$, $b$ & $c$

www.thrivingscholars.com

### Exam Tip

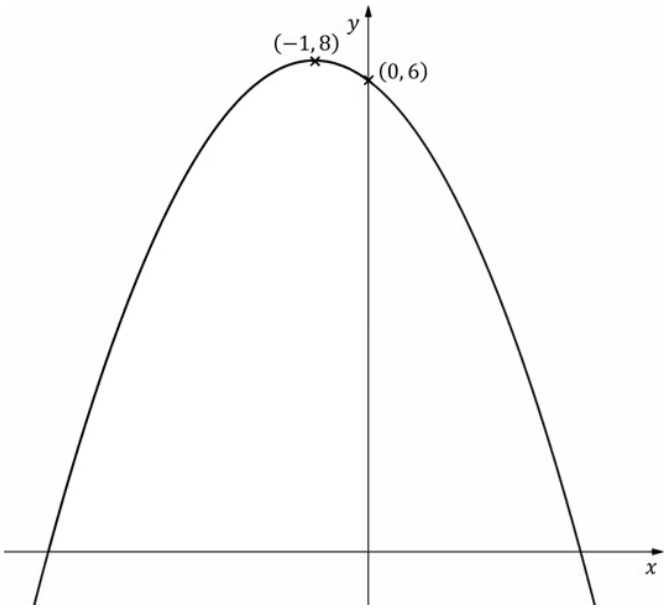
- Use your GDC to find the roots and the turning point of a quadratic function
  - You do not need to factorise or complete the square
  - It is good exam technique to sketch the graph from your GDC as part of your working

www.thrivingscholars.com

## Worked Example

The diagram below shows the graph of $y = f(x)$, where $f(x)$ is a quadratic function.

The intercept with the $y$-axis and the vertex have been labelled.




Write down an expression for $y = f(x)$.

We have the vertex so use $y = a(x-h)^2 + k$

Vertex $(-1, 8)$: $y = a(x-(-1))^2 + 8$

$y = a(x+1)^2 + 8$

Substitute the second point

$x = 0, y = 6$: $6 = a(0+1)^2 + 8$

$6 = a + 8$

$a = -2$

$y = -2(x+1)^2 + 8$

www.thrivingscholars.com





# Factorising Quadratics

## Why is factorising quadratics useful?

- Factorising gives **roots (zeroes or solutions)** of a quadratic
- It gives the ***x*-intercepts** when drawing the graph

## How do I factorise a monic quadratic of the form $x^2 + bx + c$?

- You might be able to spot the factors by **inspection**
  - Especially if c is a **prime number**
- Otherwise find two numbers *m* and *n* ..
  - A sum equal to *b*
    - $m + n = b$
  - A product equal to c
    - $mn = c$
- Rewrite *bx* as $mx + nx$
- Use this to factorise $x^2 + mx + nx + c$
- A shortcut is to write:
  - $(x + m)(x + n)$

## How do I factorise a non-monic quadratic of the form $ax^2 + bx + c$?

- If *a*, *b* & *c* have a common factor then first factorise that out to leave a quadratic with coefficients that have **no common factors**
- You might be able to spot the factors by **inspection**
  - Especially if *a* and/or c are **prime numbers**
- Otherwise find two numbers *m* and *n* ..
  - A sum equal to *b*
    - $m + n = b$
  - A product equal to *ac*
    - $mn = ac$
- Rewrite *bx* as $mx + nx$
- Use this to factorise $ax^2 + mx + nx + c$
- A shortcut is to write:
  - $\frac{(ax + m)(ax + n)}{a}$
  - Then factorise common factors from numerator to cancel with the *a* on the denominator

## How do I use the difference of two squares to factorise a quadratic of the form $A^2x^2 - C^2$?

- The **difference of two squares** can be used when...
  - There is **no *x* term**
  - The **constant term is a negative**
- Square root the two terms $A^2x^2$ & $C^2$
- The two factors are the **sum of square roots** and the **difference of the square roots**
- A shortcut is to write:

www.thrivingscholars.com

- $(Ax + C)(Ax - C)$

**Exam Tip**

- You can deduce the factors of a quadratic function by using your GDC to find the solutions of a quadratic equation
  - Using your GDC, the quadratic equation $6x^2 + x - 2 = 0$ has solutions $x = -\frac{2}{3}$ and $x = \frac{1}{2}$
  - Therefore the factors would be $(3x + 2)$ and $(2x - 1)$
  - i.e. $6x^2 + x - 2 = (3x + 2)(2x - 1)$

www.thrivingscholars.com

## Worked Example

Factorise fully:

a)

$x^2 - 7x + 12$.

Find two numbers m and n such that

$m+n = b = -7$ $\quad mn = c = 12$

$-4 + -3 = -7$ $\quad -4 \times -3 = 12$

Split $-7x$ up and factorise

$x^2 - 4x - 3x + 12$

$x(x-4) - 3(x-4)$

$(x-3)(x-4)$

Shortcut

$(x+m)(x+n)$

$(x-3)(x-4)$

b)

$4x^2 + 4x - 15$.

Find two numbers m and n such that

$m+n = b = 4$ $\quad mn = ac = 4 \times -15 = -60$

$10 + -6 = 4$ $\quad 10 \times -6 = -60$

Split $4x$ up and factorise

$4x^2 + 10x - 6x - 15$

$2x(2x+5) - 3(2x+5)$

$(2x-3)(2x+5)$

Shortcut

$\frac{(ax+m)(ax+n)}{a}$

$\frac{(4x+10)(4x-6)}{4}$

$\frac{2(2x+5)\,2(2x-3)}{4}$

$(2x-3)(2x+5)$

c)

$18 - 50x^2$.

www.thrivingscholars.com

Factorise the common factor

$2(9-25x^2)$

Use difference of two squares

$$2(3-5x)(3+5x)$$

www.thrivingscholars.com

# Completing the Square

## Why is completing the square for quadratics useful?

- Completing the square gives the **maximum/minimum** of a quadratic function
  - This can be used to define the **range** of the function
- It gives the **vertex** when drawing the graph
- It can be used to **solve quadratic equations**
- It can be used to derive the **quadratic formula**

## How do I complete the square for a monic quadratic of the form $x^2 + bx + c$?

- **Half the value of *b*** and write $\left(x + \frac{b}{2}\right)^2$
  - This is because $\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}$
- **Subtract the unwanted $\frac{b^2}{4}$ term** and **add on the constant *c***
  - $\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c$

## How do I complete the square for a non-monic quadratic of the form $ax^2 + bx + c$?

- **Factorise out the *a*** from the **terms involving *x***
  - $a\left(x^2 + \frac{b}{a}x\right) + x$
  - Leaving the c alone will **avoid working with lots of fractions**
- **Complete the square** on the quadratic term
  - **Half** $\frac{b}{a}$ and write $\left(x + \frac{b}{2a}\right)^2$
    - This is because $\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$
  - **Subtract the unwanted $\frac{b^2}{4a^2}$ term**
- **Multiply by *a*** and **add the constant *c***
  - $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c$
  - $a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$

### Exam Tip

- Some questions may not use the phrase "completing the square" so ensure you can recognise a quadratic expression or equation written in this form
  - $a(x-h)^2 + k\ (=0)$

www.thrivingscholars.com

## Worked Example

Complete the square:

a)

$x^2 - 8x + 3$.

Half b and subtract its square

$(x-4)^2 - 4^2 + 3$

$\boxed{(x-4)^2 - 13}$

b)

$3x^2 + 12x - 5$.

Factorise the 3 from the $x$ terms

$3(x^2+4x) - 5$

Complete the square on $x^2+4x$

$3\left((x+2)^2 - 2^2\right) - 5$

Simplify

$3\left((x+2)^2 - 4\right) - 5$

$3(x+2)^2 - 12 - 5$

$\boxed{3(x+2)^2 - 17}$

www.thrivingscholars.com





# Solving Quadratics Equations

## How do I decide the best method to solve a quadratic equation?

- A quadratic equation is of the form $ax^2 + bx + c = 0$
- If it is a calculator paper then use your GDC to solve the quadratic
- If it is a non-calculator paper then...
  - you can always use the **quadratic formula**
  - you can **factorise** if it can be factorised with integers
  - you can always **complete the square**

## How do I solve a quadratic equation by the quadratic formula?

- If necessary **rewrite** in the form $ax^2 + bx + c = 0$
- **Clearly identify** the values of *a*, *b* & c
- **Substitute** the values into the formula
  - $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
    - This is given in the **formula booklet**
- **Simplify** the solutions as much as possible

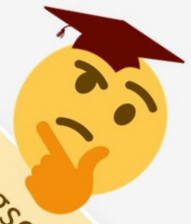
## How do I solve a quadratic equation by factorising?

- **Factorise** to rewrite the quadratic equation in the form $a(x - p)(x - q) = 0$
- Set each factor to zero and **solve**
  - $x - p = 0 \Rightarrow x = p$
  - $x - q = 0 \Rightarrow x = q$

## How do I solve a quadratic equation by completing the square?

- **Complete the square** to rewrite the quadratic equation in the form $a(x - h)^2 + k = 0$
- Get the squared term by itself
  - $(x - h)^2 = -\dfrac{k}{a}$
- If $\left(-\dfrac{k}{a}\right)$ is **negative** then there will be **no solutions**
- If $\left(-\dfrac{k}{a}\right)$ is **positive** then there will be **two values** for $x - h$
  - $x - h = \pm\sqrt{-\dfrac{k}{a}}$
- **Solve** for x
  - $x = h \pm \sqrt{-\dfrac{k}{a}}$

www.thrivingscholars.com



### Exam Tip

- When using the quadratic formula with awkward values or fractions you may find it easier to deal with the "$b^2 - 4ac$" (**discriminant**) first
  - This can help avoid numerical and negative errors, improving accuracy

www.thrivingscholars.com

## Worked Example

Solve the equations:

a)

$4x^2 + 4x - 15 = 0.$

This can be factorised

$(2x + 5)(2x - 3) = 0$

$2x + 5 = 0$ or $2x - 3 = 0$

$x = -\frac{5}{2}$ or $x = \frac{3}{2}$

b)

$3x^2 + 12x - 5 = 0.$

This can not be factorised but $3x^2$ and $12x$ have a common factor so complete the square

$3(x+2)^2 - 17 = 0$

$(x+2)^2 = \frac{17}{3}$ Rearrange

$x + 2 = \pm\sqrt{\frac{17}{3}}$ Remember $\pm$

$x = -2 \pm \sqrt{\frac{17}{3}}$

c)

$7 - 3x - 5x^2 = 0.$

This can not be factorised so use formula

Formula booklet

| Solutions of a quadratic equation | $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$ |
|---|---|

$a = -5$ $b = -3$ $c = 7$

$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-5)(7)}}{2(-5)}$

$= \frac{3 \pm \sqrt{9 + 140}}{-10}$

$x = -\frac{3 \pm \sqrt{149}}{10}$

www.thrivingscholars.com





# Quadratic Inequalities

## What affects the inequality sign when rearranging a quadratic inequality?

- The inequality sign is **unchanged** by...
  - **Adding/subtracting** a term to both sides
  - **Multiplying/dividing** both sides by a **positive term**
- The inequality sign **flips** (< changes to >) when...
  - **Multiplying/dividing** both sides by a **negative term**

## How do I solve a quadratic inequality?

- **STEP 1**: **Rearrange** the inequality into quadratic form with a **positive squared term**
  - $ax^2 + bx + c > 0$
  - $ax^2 + bx + c \geq 0$
  - $ax^2 + bx + c < 0$
  - $ax^2 + bx + c \leq 0$
- **STEP 2**: Find the **roots** of the quadratic equation
  - Solve $ax^2 + bx + c = 0$ to get $x_1$ and $x_2$ where $x_1 < x_2$
- **STEP 3**: **Sketch** a graph of the quadratic and label the roots
  - As the squared term is positive it will be **concave up** so "U" shaped
- **STEP 4**: **Identify** the **region** that satisfies the inequality
  - If you want the graph to be **above the *x*-axis** then choose the region to be the **two intervals outside** of the two roots
  - If you want the graph to be **below the *x*-axis** then choose the region to be the **interval between** the two roots
  - For $ax^2 + bx + c > 0$
    - The solution is $\boldsymbol{x < x_1}$ **or** $\boldsymbol{x > x_2}$
  - For $ax^2 + bx + c \geq 0$
    - The solution is $\boldsymbol{x \leq x_1}$ **or** $\boldsymbol{x \geq x_2}$
  - For $ax^2 + bx + c < 0$
    - The solution is $\boldsymbol{x_1 < x < x_2}$
  - For $ax^2 + bx + c \leq 0$
    - The solution is $\boldsymbol{x_1 \leq x \leq x_2}$

## How do I solve a quadratic inequality of the form $(x - h)^2 < n$ or $(x - h)^2 > n$?

- The safest way is by following the steps above
  - Expand and rearrange
- A **common mistake** is writing $x - h < \pm\sqrt{n}$ or $x - h > \pm\sqrt{n}$
  - This is **NOT correct**!
- The correct solution to $\boldsymbol{(x - h)^2 < n}$ is
  - $|x - h| < \sqrt{n}$ which can be written as $-\sqrt{n} < x - h < \sqrt{n}$
  - The **final solution** is $h - \sqrt{n} < x < h + \sqrt{n}$
- The correct solution to $\boldsymbol{(x - h)^2 > n}$ is
  - $|x - h| > \sqrt{n}$ which can be written as $x - h < -\sqrt{n}$ or $x - h > \sqrt{n}$

www.thrivingscholars.com

- The **final solution** is $x < h - \sqrt{n}$ **or** $x > h + \sqrt{n}$

## Exam Tip

- It is easiest to sketch the graph of a quadratic when it has a positive $x^2$ term, so rearrange first if necessary
- Use your GDC to help select the correct region(s) for the inequality
- Some makes/models of GDC may have the ability to solve inequalities directly
  - However unconventional notation may be used to display the answer (e.g. $6 > x > 3$ rather than $3 < x < 6$)
  - The safest method is to **always** sketch the graph

## Worked Example

Find the set of values which satisfy $3x^2 + 2x - 6 > x^2 + 4x - 2$.

Step 1: Rearrange

$(3x^2 + 2x - 6) - (x^2 + 4x - 2) > 0$ This way gives $a > 0$

$2x^2 - 2x - 4 > 0$

$x^2 - x - 2 > 0$ Divide by factor of 2

Step 2: Find the roots

$x^2 - x - 2 = 0$

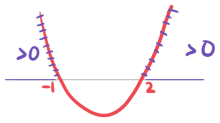
$(x - 2)(x + 1) = 0$

$x = 2$ or $x = -1$

Step 3: Sketch




Step 4: Identify region




$x < -1$ or $x > 2$

www.thrivingscholars.com





# Discriminants

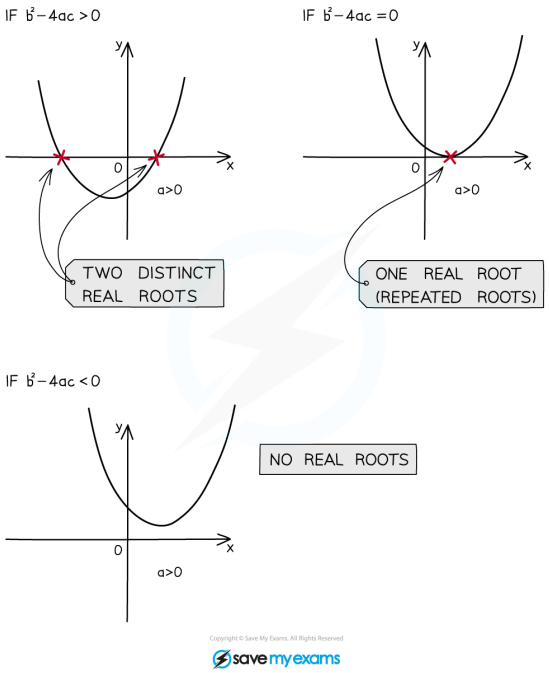
## What is the discriminant of a quadratic function?

- The discriminant of a quadratic is denoted by the Greek letter $\Delta$ (upper case delta)
- For the quadratic function the discriminant is given by
  - $\Delta = b^2 - 4ac$
    - This is given in the **formula booklet**
- The discriminant is the expression that is square rooted in the **quadratic formula**

## How does the discriminant of a quadratic function affect its graph and roots?

- If $\Delta > 0$ then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **two distinct values**
  - The equation $ax^2 + bx + c = 0$ has **two distinct real solutions**
  - The graph of $y = ax^2 + bx + c$ has **two distinct real roots**
    - This means the graph **crosses** the $x$-axis **twice**
- If $\Delta = 0$ then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **both zero**
  - The equation $ax^2 + bx + c = 0$ has **one repeated real solution**
  - The graph of $y = ax^2 + bx + c$ has **one repeated real root**
    - This means the graph **touches** the $x$-axis at **exactly one point**
    - This means that the **$x$-axis** is a **tangent** to the graph
- If $\Delta < 0$ then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **both undefined**
  - The equation $ax^2 + bx + c = 0$ has **no real solutions**
  - The graph of $y = ax^2 + bx + c$ has **no real roots**
    - This means the graph **never touches** the **$x$-axis**
    - This means that graph is **wholly above** (or **below**) the **$x$-axis**

www.thrivingscholars.com




## Forming equations and inequalities using the discriminant

- Often at least one of the coefficients of a quadratic is **unknown**
  - Questions usually use the letter *k* for the unknown constant
- You will be given a fact about the quadratic such as:
  - The **number of solutions** of the equation
  - The **number of roots** of the graph
- To find the **value or range of values** of *k*
  - Find an **expression for the discriminant**
    - Use $\Delta = b^2 - 4ac$
  - Decide whether $\Delta > 0$, $\Delta = 0$ or $\Delta < 0$
    - If the question says there are **real roots** but does not specify how many then use $\Delta \geq 0$
  - **Solve** the resulting equation or inequality

### Exam Tip

- Questions will rarely use the word discriminant so it is important to recognise when its use is required
  - Look for
    - a number of roots or solutions being stated
    - whether and/or how often the graph of a quadratic function intercepts the $x$-axis
- Be careful setting up inequalities that concern "two real roots" ($\Delta \geq 0$) as opposed to "two real distinct roots" ($\Delta > 0$)

www.thrivingscholars.com

## Worked Example

A function is given by $f(x) = 2kx^2 + kx - k + 2$, where $k$ is a constant. The graph of $y = f(x)$ has two distinct real roots.

a)

Show that $9k^2 - 16k > 0$.

Two distinct real roots => $\Delta > 0$

Formula booklet

| Discriminant | $\Delta = b^2 - 4ac$ |
|---|---|

$a = 2k \quad b = k \quad c = (-k+2)$

$\Delta = k^2 - 4(2k)(-k+2)$

$= k^2 + 8k^2 - 16k$

$= 9k^2 - 16k$

$\Delta > 0 \Rightarrow 9k^2 - 16k > 0$

b)

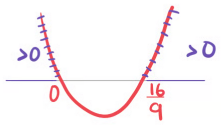
Hence find the set of possible values of $k$.

Solve the inequality

$9k^2 - 16k = 0$

$k(9k - 16) = 0$

$k = 0$ or $k = \frac{16}{9}$




$k < 0$ or $k > \frac{16}{9}$

www.thrivingscholars.com

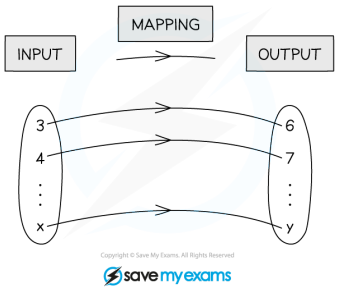
# 2.3 Functions Toolkit

## 2.3.1 Language of Functions

### Language of Functions

### What is a mapping?

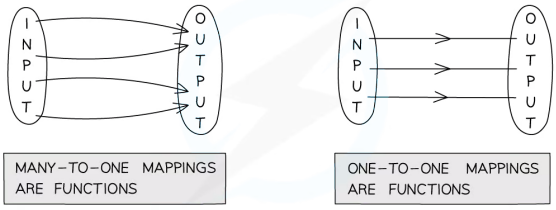
- A **mapping transforms** one set of values (**inputs**) into another set of values (**outputs**)
- Mappings can be:
  - **One-to-one**
    - Each input gets mapped to **exactly one unique** output
    - No two inputs are mapped to the same output
    - For example: A mapping that cubes the input
  - **Many-to-one**
    - Each input gets mapped to **exactly one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that squares the input
  - **One-to-many**
    - An input can be mapped to **more than one** output
    - No two inputs are mapped to the same output
    - For example: A mapping that gives the numbers which when squared equal the input
  - **Many-to-many**
    - An input can be mapped to **more than one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that gives the factors of the input




### What is a function?

- A **function** is a mapping between two sets of numbers where **each input** gets mapped to **exactly one output**
  - The output does not need to be unique
- **One-to-one** and **many-to-one** mappings are functions
- A mapping is a function if its graph passes the **vertical line test**
  - Any **vertical line** will intersect with the graph **at most once**

www.thrivingscholars.com



Copyright © Save My Exams. All Rights Reserved
save my exams

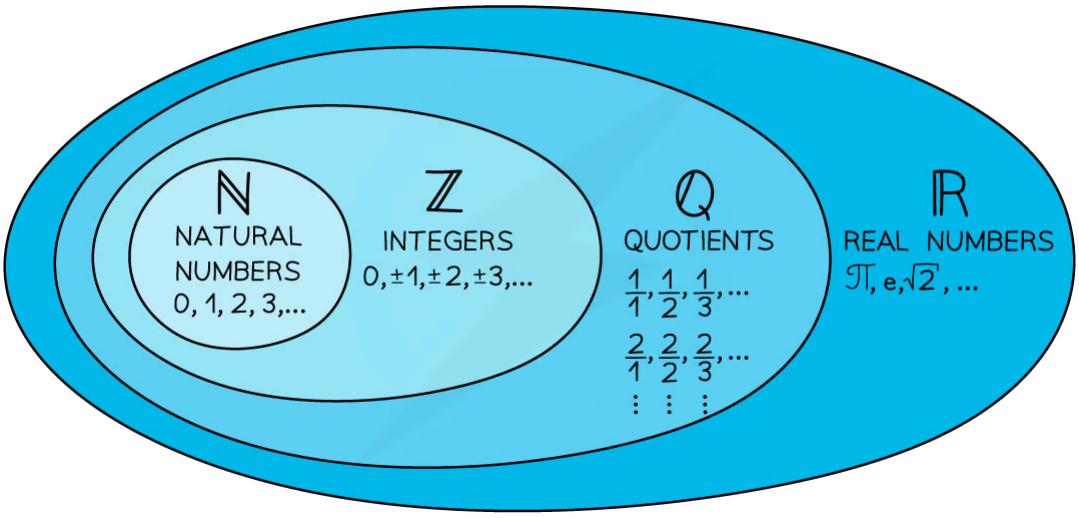
## What notation is used for functions?

- Functions are denoted using letters (such as $f$, $v$, $g$, etc)
  - A function is followed by a variable in a bracket
  - This shows the input for the function
  - The letter $f$ is used most commonly for functions and will be used for the remainder of this revision note
- $f(x)$ represents an expression for the value of the function $f$ when evaluated for the variable $x$
- Function notation gets rid of the need for words which makes it **universal**
  - $f = 5$ when $x = 2$ can simply be written as $f(2) = 5$

## What are the domain and range of a function?

- The **domain** of a function is the set of values that are used as **inputs**
- A domain should be stated with a function
  - If a domain is not stated then it is assumed the domain is all the real values which would work as inputs for the function
  - Domains are expressed in terms of the input
    - $x \leq 2$
- The **range** of a function is the set of values that are given as **outputs**
  - The range depends on the domain
  - Ranges are expressed in terms of the output
    - $f(x) \geq 0$
- To graph a function we use the **inputs as the *x*-coordinates** and the **outputs as the *y*-coordinates**
  - $f(2) = 5$ corresponds to the coordinates (2, 5)
- Graphing the function can help you visualise the range
- Common sets of numbers have special symbols:
  - $\mathbb{R}$ represents all the real numbers that can be placed on a number line
    - $x \in \mathbb{R}$ means $x$ is a real number
  - $\mathbb{Q}$ represents all the rational numbers $\frac{a}{b}$ where $a$ and $b$ are integers and $b \neq 0$
  - $\mathbb{Z}$ represents all the integers (positive, negative and zero)
    - $\mathbb{Z}^+$ represents positive integers
  - $\mathbb{N}$ represents the natural numbers (0,1,2,3...)

www.thrivingscholars.com



Copyright © Save My Exams. All Rights Reserved

## What are piecewise functions?

- **Piecewise functions** are defined by different functions depending on which interval the input is in
  - E.g. $f(x)=\begin{cases} x+1 & x\leq 5 \\ 2x-4 & 5<x<10 \\ x^2 & 10\leq x\leq 20 \end{cases}$
- The region for the individual functions **cannot overlap**
- To evaluate a piecewise function for a particular value $x=k$
  - Find which interval includes $k$
  - Substitute $x=k$ into the corresponding function
- The function **may or may not be continuous** at the ends of the intervals
  - In the example above the function is
    - continuous at $x=5$ as $5+1=2(5)-4$
    - not continuous at $x=10$ as $2(10)-4\neq 10^2$

### Exam Tip

- Questions may refer to "the largest possible domain"
  - This would usually be $x\in\mathbb{R}$ unless $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$ has already been stated
  - There are usualy some exceptions
    - e.g. square roots; $x\geq 0$ for a function involving $\sqrt{x}$
    - e.g. reciprocal functions; $x\neq 2$ for a function with denominator $(x-2)$

www.thrivingscholars.com

## Worked Example

For the function $f(x) = x^3 + 1,\ 2 \le x \le 10$:

a)
write down the value of $f(7)$.

Substitute $x = 7$

$f(7) = 7^3 + 1$

$f(7) = 344$

b)
find the range of $f(x)$.

Find the values of $x^3+1$ when $2 \le x \le 10$

$2 \le x \le 10$

$8 \le x^3 \le 1000$

$9 \le x^3+1 \le 1001$

$9 \le f(x) \le 1001$

www.thrivingscholars.com





# Composite Functions

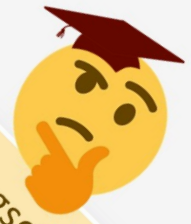
## What is a composite function?

- A **composite function** is where a function is applied to another function
- A composite function can be denoted
  - $(f \circ g)(x)$
  - $fg(x)$
  - $f(g(x))$
- The order matters
  - $(f \circ g)(x)$ means:
    - First apply *g* to *x* to get $g(x)$
    - Then apply *f* to the previous output to get $f(g(x))$
    - Always start with the function **closest to the variable**
  - $(f \circ g)(x)$ is not usually equal to $(g \circ f)(x)$

## How do I find the domain and range of a composite function?

- The domain of $f \circ g$ is the set of values of $x$...
  - which are a **subset** of the **domain of *g***
  - which maps *g* to a value that is in the **domain of *f***
- The range of $f \circ g$ is the set of values of $x$...
  - which are a **subset** of the **range of *f***
  - found by **applying *f*** to the **range of *g***
- To find the **domain** and **range** of $f \circ g$
  - First find the **range of *g***
  - **Restrict** these values to the values that are **within the domain** of ***f***
    - The **domain** is the set of values that **produce the restricted range** of *g*
    - The **range** is the set of values that are **produced using the restricted range** of *g* as the domain for *f*
- For example: let $f(x) = 2x + 1,\ -5 \le x \le 5$ and $g(x) = \sqrt{x},\ 1 \le x \le 49$
  - The **range of *g*** is $1 \le g(x) \le 7$
    - **Restricting** this to fit the **domain of *f*** results in $1 \le g(x) \le 5$
  - The **domain** of $f \circ g$ is therefore $1 \le x \le 25$
    - These are the values of *x* which map to $1 \le g(x) \le 5$
  - The **range** of $f \circ g$ is therefore $3 \le (f \circ g)(x) \le 11$
    - These are the values which *f* maps $1 \le g(x) \le 5$ to

www.thrivingscholars.com



### Exam Tip

- Make sure you know what your GDC is capable of with regard to functions
  - You may be able to store individual functions and find composite functions and their values for particular inputs
  - You may be able to graph composite functions directly and so deduce their domain and range from the graph
- The link between the domains and ranges of a function and its inverse can act as a check for your solution
- $ff(x)$ is not the same as $[f(x)]^2$

www.thrivingscholars.com

## Worked Example

Given $f(x)=\sqrt{x+4}$ and $g(x)=3+2x$:

a)
Write down the value of $(g \circ f)(12)$.

First apply function closest to input

$(g \circ f)(12) = g(f(12))$

$f(12) = \sqrt{12+4} = \sqrt{16} = 4$

$g(4) = 3 + 2(4) = 11$

$\boxed{(g \circ f)(12) = 11}$

b)
Write down an expression for $(f \circ g)(x)$.

First apply function closest to input

$(f \circ g)(x) = f(g(x))$

$= f(3+2x)$

$= \sqrt{3+2x+4}$

$\boxed{(f \circ g)(x) = \sqrt{7+2x}}$

c)
Write down an expression for $(g \circ g)(x)$.

$(g \circ g)(x) = g(g(x))$

$= g(3+2x)$

$= 3 + 2(3+2x)$

$= 3 + 6 + 4x$

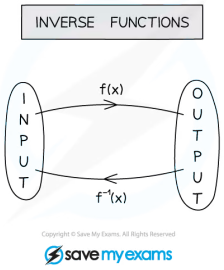
$\boxed{(g \circ g)(x) = 9 + 4x}$

www.thrivingscholars.com

# Inverse Functions

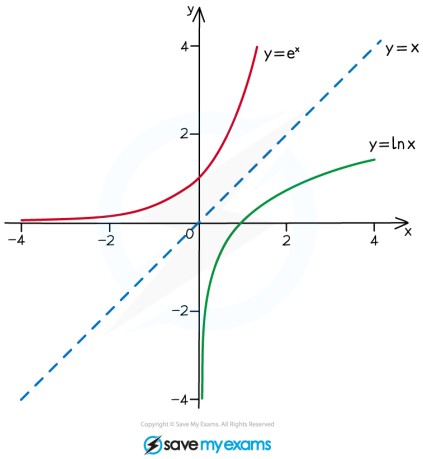
## What is an inverse function?

- **Only one-to-one** functions have inverses
- A function has an inverse if its graph passes the **horizontal line test**
  - Any **horizontal line** will intersect with the graph **at most once**
- The **identity function** $\mathrm{id}$ maps each value to itself
  - $\mathrm{id}(x) = x$
- If $f \circ g$ and $g \circ f$ have the **same effect as the identity function** then $f$ and $g$ are **inverses**
- Given a function $f(x)$ we denote the **inverse function** as $f^{-1}(x)$
- An inverse function **reverses the effect** of a function
  - $f(2) = 5$ means $f^{-1}(5) = 2$
- Inverse functions are used to solve equations
  - The solution of $f(x) = 5$ is $x = f^{-1}(5)$
- A composite function made of $f$ and $f^{-1}$ has the **same effect as the identity function**
  - $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$




## What are the connections between a function and its inverse function?

- The **domain of a function** becomes the **range of its inverse**
- The **range of a function** becomes the **domain of its inverse**
- The graph of $y = f^{-1}(x)$ is a **reflection** of the graph $y = f(x)$ in the line $y = x$
  - Therefore solutions to $f(x) = x$ or $f^{-1}(x) = x$ will also be solutions to $f(x) = f^{-1}(x)$
    - There could be other solutions to $f(x) = f^{-1}(x)$ that don't lie on the line $y = x$




## How do I find the inverse of a function?

www.thrivingscholars.com



- STEP 1: **Swap** the *x* and *y* in $y = f(x)$
  - If $y = f^{-1}(x)$ then $x = f(y)$
- STEP 2: **Rearrange** $x = f(y)$ to make $y$ the subject
- Note this can be done in any order
  - Rearrange $y = f(x)$ to make $x$ the subject
  - Swap $x$ and $y$

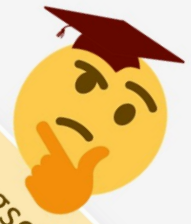
## Can many-to-one functions ever have inverses?

- You can **restrict the domain** of a many-to-one function so that it has an inverse
- Choose a subset of the domain where the function is one-to-one
  - The inverse will be determined by the restricted domain
  - Note that a many-to-one function can **only** have an inverse if its domain is restricted first
- For **quadratics** – use the **vertex** as the upper or lower bound for the **restricted domain**
  - For $f(x) = x^2$ restrict the domain so 0 is either the maximum or minimum value
    - For example: $x \geq 0$ or $x \leq 0$
  - For $f(x) = a(x-h)^2 + k$ restrict the domain so *h* is either the maximum or minimum value
    - For example: $x \geq h$ or $x \leq h$
- For **trigonometric functions** – use part of a cycle as the **restricted domain**
  - For $f(x) = \sin x$ restrict the domain to half a cycle between a maximum and a minimum
    - For example: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
  - For $f(x) = \cos x$ restrict the domain to half a cycle between maximum and a minimum
    - For example: $0 \leq x \leq \pi$
  - For $f(x) = \tan x$ restrict the domain to one cycle between two asymptotes
    - For example: $-\frac{\pi}{2} < x < \frac{\pi}{2}$

## How do I find the inverse function after restricting the domain?

- The range of the inverse is the same as the restricted domain of the original function
- The inverse function is determined by the restricted domain
  - Restricting the domain differently will change the inverse function
- Use the range of the inverse to help find the inverse function
  - Restricting the domain of $f(x) = x^2$ to $x \geq 0$ means the range of the inverse is $f^{-1}(x) \geq 0$
    - Therefore $f^{-1}(x) = \sqrt{x}$
  - Restricting the domain of $f(x) = x^2$ to $x \leq 0$ means the range of the inverse is $f^{-1}(x) \leq 0$
    - Therefore $f^{-1}(x) = -\sqrt{x}$

www.thrivingscholars.com



### Exam Tip

- Remember that an inverse function is a reflection of the original function in the line $y = x$
  - Use your GDC to plot the function and its inverse on the same graph to visually check this
- $f^{-1}(x)$ is not the same as $\frac{1}{f(x)}$

www.thrivingscholars.com

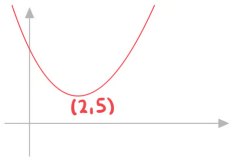
## Worked Example

The function $f(x) = (x-2)^2 + 5, \ x \leq m$ has an inverse.

a)
Write down the largest possible value of $m$.

Sketch $y = f(x)$

The graph is one-to-one for $x \leq 2$




$m = 2$

b)
Find the inverse of $f(x)$.

Let $y = f^{-1}(x)$ and rearrange $x = f(y)$

$x = (y-2)^2 + 5$

$x - 5 = (y-2)^2$

$\pm\sqrt{x-5} = y - 2$

$2 \pm \sqrt{x-5} = y$

Range of $f^{-1}$ is the domain of $f$

$f^{-1}(x) \leq 2 \quad \therefore \ y = 2 - \sqrt{x-5}$

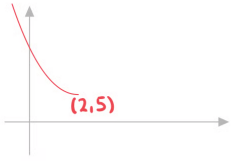
$f^{-1}(x) = 2 - \sqrt{x-5}$

c)
Find the domain of $f^{-1}(x)$.

Domain of $f^{-1}$ is the range of $f$

Sketch $y = f(x)$ to see range

For $x \leq 2$, $f(x) \geq 5$




Domain of $f^{-1}$ : $x \geq 5$

d)
Find the value of $k$ such that $f(k) = 9$.

www.thrivingscholars.com



Use inverse $f(a) = b \iff a = f^{-1}(b)$

$k = f^{-1}(9) = 2 - \sqrt{9-5}$

$\boxed{k = 0}$

www.thrivingscholars.com





# Odd & Even Functions

## What are odd functions?

- A function $f(x)$ is called **odd** if
  - $f(-x) = -f(x)$ for all values of $x$
- Examples of odd functions include:
  - Power functions with **odd powers**: $x^{2n+1}$ where $n \in \mathbb{Z}$
    - For example: $(-x)^3 = -x^3$
  - Some **trig functions**: $\sin x$, $\operatorname{cosec} x$, $\tan x$, $\cot x$
    - For example: $\sin(-x) = -\sin x$
  - **Linear combinations** of odd functions
    - For example: $f(x) = 3x^5 - 4\sin x + \frac{6}{x}$

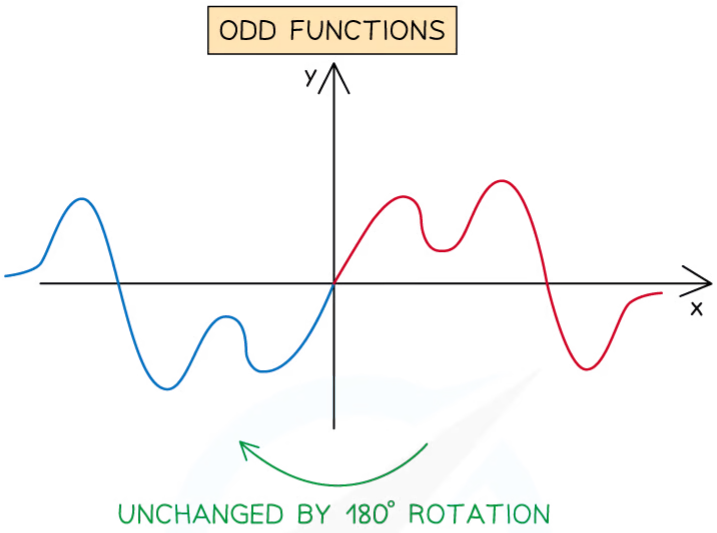
## What are even functions?

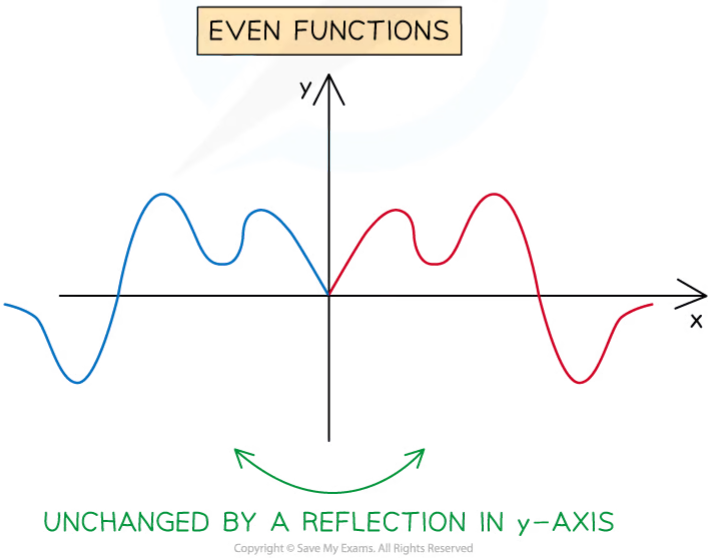
- A function $f(x)$ is called **even** if
  - $f(-x) = f(x)$ for all values of $x$
- Examples of even functions include:
  - Power functions with **even powers**: $x^{2n}$ where $n \in \mathbb{Z}$
    - For example: $(-x)^4 = x^4$
  - Some **trig functions**: $\cos x$, $\sec x$
    - For example: $\cos(-x) = \cos x$
  - **Modulus function**: $|x|$
  - **Linear combinations** of even functions
    - For example: $f(x) = 7x^6 + 3|x| - 8\cos x$

## What are the symmetries of graphs of odd & even functions?

- The graph of an **odd** function has **rotational symmetry**
  - The graph is unchanged by a **180° rotation** about the origin
- The graph of an **even** function has **reflective symmetry**
  - The graph is unchanged by a **reflection** in the **y-axis**

www.thrivingscholars.com




Copyright © Save My Exams. All Rights Reserved

## Exam Tip

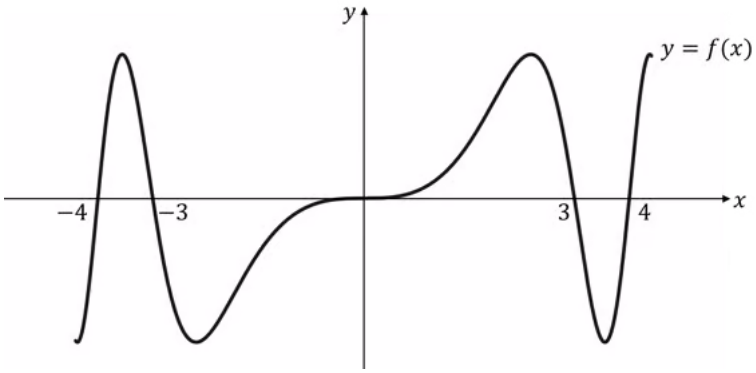
- Turn your GDC upside down for a quick visual check for an odd function!
  - Ignoring axes, etc, if the graph looks exactly the same both ways, it's odd

www.thrivingscholars.com

## Worked Example

a)

The graph $y = f(x)$ is shown below. State, with a reason, whether the function $f$ is odd, even or neither.




f is an odd function as its graph has rotational symmetry - it is unchanged by a 180° rotation about the origin.

b)

Use algebra to show that $g(x) = x^3\sin(x) + 5\cos(x^5)$ is an even function.

g is even if $g(-x) = g(x)$ for all $x$

$g(-x) = (-x)^3\sin(-x) + 5\cos((-x)^5)$

$= (-x^3)(-\sin(x)) + 5\cos(-x^5)$ — $x^3$, $x^5$, $\sin x$ are odd

$= x^3\sin(x) + 5\cos(x^5)$ — $\cos x$ is even

$= g(x)$

g is even as $g(-x) = g(x)$ for all $x$

www.thrivingscholars.com

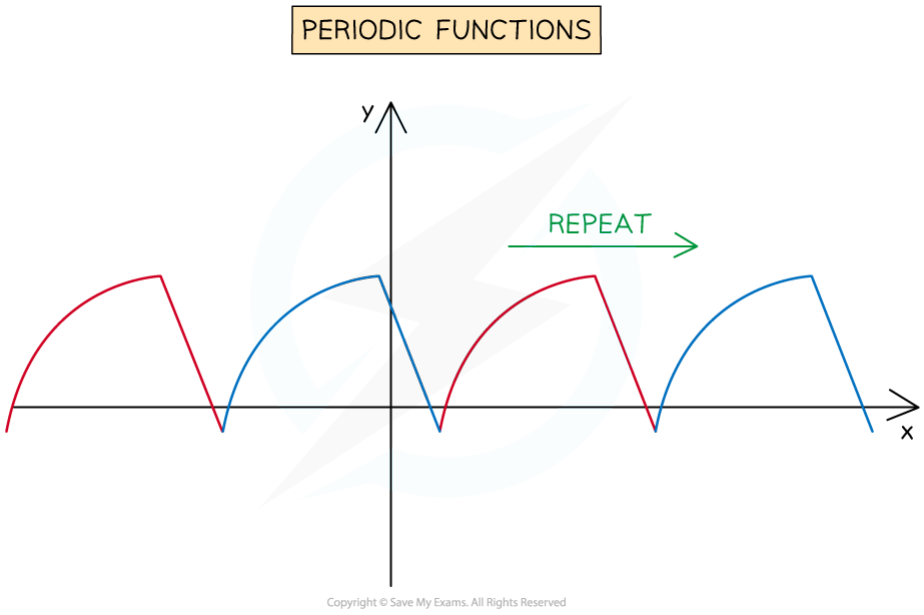
# Periodic Functions

## What are periodic functions?

- A function $f(x)$ is called **periodic**, with **period *k***, if
  - $f(x+k)=f(x)$ for all values of $x$
- Examples of periodic functions include:
  - sin *x* & cos *x*: The period is 2π or 360°
  - tan *x*: The period is π or 180°
  - **Linear combinations** of periodic functions with the **same period**
    - For example: $f(x)=2\sin(3x)-5\cos(3x+2)$

## What are the symmetries of graphs of periodic functions?

- The graph of a **periodic** function has **translational symmetry**
  - The graph is unchanged by **translations** that are **integer multiples of** $\begin{pmatrix} k \\ 0 \end{pmatrix}$
  - The means that the graph appears to **repeat** the same section (cycle) infinitely




Copyright © Save My Exams. All Rights Reserved

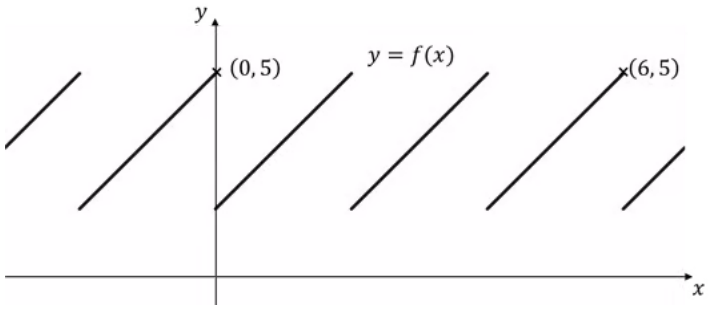
### Exam Tip

- There may be several intersections between the graph of a periodic function and another function
  - i.e. Equations may have several solutions so only answers within a certain range of $x$-values may be required
    - e.g. Solve $\tan x=\sqrt{3}$ for $0° \leq x \leq 360°$
    - $x=60°, 240°$
  - Alternatively you may have to write **all** solutions in a general form
    - e.g. $x=60(3k+1)°, \quad k=0, \pm 1, \pm 2, \ldots$

www.thrivingscholars.com

## Worked Example

The graph $y = f(x)$ is shown below. Given that $f$ is periodic, write down the period.




Period is the length of the interval of a single cycle

Between $x=0$ and $x=6$ there are 3 cycles

Period = $\frac{6-0}{3}$

Period = 2

www.thrivingscholars.com

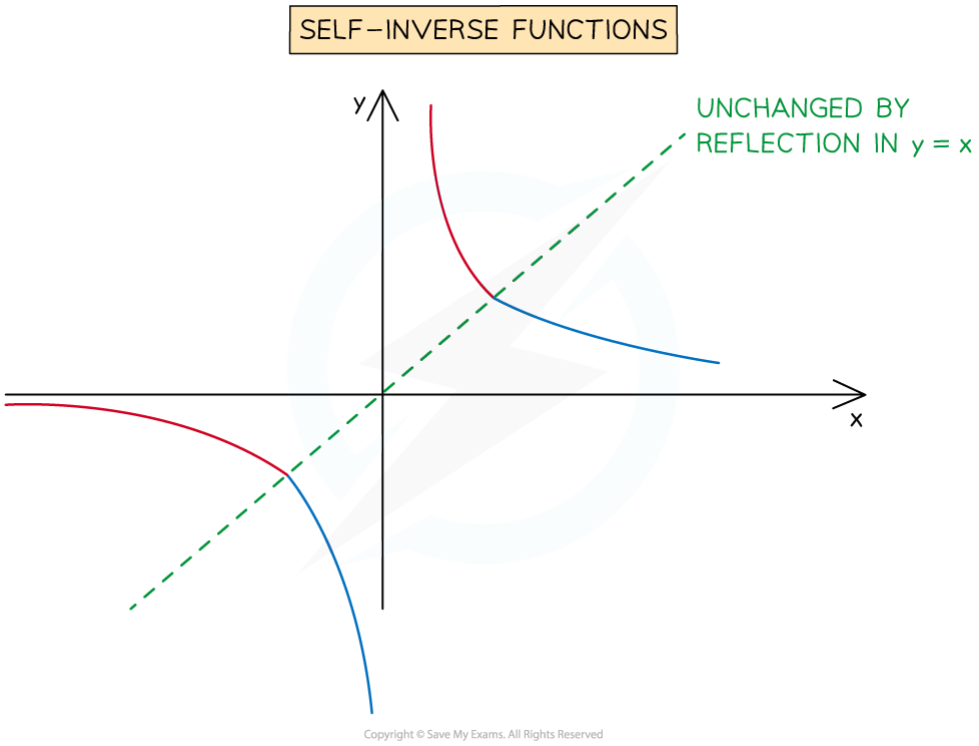
# Self-Inverse Functions

## What are self-inverse functions?

- A function $f(x)$ is called **self-inverse** if
  - $(f \circ f)(x) = x$ for all values of $x$
  - $f^{-1}(x) = f(x)$
- Examples of self-inverse functions include:
  - **Identity function**: $f(x) = x$
  - **Reciprocal function**: $f(x) = \frac{1}{x}$
  - **Linear functions** with a **gradient of -1**: $f(x) = -x + c$

## What are the symmetries of graphs of self-inverse functions?

- The graph of a **self-inverse** function has **reflective symmetry**
  - The graph is unchanged by a **reflection** in the line ***y = x***



Copyright © Save My Exams. All Rights Reserved

### Exam Tip

- If your expression for $f^{-1}(x)$ is not the same as the expression for $f(x)$ you can check their equivalence by plotting both on your GDC
  - If equivalent the graphs will sit on top of one another and appear as one
  - This will indicate if you have made an error in your algebra, before trying to simplify/rewrite to make the two expressions identical
- It is sometimes easier to consider self inverse functions geometrically rather than algebraically

www.thrivingscholars.com

## Worked Example

Use algebra to show the function defined by $f(x)=\dfrac{7x-5}{x-7}$, $x\neq 7$ is self-inverse.

**Method 1: $f^{-1}(x)$**

Let $y=f^{-1}(x)$ so $x=f(y)$

$x=\dfrac{7y-5}{y-7}$

$(y-7)x=7y-5$

$xy-7x=7y-5$

$xy-7y=7x-5$ — Isolate $y$ on one side

$(x-7)y=7x-5$

$y=\dfrac{7x-5}{x-7}$

$f^{-1}(x)=\dfrac{7x-5}{x-7}=f(x)$

$\therefore f$ is self-inverse

**Method 2: $(f\circ f)(x)$**

$(f\circ f)(x)=f(f(x))$

$f(f(x))=\dfrac{7f(x)-5}{f(x)-7}$

$=\dfrac{7\left(\frac{7x-5}{x-7}\right)-5}{\frac{7x-5}{x-7}-7}$ — $\times \frac{x-7}{x-7}$

$=\dfrac{7(7x-5)-5(x-7)}{7x-5-7(x-7)}$

$=\dfrac{49x-35-5x+35}{7x-5-7x+49}$

$=\dfrac{44x}{44}$

$(f\circ f)(x)=x$

$\therefore f$ is self-inverse

www.thrivingscholars.com



# Graphing Functions

## How do I graph the function $y = f(x)$?

- A point $(a, b)$ lies on the graph $y = f(x)$ if $f(a) = b$
- The **horizontal axis** is used for the **domain**
- The **vertical axis** is used for the **range**
- You will be able to graph some functions by hand
- For some functions you will need to use your GDC
- You might be asked to graph the **sum** or **difference** of two functions
  - Use your GDC to graph $y = f(x) + g(x)$ or $y = f(x) - g(x)$
  - Just type the functions into the graphing mode

## What is the difference between "draw" and "sketch"?

- If asked to sketch you should:
  - Show the general shape
  - Label any key points such as the intersections with the axes
  - Label the axes
- If asked to draw you should:
  - Use a pencil and ruler
  - Draw to scale
  - Plot any points **accurately**
  - Join points with a straight line or smooth curve
  - Label any key points such as the intersections with the axes
  - Label the axes

## How can my GDC help me sketch/draw a graph?

- You use your GDC to plot the graph
  - Check the scales on the graph to make sure you see the full shape
- Use your GDC to find any key points
- Use your GDC to check specific points to help you plot the graph

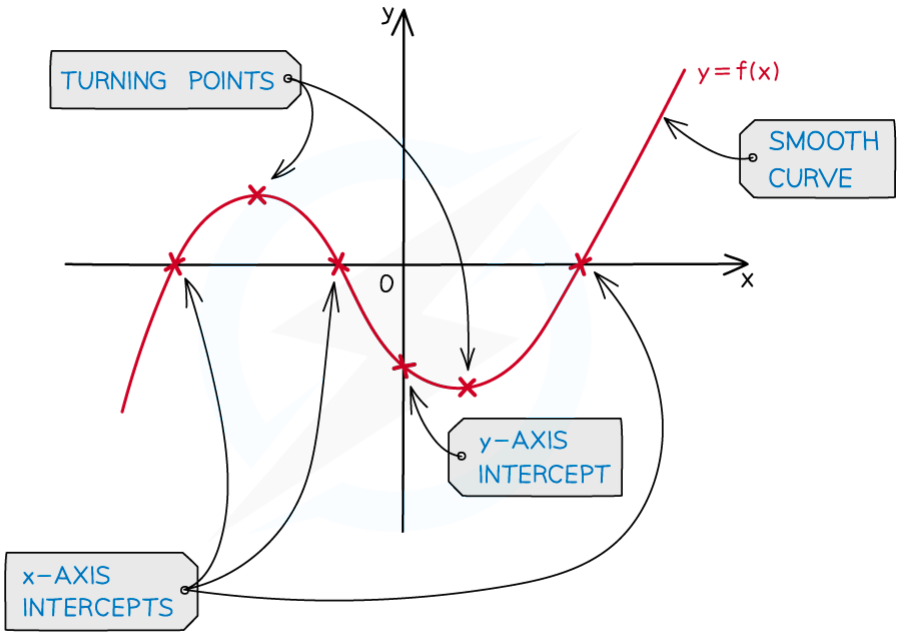
www.thrivingscholars.com

# Key Features of Graphs

## What are the key features of graphs?

- You should be familiar with the following key features and know how to use your GDC to find them
- Local minimums/maximums
  - These are points where the graph has a minimum/maximum for a small region
  - They are also called **turning points**
    - This is where the graph changes its direction between upwards and downwards directions
  - A graph can have multiple local minimums/maximums
  - A local minimum/maximum is not necessarily the minimum/maximum of the whole graph
    - This would be called the **global** minimum/maximum
  - For quadratic graphs the minimum/maximum is called the **vertex**
- Intercepts
  - $y$ – intercepts are where the graph crosses the $y$-axis
    - At these points $x = 0$
  - $x$ – intercepts are where the graph crosses the $x$-axis
    - At these points $y = 0$
    - These points are also called the **zeros of the function** or **roots of the equation**
- Symmetry
  - Some graphs have lines of symmetry
    - A quadratic will have a vertical line of symmetry
- Asymptotes
  - These are lines which the graph will get closer to but not cross
  - These can be horizontal or vertical
    - Exponential graphs have horizontal asymptotes
    - Graphs of variables which vary inversely can have vertical and horizontal asymptotes

www.thrivingscholars.com




Copyright © Save My Exams. All Rights Reserved
save my exams

## Exam Tip

- Most GDC makes/models will not plot/show asymptotes just from inputting a function
  - Add the asymptotes as additional graphs for your GDC to plot
  - You can then check the equations of your asymptotes visually
  - You may have to zoom in or change the viewing window options to confirm an asymptote
- Even if using your GDC to plot graphs and solve problems sketching them as part of your working is good exam technique
  - Label the key features of the graph and anything else relevant to the question on your sketch

www.thrivingscholars.com

## Worked Example

Two functions are defined by

$$f(x) = x^2 - 4x - 5 \text{ and } g(x) = 2 + \frac{1}{x+1}.$$

a)

Draw the graph $y = f(x)$.

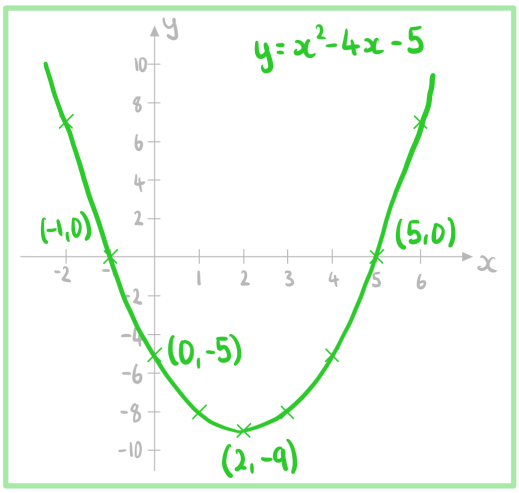
Draw means accurately

Use GDC to find vertex, roots and y-intercepts

Vertex = (2, -9)

Roots = (-1, 0) and (5, 0)

y-intercept = (0, -5)




b)

Sketch the graph $y = g(x)$.

www.thrivingscholars.com

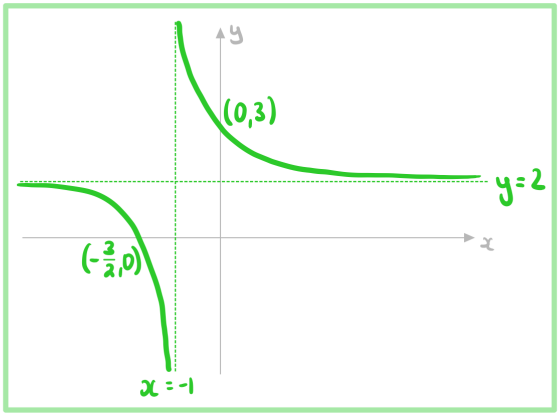
Sketch means rough but showing key points

Use GDC to find $x$ and $y$-intercepts and asymptotes

$x$-intercept $= \left(-\frac{3}{2}, 0\right)$

$y$-intercept $= (0, 3)$

Asymptotes : $x = -1$ and $y = 2$

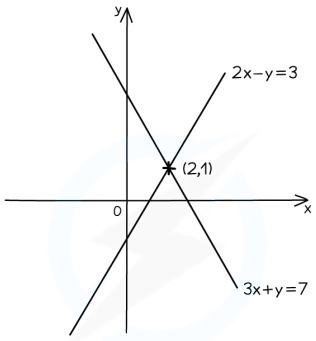


www.thrivingscholars.com

# Intersecting Graphs

## How do I find where two graphs intersect?

- Plot both graphs on your GDC
- Use the intersect function to find the intersections
- Check if there is more than one point of intersection




- LINES INTERSECT AT (2,1)
- SOLVING 2x−y=3 AND 3x+y=7 SIMULTANEOUSLY IS x=2, y=1

Copyright © Save My Exams. All Rights Reserved
save my exams

## How can I use graphs to solve equations?

- One method to solve equations is to use graphs
- To solve $f(x) = a$
  - Plot the two graphs $y = f(x)$ and $y = a$ on your GDC
  - Find the points of intersections
  - The **x-coordinates** are the **solutions** of the equation
- To solve $f(x) = g(x)$
  - Plot the two graphs $y = f(x)$ and $y = g(x)$ on your GDC
  - Find the points of intersections
  - The **x-coordinates** are the **solutions** of the equation
- Using graphs makes it easier to see **how many solutions** an equation will have

### Exam Tip

- You can use graphs to solve equations
  - Questions will not necessarily ask for a drawing/sketch or make reference to graphs
  - Use your GDC to plot the equations and find the intersections between the graphs

www.thrivingscholars.com

## Worked Example

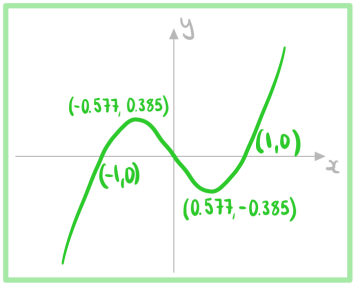
Two functions are defined by

$$f(x) = x^3 - x \text{ and } g(x) = \frac{4}{x}.$$

a)

Sketch the graph $y = f(x)$.

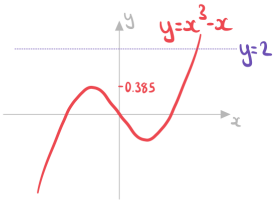
Use GDC to find max, min, intercepts




b)

Write down the number of real solutions to the equation $x^3 - x = 2$.

Identify the number of intersections between $y = x^3 - x$ and $y = 2$




1 intersection

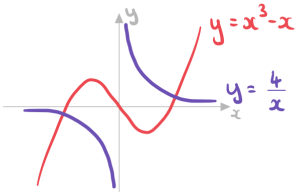
1 solution

c)

Find the coordinates of the points where $y = f(x)$ and $y = g(x)$ intersect.

www.thrivingscholars.com

Use GDC to sketch both graphs




$(-1.60, -2.50)$ and $(1.60, 2.50)$

d)

Write down the solutions to the equation $x^3 - x = \frac{4}{x}$.

Solutions to $x^3 - x = \frac{4}{x}$ are the $x$ coordinates of the points of intersection.

$x = -1.60$ and $x = 1.60$

www.thrivingscholars.com

# 2.4 Other Functions & Graphs

## 2.4.1 Exponential & Logarithmic Functions

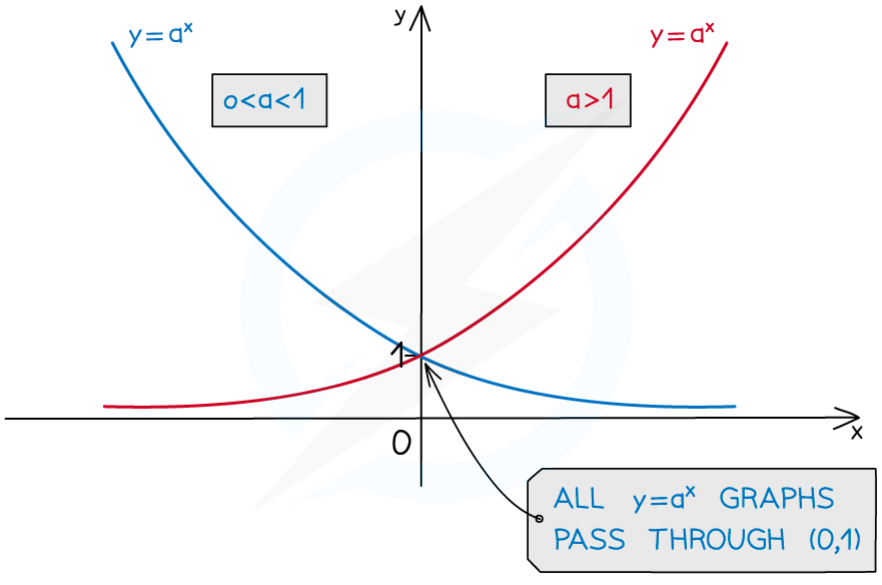
### Exponential Functions & Graphs

### What is an exponential function?

- An **exponential function** is defined by $f(x) = a^x,\ a > 0$
- Its **domain** is the set of **all real values**
- Its **range** is the set of **all positive real values**
- An important exponential function is $f(x) = e^x$
  - Where e is the mathematical constant 2.718...
- Any exponential function can be written using e
  - $a^x = e^{x \ln a}$
    - This is given in the **formula booklet**

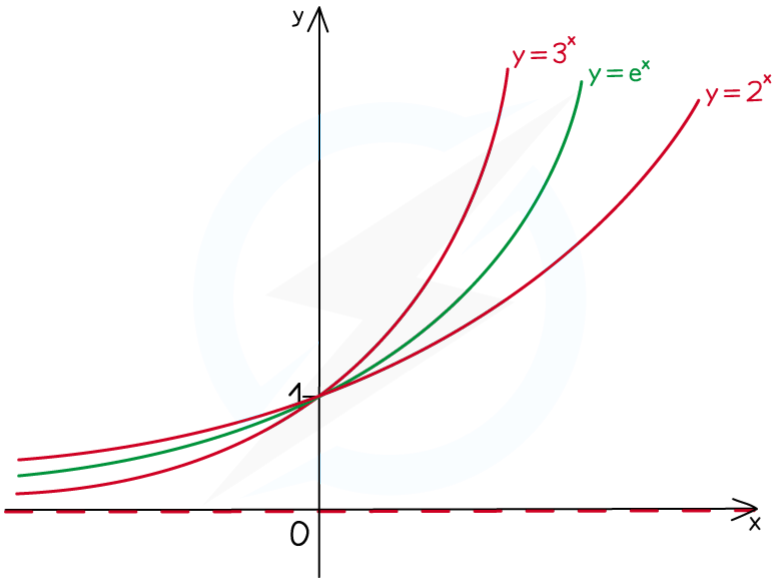
### What are the key features of exponential graphs?

- The graphs have a **y-intercept** at (0, 1)
- The graphs **do not have any roots**
- The graphs have **a horizontal asymptote** at the *x*-axis: $y = 0$
  - For ***a* > 1** this is the **limiting value** when *x* tends to **negative infinity**
  - For **0 < *a* < 1** this is the **limiting value** when *x* tends to **positive infinity**
- The graphs **do not have any minimum or maximum points**

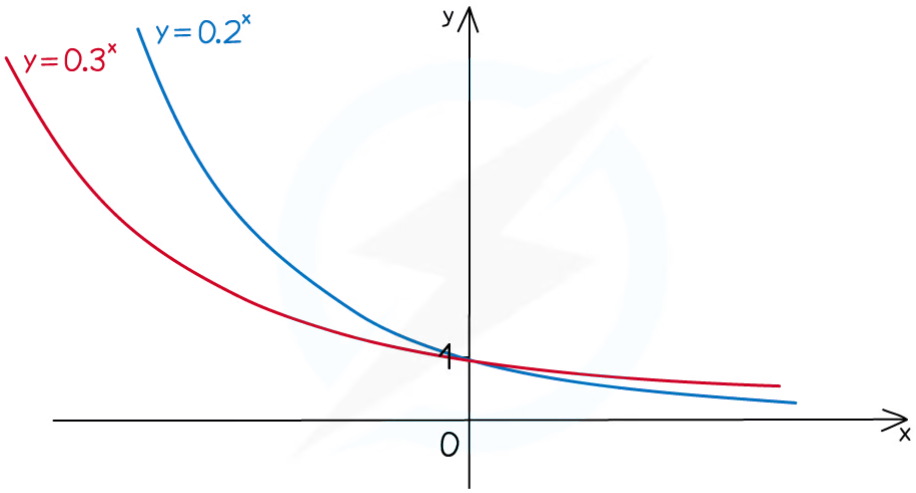



Copyright © Save My Exams. All Rights Reserved

www.thrivingscholars.com




Copyright © Save My Exams. All Rights Reserved




Copyright © Save My Exams. All Rights Reserved

www.thrivingscholars.com

# Logarithmic Functions & Graphs

## What is a logarithmic function?

- A **logarithmic function** is of the form $f(x) = \log_a x,\ x > 0$
- Its **domain** is the set of all **positive real values**
  - You can't take a log of zero or a negative number
- Its **range** is set of **all real values**
- $\log_a x$ and $a^x$ are **inverse** functions
- An important logarithmic function is $f(x) = \ln x$
  - This is the natural logarithmic function $\ln x \equiv \log_e x$
  - This is the inverse of $e^x$
    - $\ln e^x = x$ and $e^{\ln x} = x$
- Any logarithmic function can be written using ln
  - $\log_a x = \dfrac{\ln x}{\ln a}$ using the change of base formula

## What are the key features of logarithmic graphs?

- The graphs **do not have a *y*-intercept**
- The graphs have **one root** at (1, 0)
- The graphs have a **vertical asymptote** at the *y*-axis: $x = 0$
- The graphs **do not have any minimum or maximum points**

www.thrivingscholars.com

## Worked Example

The function $f$ is defined by $f(x) = \log_5 x$ for $x > 0$.

a)

Write down the inverse of $f$. Give your answer in the form $e^{g(x)}$.

Formula booklet

| Exponents & logarithms | $a^x = b \Leftrightarrow x = \log_a b$ $\quad a > 0, b > 0, a \neq 1$ |
|---|---|

$x = \log_5 y \Leftrightarrow y = 5^x$

Formula booklet

| Exponential & logarithmic functions | $a^x = e^{x \ln a}$ |
|---|---|

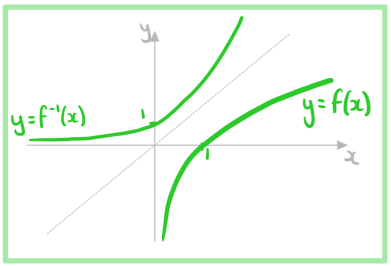
$5^x = e^{x \ln 5}$

$f^{-1}(x) = e^{x \ln 5}$

b)

Sketch the graphs of $f$ and its inverse on the same set of axes.

$f$ and $f^{-1}$ are reflections in line $y = x$



www.thrivingscholars.com





# Solving Equations Analytically

## How can I solve equations analytically where the unknown appears only once?

- These equations can be **solved by rearranging**
- For **one-to-one functions** you can just apply the **inverse**
  - Addition and subtraction are inverses
    - $y = x + k \Leftrightarrow x = y - k$
  - Multiplication and division are inverses
    - $y = kx \Leftrightarrow x = \frac{y}{k}$
  - Taking the reciprocal is a self-inverse
    - $y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}$
  - Odd powers and roots are inverses
    - $y = x^n \Leftrightarrow x = \sqrt[n]{y}$
    - $y = x^n \Leftrightarrow x = y^{\frac{1}{n}}$
  - Exponentials and logarithms are inverses
    - $y = a^x \Leftrightarrow x = \log_a y$
    - $y = \mathrm{e}^x \Leftrightarrow x = \ln y$
- For **many-to-one functions** you will need to use your knowledge of the functions to find the **other solutions**
  - Even powers lead to positive and negative solutions
    - $y = x^n \Leftrightarrow x = \pm\sqrt[n]{y}$
  - Modulus functions lead to positive and negative solutions
    - $y = |x| \Leftrightarrow x = \pm y$
  - Trigonometric functions lead to infinite solutions using their symmetries
    - $y = \sin x \Leftrightarrow x = 2k\pi + \arcsin y \text{ or } x = (1 + 2k)\pi - \arcsin y$
    - $y = \cos x \Leftrightarrow x = 2k\pi \pm \arccos y$
    - $y = \tan x \Leftrightarrow x = k\pi + \arctan y$
- Take care when you apply **many-to-one functions** to **both sides** of an equation as this can create **additional solutions** which are incorrect
  - For example: squaring both sides
    - $x + 1 = 3$ has one solution $x = 2$
    - $(x + 1)^2 = 3^2$ has two solutions $x = 2$ and $x = -4$
- Always **check your solutions** by substituting back into the **original equation**

## How can I solve equations analytically where the unknown appears more than once?

- Sometimes it is possible to **simplify expressions** to make the **unknown appear only once**
- **Collect all terms** involving $x$ on **one side** and try to simplify into one term
  - For **exponents** use
    - $a^{f(x)} \times a^{g(x)} = a^{f(x)+g(x)}$

www.thrivingscholars.com



- $\frac{a^{f(x)}}{a^{g(x)}} = a^{f(x)-g(x)}$
- $(a^{f(x)})g(x) = a^{f(x)\times g(x)}$
- $a^{f(x)} = e^{f(x)\ln a}$

- For **logarithms** use
  - $\log_a f(x) + \log_a g(x) = \log_a (f(x) \times g(x))$
  - $\log_a f(x) - \log_a g(x) = \log_a \left(\frac{f(x)}{g(x)}\right)$
  - $n\log_a f(x) = \log_a (f(x))^n$

## How can I solve equations analytically when the equation can't be simplified?

- Sometimes it is **not possible to simplify** equations
- Most of these equations **cannot be solved analytically**
- A **special case** that can be solved is where the equation can be **transformed into a quadratic** using a substitution
  - These will have **three terms** and involve the same type of function
- **Identify** the **suitable substitution** by considering which **function is a square of another**
  - For example: the following can be transformed into $2y^2 + 3y - 4 = 0$
    - $2x^4 + 3x^2 - 4 = 0$ using $y = x^2$
    - $2x + 3\sqrt{x} - 4 = 0$ using $y = \sqrt{x}$
    - $\frac{2}{x^6} + \frac{3}{x^3} - 4 = 0$ using $y = \frac{1}{x^3}$
    - $2e^{2x} + 3e^x - 4 = 0$ using $y = e^x$
    - $2 \times 25^x + 3 \times 5^x - 4 = 0$ using $y = 5^x$
    - $2^{2x+1} + 3 \times 2^x - 4 = 0$ using $y = 2^x$
    - $2(x^3 - 1)^2 + 3(x^3 - 1) - 4 = 0$ using $y = x^3 - 1$
- To **solve**:
  - Make the **substitution** $y = f(x)$
  - **Solve** the **quadratic equation** $ay^2 + by + c = 0$ to get $y_1$ & $y_2$
  - **Solve** $f(x) = y_1$ and $f(x) = y_2$
    - Note that some equations might have **zero or several solutions**

## Can I divide both sides of an equation by an expression?

- When dividing by an expression you must consider whether the **expression could be zero**
- Dividing by an expression that could be zero could result in you **losing solutions to the original equation**
  - For example: $(x + 1)(2x - 1) = 3(x + 1)$
    - If you divide both sides by $(x + 1)$ you get $2x - 1 = 3$ which gives $x = 2$
    - However $x = -1$ is also a solution to the original equation
- To ensure you **do not lose solutions** you can:
  - **Split the equation into two equations**
    - One where the dividing expression equals zero: $x + 1 = 0$
    - One where the equation has been divided by the expression: $2x - 1 = 3$
  - **Make the equation equal zero and factorise**

www.thrivingscholars.com



- $(x+1)(2x-1)-3(x+1)=0$
- $(x+1)(2x-1-3)=0$ which gives $(x+1)(2x-4)=0$
- Set each factor equal to zero and solve: $x+1=0$ and $2x-4=0$



### Exam Tip

- A common mistake that students make in exams is applying functions to each term rather than to each side
  - For example: Starting with the equation $\ln x+\ln(x-1)=5$ it would be incorrect to write $e^{\ln x}+e^{\ln(x-1)}=e^5$ or $x+(x-1)=e^5$
  - Instead it would be correct to write $e^{\ln x+\ln(x-1)}=e^5$ and then simplify from there

www.thrivingscholars.com

## Worked Example

Find the exact solutions for the following equations:

a)

$5-2\log_4 x=0.$

Rearrange using inverse functions

$5-2\log_4 x=0$

$2\log_4 x=5$ $\quad y=x-k \Leftrightarrow x=y+k$

$\log_4 x=\frac{5}{2}$ $\quad y=kx \Leftrightarrow x=\frac{y}{k}$

$x=4^{5/2}$ $\quad y=\log_a x \Leftrightarrow x=a^y$

$x=(\sqrt{4})^5$ $\quad a^{\frac{m}{n}}=(\sqrt[n]{a})^m$

$x=32$

b)

$x=\sqrt{x+2}.$

Square both sides (Many-to-one function)

$x^2=x+2 \Rightarrow x^2-x-2=0$

$(x-2)(x+1)=0 \Rightarrow x=2$ or $x=-1$

Check whether each solution is valid

$x=2$: LHS $=2$ RHS $=\sqrt{2+2}=2$ ✓

$x=-1$: LHS $=-1$ RHS $=\sqrt{-1+2}=1$ ×

$x=2$

c)

$e^{2x}-4e^x-5=0.$

www.thrivingscholars.com



Notice $e^{2x} = (e^x)^2$, let $y = e^x$

$y^2 - 4y - 5 = 0 \Rightarrow (y+1)(y-5) = 0$

$y = -1$ or $y = 5$

Solve using $y = e^x$

$e^x = -1$ has no solutions as $e^x > 0$

$e^x = 5 \quad \therefore \quad x = \ln 5$

$\boxed{x = \ln 5}$

www.thrivingscholars.com

# Solving Equations Graphically

## How can I solve equations graphically?

- To solve $f(x) = g(x)$
  - One method is to **draw the graphs** $y = f(x)$ and $y = g(x)$
    - The **solutions** are the **x-coordinates** of the points of **intersection**
  - Another method is to **draw the graph** $y = f(x) - g(x)$ or $y = g(x) - f(x)$
    - The **solutions** are the **roots (zeros)** of this graph
      - This method is sometimes quicker as it involves **drawing only one graph**

## Why do I need to solve equations graphically?

- Some equations **cannot be solved analytically**
  - **Polynomials** of degree higher than 4
    - $x^5 - x + 1 = 0$
  - Equations involving **different types of functions**
    - $e^x = x^2$

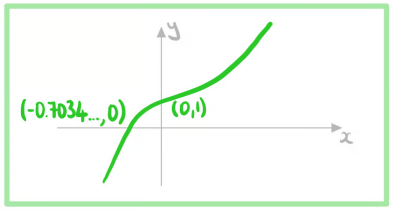
### Exam Tip

- On a calculator paper you are allowed to solve equations using your GDC unless the question asks for an algebraic method
- If your answer needs to be an exact value then you might need to solve analytically to get the exact value

### Worked Example

a)
Sketch the graph $y = e^x - x^2$.

Sketch using GDC




b)
Hence find the solution to $e^x = x^2$.

$e^x = x^2$ when $e^x - x^2 = 0$

Solution is the $x$-intercept of $y = e^x - x^2$

$x = -0.703$ (3sf)

www.thrivingscholars.com

# Modelling with Functions

## What is a mathematical model?

- A **mathematical model** simplifies a real-world situation so it can be described using mathematics
  - The model can then be used to make predictions
- **Assumptions** about the situation are made in order to simplify the mathematics
- Models can be **refined** (improved) if further information is available or if the model is compared to real-world data

## How do I set up the model?

- The question could:
  - give you the equation of the model
  - tell you about the relationship
    - It might say the relationship is linear, quadratic, etc
  - ask you to suggest a **suitable model**
    - Use your knowledge of each model
    - E.g. if it is compound interest then an exponential model is the most appropriate
- You may have to determine a **reasonable domain**
  - Consider real-life context
    - E.g. if dealing with hours in a day then
    - E.g. if dealing with physical quantities (such as length) then
  - Consider the **possible ranges**
    - If the outcome cannot be negative then you want to choose a domain which corresponds to a range with no negative values
    - **Sketching the graph** is helpful to determine a suitable domain

## Which models might I need to use?

- You could be given any model and be expected to use it
- Common models include:
  - **Linear**
    - Arithmetic sequences
    - Linear regression
  - **Quadratic**
    - Projectile motion
    - The height of a cable supporting a bridge
    - Profit
  - **Exponential**
    - Geometric sequences
    - Exponential growth and decay
    - Compound interest
  - **Logarithmic**
    - Richter scale for the magnitude of earthquakes
  - **Rational**
    - Temperature of a cup of coffee

www.thrivingscholars.com



    - **Trigonometric**
        - The depth of a tide

## How do I use a model?

- You can use a model by substituting in values for the variable to **estimate outputs**
    - For example: Let $h(t)$ be the height of a football $t$ seconds after being kicked
        - $h(3)$ will be an estimate for the height of the ball 3 seconds after being kicked
- Given an **output** you can **form an equation** with the model to **estimate the input**
    - For example: Let $P(n)$ be the profit made by selling $n$ items
        - Solving $P(n) = 100$ will give you an estimate for the number of items needing to be sold to make a profit of 100
- If your variable is **time** then substituting $t = 0$ will give you the **initial value** according to the model
- Fully understand the **units for the variables**
    - If the units of $P$ are measured in **thousand dollars** then $P = 3$ represents \$3000
- Look out for **key words** such as:
    - Initially
    - Minimum/maximum
    - Limiting value

## What do I do if some of the parameters are unknown?

- A general method is to **form equations** by substituting in given values
    - You can form **multiple equations** and **solve them simultaneously** using your GDC
    - This method **works for all models**
- The **initial value** is the value of the function when the variable is 0
    - This is **normally one of the parameters** in the equation of the model

www.thrivingscholars.com

## Worked Example

The temperature, $T$°C, of a cup of coffee is monitored. Initially the temperature is 80°C and 5 minutes later it is 40°C. It is suggested that the temperature follows the model:

$$T(t) = Ae^{kt} + 16,\ t \geq 0.$$

where $t$ is the time, in minutes, after the coffee has been made.

a)
State the value of $A$.

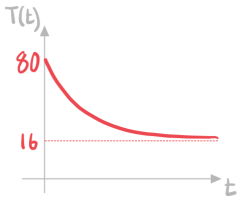
Initially temperature is 80°C

$T(0) = 80$

$Ae^{-k(0)} + 16 = 80$

$A + 16 = 80$

$\boxed{A = 64}$




b)
Find the exact value of $k$.

$t = 5,\ T = 40$

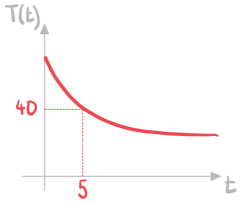
$40 = 64e^{5k} + 16$

$64e^{5k} = 24$

$e^{5k} = \frac{3}{8}$

$5k = \ln\frac{3}{8}$

$\boxed{k = \frac{1}{5}\ln\frac{3}{8}}$




c)
Find the time taken for the temperature of the coffee to reach 30°C.

www.thrivingscholars.com

Find $t$ such that $T(t) = 30$

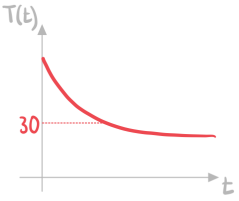
$30 = 64e^{kt} + 16$ Leave as $k$ until the end to save writing $\frac{1}{5}\ln\frac{3}{8}$ each time

$64e^{kt} = 14$

$e^{kt} = \frac{7}{32}$

$kt = \ln\frac{7}{32}$

$t = \frac{\ln\frac{7}{32}}{k} = \frac{\ln\frac{7}{32}}{\frac{1}{5}\ln\frac{3}{8}} = 7.7476..$




**7.75 minutes (3sf)**

www.thrivingscholars.com

# 2.5 Reciprocal & Rational Functions

## 2.5.1 Reciprocal & Rational Functions

### Reciprocal Functions & Graphs

### What is the reciprocal function?

- The **reciprocal function** is defined by $f(x) = \frac{1}{x},\ x \neq 0$
- Its **domain** is the set of **all real values except 0**
- Its **range** is the set of **all real values except 0**
- The reciprocal function has a **self-inverse** nature
  - $f^{-1}(x) = f(x)$
  - $(f \circ f)(x) = x$

### What are the key features of the reciprocal graph?

- The graph **does not have a *y*-intercept**
- The graph **does not have any roots**
- The graph has **two asymptotes**
  - A **horizontal** asymptote at the *x*-axis: $y = 0$
    - This is the **limiting value** when the absolute value of *x* gets very large
  - A **vertical** asymptote at the *y*-axis: $x = 0$
    - This is the value that causes the **denominator to be zero**
- The graph has **two axes of symmetry**
  - $y = x$
  - $y = -x$
- The graph **does not have any minimum or maximum points**

www.thrivingscholars.com

# Linear Rational Functions & Graphs

## What is a rational function with linear terms?

- A **(linear) rational function** is of the form $f(x) = \frac{ax+b}{cx+d}, x \neq -\frac{d}{c}$
- Its **domain** is the set of **all real values except** $-\frac{d}{c}$
- Its **range** is the set of **all real values except** $\frac{a}{c}$
- The **reciprocal function** is a **special case** of a rational function

## What are the key features of linear rational graphs?

- The graph has a ***y*-intercept** at $\left(0, \frac{b}{d}\right)$ provided $d \neq 0$
- The graph has **one root** at $\left(-\frac{b}{a}, 0\right)$ provided $a \neq 0$
- The graph has **two asymptotes**
  - A **horizontal** asymptote: $y = \frac{a}{c}$
    - This is the **limiting value** when the absolute value of *x* gets very large
  - A **vertical** asymptote: $x = -\frac{d}{c}$
    - This is the value that causes the **denominator to be zero**
- The graph **does not have any minimum or maximum points**
- If you are asked to **sketch or draw** a rational graph:
  - Give the **coordinates** of any **intercepts** with the axes
  - Give the **equations** of the **asymptotes**

### Exam Tip

- If you draw a horizontal line anywhere it should only intersect this type of graph once at most
- The only horizontal line that should not intersect the graph is the horizontal asymptote
  - This can be used to check your sketch in an exam

www.thrivingscholars.com

## Worked Example

The function $f$ is defined by $f(x) = \dfrac{10-5x}{x+2}$ for $x \neq -2$.

a)
Write down the equation of
(i)
the vertical asymptote of the graph of $f$,
(ii)
the horizontal asymptote of the graph of $f$.

(i) Vertical asymptote is when denominator equals zero

$x+2=0$ $\boxed{x=-2}$

(ii) Horizontal asymptote is limiting value as $x$ gets large

$\lim\limits_{x\to\infty} \dfrac{10-5x}{x+2} = \lim\limits_{x\to\infty} \dfrac{-5x}{x}$ $\boxed{y=-5}$

b)
Find the coordinates of the intercepts of the graph of $f$ with the axes.

y-intercept occurs when $x=0$

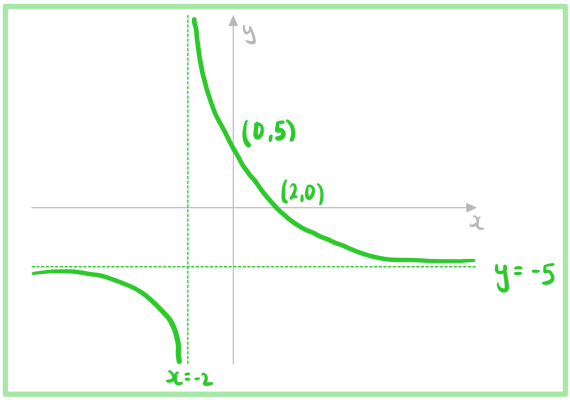
$y = \dfrac{10-5(0)}{0+2} = 5$ $\boxed{(0,5)}$

x-intercept occurs when $y=0$

$\dfrac{10-5x}{x+2} = 0 \Rightarrow 10-5x=0 \Rightarrow x=2$ $\boxed{(2,0)}$

c)
Sketch the graph of $f$.

Include asymptotes and intercepts



www.thrivingscholars.com

# Quadratic Rational Functions & Graphs

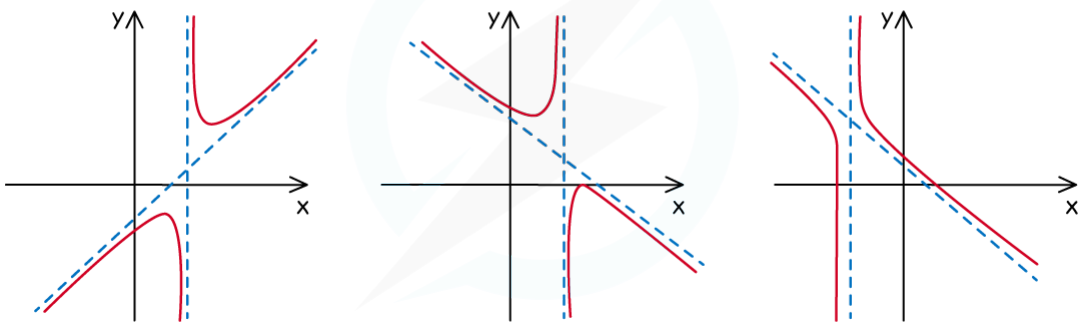
## How do I sketch the graph of a rational function where the terms are not linear?

- A rational function can be written $f(x) = \frac{g(x)}{h(x)}$
  - Where *g* and *h* are polynomials
- To find the **y-intercept** evaluate $\frac{g(0)}{h(0)}$
- To find the **x-intercept(s)** solve $g(x) = 0$
- To find the equations of the **vertical asymptote(s)** solve $h(x) = 0$
- There will also be an **asymptote** determined by what *f*(*x*) tends to as *x* approaches infinity
  - In this course it will be either:
    - **Horizontal**
    - **Oblique (a slanted line)**
  - This can be found by writing $g(x)$ in the form $h(x)Q(x) + r(x)$
    - You can do this by p**olynomial division** or **comparing coefficients**
  - The function then tends to the curve $y = Q(x)$

## What are the key features of rational graphs: quadratic over linear?

- For the rational function of the form $f(x) = \frac{ax^2 + bx + c}{dx + e}$
- The graph has a **y-intercept** at $\left(0, \frac{c}{e}\right)$ provided $e \neq 0$
- The graph can have **0, 1 or 2 roots**
  - They are the solutions to $ax^2 + bx + c = 0$
- The graph has **one vertical asymptote** $x = -\frac{e}{d}$
- The graph has an **oblique asymptote** $y = px + q$
  - Which can be found by writing $ax^2 + bx + c$ in the form $(dx + e)(px + q) + r$
    - Where *p*, *q*, *r* are constants
    - This can be done by **polynomial division** or **comparing coefficients**

$$y = \frac{ax^2 + bx + c}{dx + e}$$

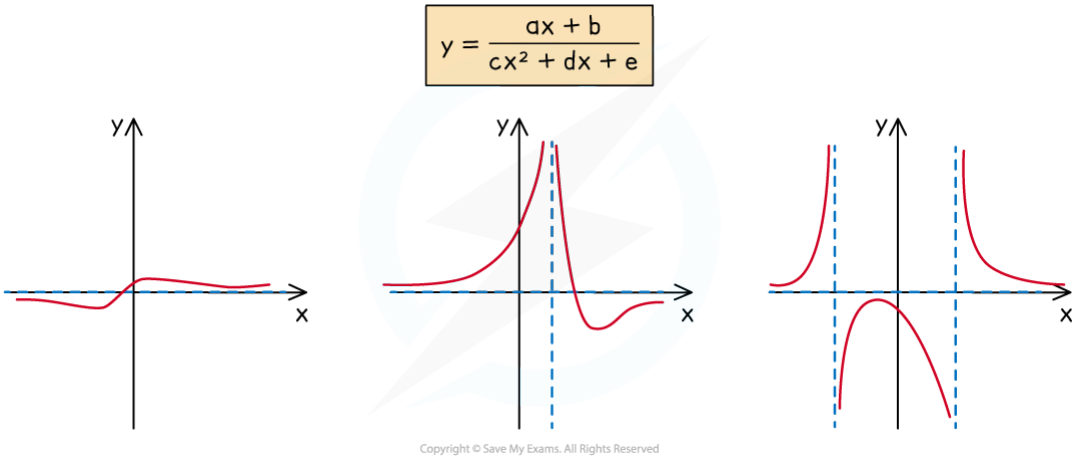



Copyright © Save My Exams. All Rights Reserved

www.thrivingscholars.com

## What are the key features of rational graphs: linear over quadratic?

- For the rational function of the form $f(x) = \dfrac{ax + b}{cx^2 + dx + e}$
- The graph has a **y-intercept** at $\left(0, \dfrac{b}{e}\right)$ provided $e \neq 0$
- The graph has **one root** at $x = -\dfrac{b}{a}$
- The graph has can have **0, 1 or 2 vertical asymptotes**
  - They are the solutions to $cx^2 + dx + e = 0$
- The graph has a **horizontal asymptote**



Copyright © Save My Exams. All Rights Reserved

### Exam Tip

- If you draw a horizontal line anywhere it should only intersect this type of graph twice at most
  - This idea can be used to check your graph or help you sketch it

www.thrivingscholars.com

## Worked Example

The function $f$ is defined by $f(x) = \dfrac{2x^2 + 5x - 3}{x + 1}$ for $x \neq -1$.

a)

(i)

Show that $\dfrac{2x^2 + 5x - 3}{x + 1} = px + q + \dfrac{r}{x + 1}$ for constants $p$, $q$ and $r$ which are to be found.

(ii)

Hence write down the equation of the oblique asymptote of the graph of $f$.

(i) Write $2x^2 + 5x - 3$ as $(x+1)(px+q) + r$

$2x^2 + 5x - 3 = px^2 + qx + px + q + r$

Compare coefficients

$x^2$: $2 = p$ $\quad\therefore p = 2$

$x$: $5 = q + p$ $\quad\therefore q = 3$

constant: $-3 = q + r$ $\quad\therefore r = -6$

$$\frac{2x^2+5x-3}{x+1} = \frac{(x+1)(2x+3)-6}{x+1} = 2x + 3 - \frac{6}{x+1}$$

(ii) $y = 2x + 3$

b)

Find the coordinates of the intercepts of the graph of $f$ with the axes.

$y$-intercept occurs when $x = 0$

$y = \dfrac{2(0)^2+5(0)-3}{(0)+1} = -3$ $\quad(0, -3)$

$x$-intercept occurs when $y = 0$

$\dfrac{2x^2+5x-3}{x+1} = 0 \Rightarrow 2x^2+5x-3=0 \Rightarrow (2x-1)(x+3) \Rightarrow x = 0.5$ or $x = -3$

$(0.5, 0)$ and $(-3, 0)$

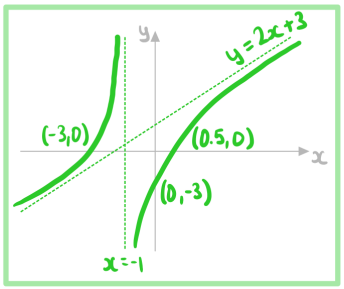
c)

Sketch the graph of $f$.

www.thrivingscholars.com

Vertical asymptote when denominator is zero $x = -1$

Include asymptotes and intercepts



www.thrivingscholars.com

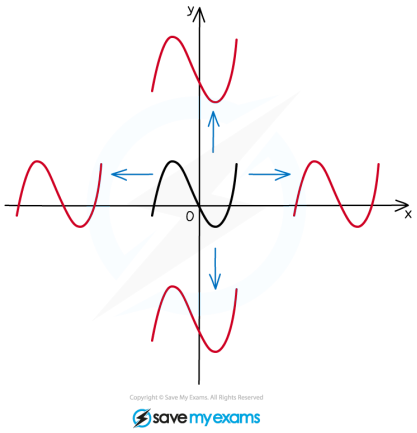
# 2.6 Transformations of Graphs

## 2.6.1 Translations of Graphs

### Translations of Graphs

### What are translations of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **translation**:
  - the graph is **moved** (up or down, left or right) in the xy plane
    - Its position **changes**
  - the shape, size, and orientation of the graph remain **unchanged**
- A particular translation (how far left/right, how far up/down) is specified by a **translation vector** $\begin{pmatrix} x \\ y \end{pmatrix}$:
  - $x$ is the **horizontal** displacement
    - **Positive** moves **right**
    - **Negative** moves **left**
  - $y$ is the **vertical** displacement
    - **Positive** moves **up**
    - **Negative** moves **down**



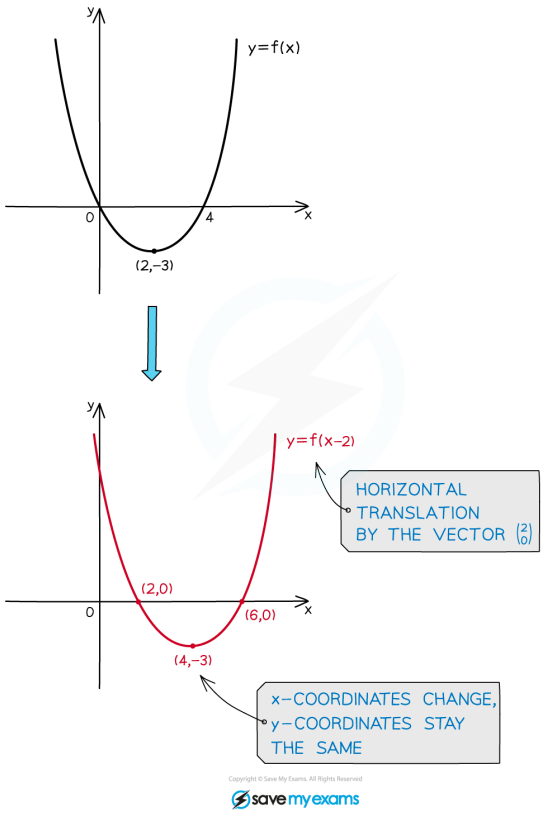
Copyright © Save My Exams. All Rights Reserved
save my exams

### What effects do horizontal translations have on the graphs and functions?

- A **horizontal translation** of the graph $y = f(x)$ by the vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is represented by
  - $y = f(x - a)$
- The ***x*-coordinates change**
  - The value $a$ is **subtracted** from them
- The ***y*-coordinates stay the same**
- The coordinates $(x, y)$ become $(x - a, y)$
- **Horizontal** asymptotes **stay the same**

www.thrivingscholars.com

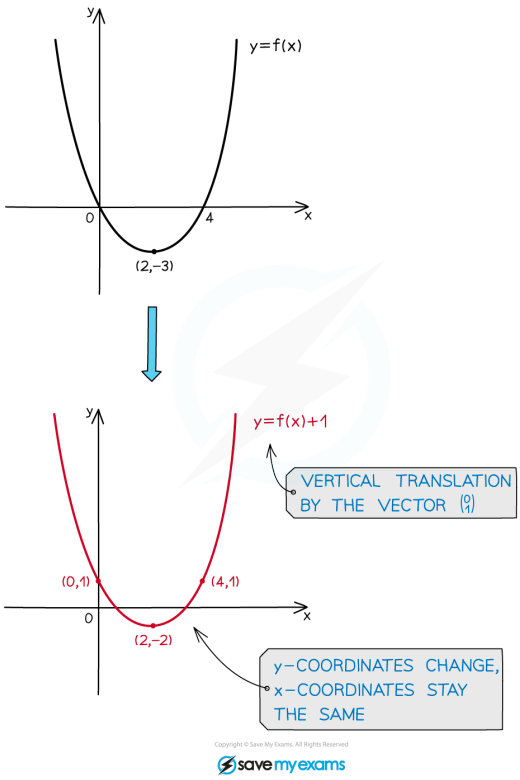
- **Vertical** asymptotes **change**
  - $x = k$ becomes $x = k - a$




## What effects do vertical translations have on the graphs and functions?

- A **vertical translation** of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ b \end{pmatrix}$ is represented by
  - $y - b = f(x)$
  - This is often rearranged to $y = f(x) + b$
- The ***x*-coordinates stay the same**
- The ***y*-coordinates change**
  - The value $b$ is **added** to them
- The coordinates $(x, y)$ become $(x, y + b)$
- **Horizontal** asymptotes **change**
  - $y = k$ becomes $y = k + b$
- **Vertical** asymptotes **stay the same**

www.thrivingscholars.com



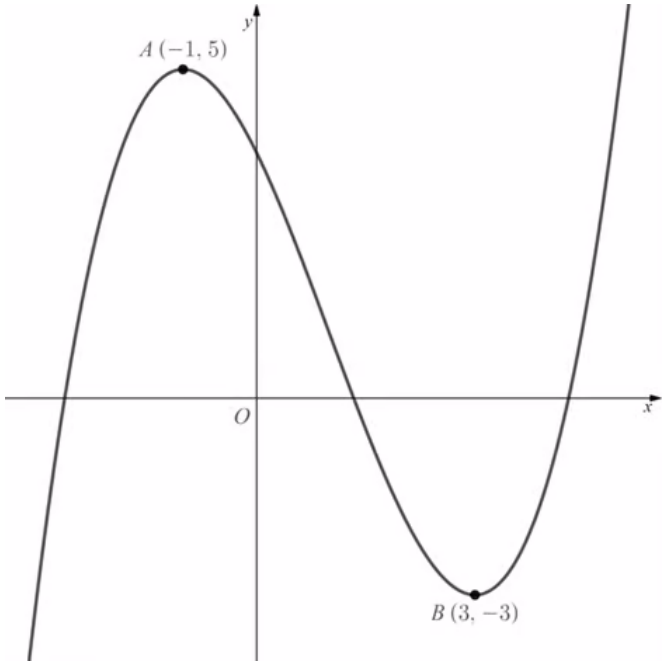

## Exam Tip

- To get full marks in an exam make sure you use correct mathematical terminology
  - For example: Translate by the vector $\begin{pmatrix}2\\-4\end{pmatrix}$

www.thrivingscholars.com

## Worked Example

The diagram below shows the graph of $y = f(x)$.




a)

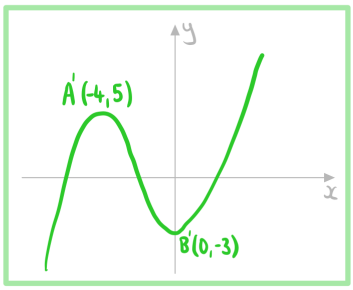
Sketch the graph of $y = f(x+3)$.

$y = f(x+k)$ translation by $\begin{pmatrix} -k \\ 0 \end{pmatrix}$

Translate $y = f(x)$ by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

A becomes $(-4, 5)$

B becomes $(0, -3)$




b)

Sketch the graph of $y = f(x) + 3$.

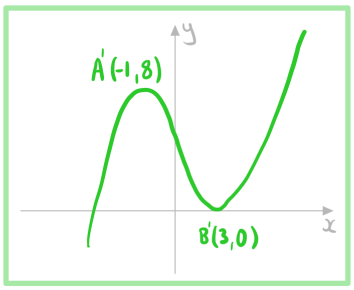
www.thrivingscholars.com

$y = f(x) + k$ translation by $\begin{pmatrix} 0 \\ k \end{pmatrix}$

Translate $y = f(x)$ by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

A becomes $(-1, 8)$

B becomes $(3, 0)$



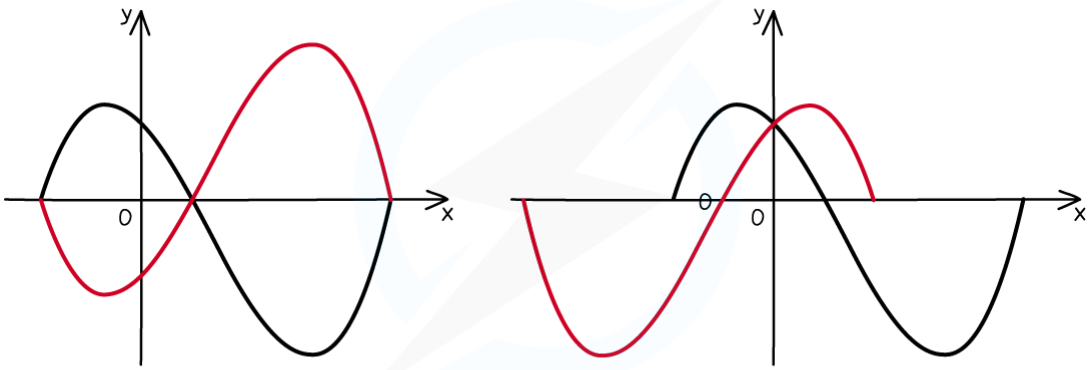
www.thrivingscholars.com



# Reflections of Graphs

## What are reflections of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **reflection**:
  - the graph is **flipped** about one of the coordinate axes
    - Its orientation **changes**
  - the size of the graph remains **unchanged**
- A particular reflection is specified by an **axis of symmetry**:
  - $y = 0$
    - This is the $x$-axis
  - $x = 0$
    - This is the $y$-axis

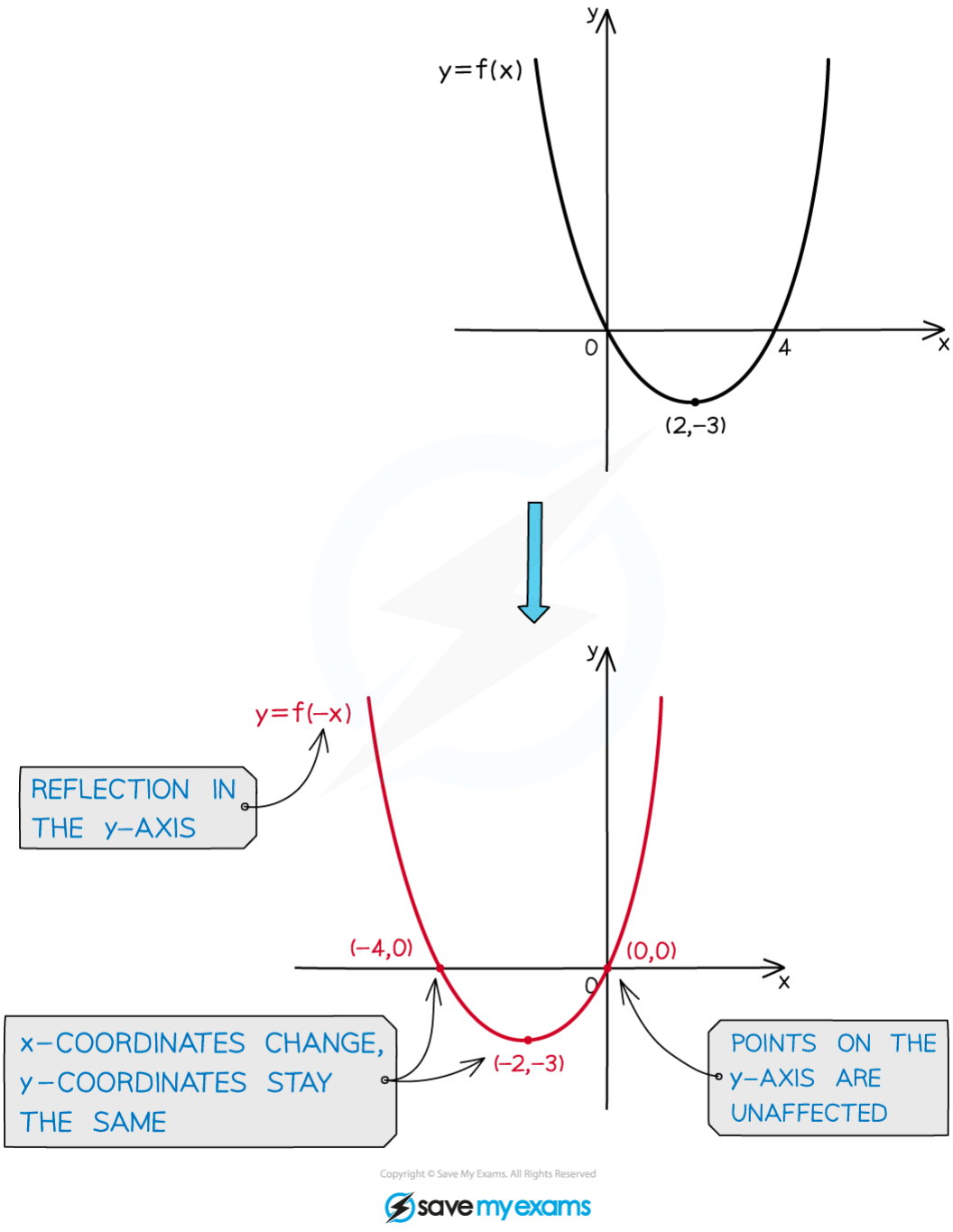



Copyright © Save My Exams. All Rights Reserved
save my exams

## What effects do horizontal reflections have on the graphs and functions?

- A **horizontal reflection** of the graph $y = f(x)$ about the $y$-axis is represented by
  - $y = f(-x)$
- The **$x$-coordinates change**
  - Their **sign** changes
- The **$y$-coordinates stay the same**
- The coordinates $(x, y)$ become $(-x, y)$
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
  - $x = k$ becomes $x = -k$

www.thrivingscholars.com



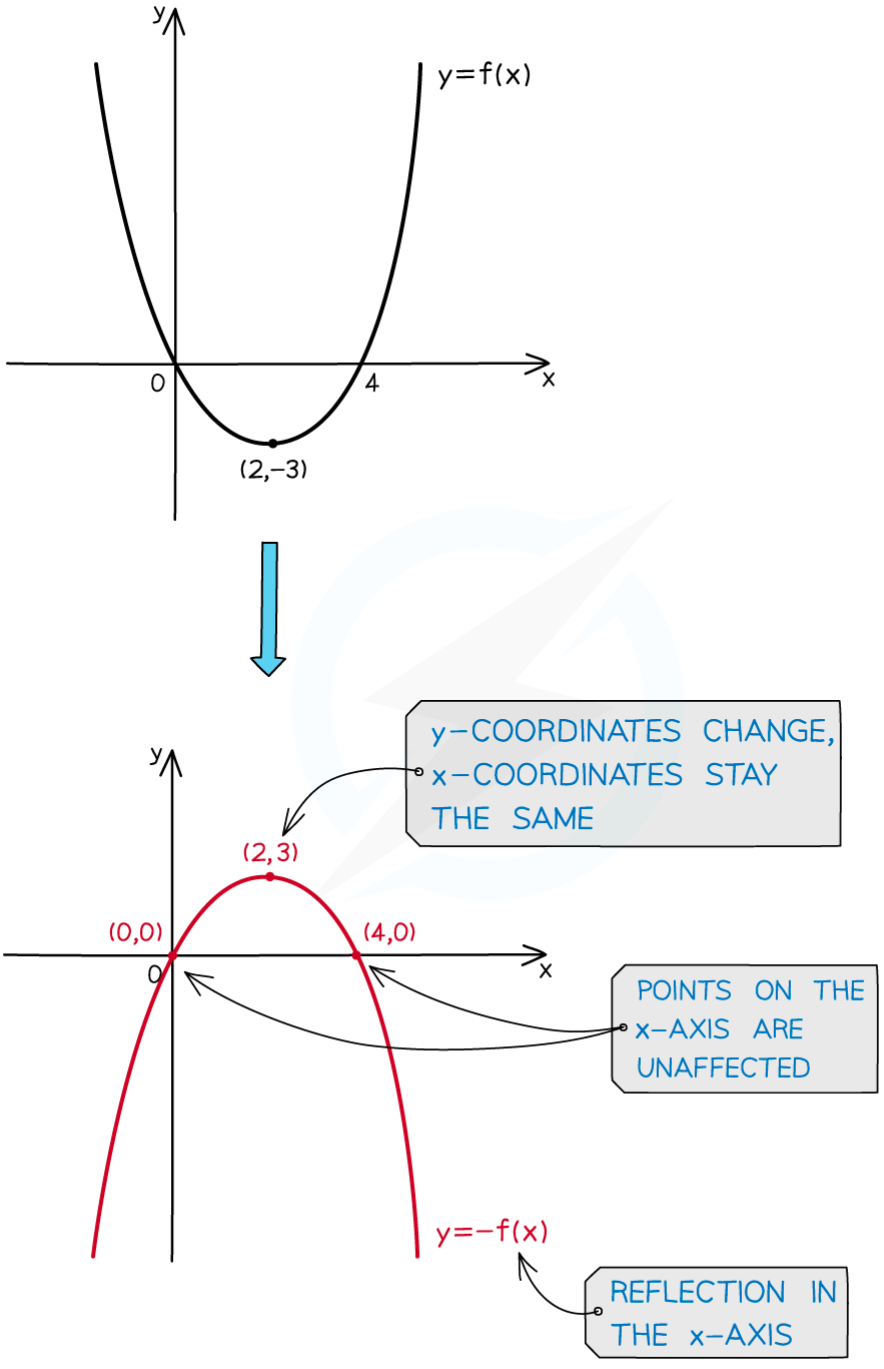

Copyright © Save My Exams. All Rights Reserved
savemyexams

## What effects do vertical reflections have on the graphs and functions?

- A **vertical reflection** of the graph $y = f(x)$ about the $x$-axis is represented by
  - $-y = f(x)$
  - This is often rearranged to $y = -f(x)$
- The **$x$-coordinates stay the same**
- The **$y$-coordinates change**
  - Their **sign** changes

www.thrivingscholars.com

- The coordinates $(x, y)$ become $(x, -y)$
- **Horizontal** asymptotes **change**
  - $y = k$ becomes $y = -k$
- **Vertical** asymptotes **stay the same**

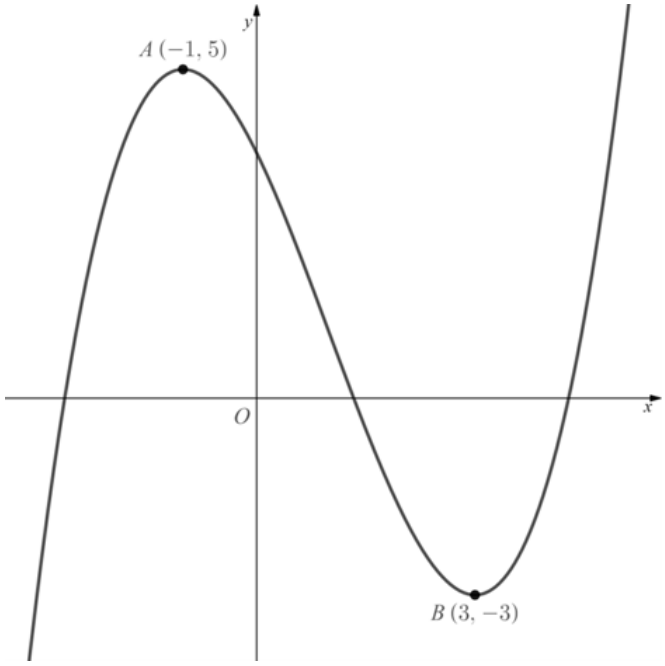



Copyright © Save My Exams. All Rights Reserved
save my exams

www.thrivingscholars.com

## Worked Example

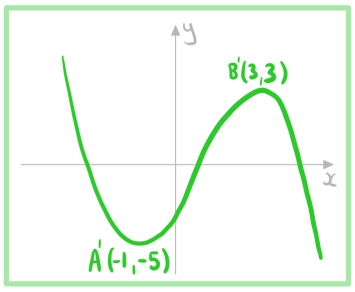
The diagram below shows the graph of $y = f(x)$.




a)
Sketch the graph of $y = -f(x)$.

$y = -f(x)$ reflection in $x$-axis

A becomes $(-1, -5)$
B becomes $(3, 3)$



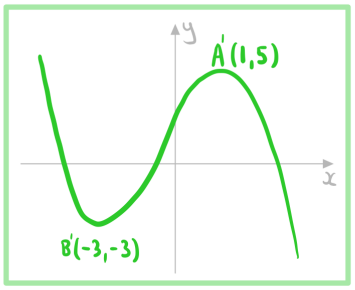

b)
Sketch the graph of $y = f(-x)$.

www.thrivingscholars.com

$y = f(-x)$ reflection in y-axis

A becomes $(1, 5)$

B becomes $(-3, -3)$



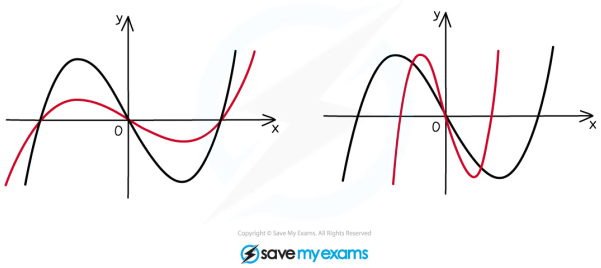
www.thrivingscholars.com



# Stretches of Graphs

## What are stretches of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **stretch**:
  - the graph is **stretched** about one of the coordinate axes by a scale factor
    - Its size **changes**
  - the orientation of the graph remains **unchanged**
- A particular stretch is specified by a **coordinate axis** and a **scale factor**:
  - The **distance** between a **point** on the graph and the **specified coordinate axis** is **multiplied** by the **constant scale factor**
  - The graph is stretched in the **direction** which is **parallel** to the **other coordinate axis**
  - For scale factors **bigger than 1**
    - the points on the graph get **further away** from the **specified coordinate axis**
  - For scale factors **between 0 and 1**
    - the points on the graph get **closer** to the **specified coordinate axis**
    - This is also called a **compression**

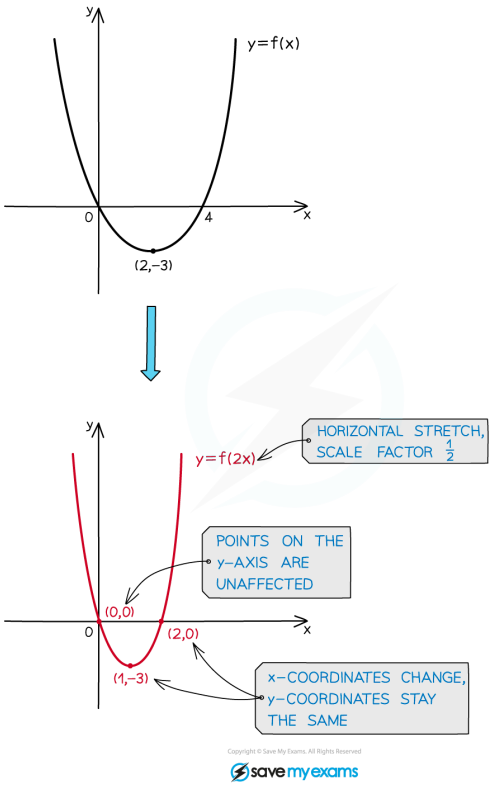



Copyright © Save My Exams. All Rights Reserved
savemyexams

## What effects do horizontal stretches have on the graphs and functions?

- A **horizontal stretch** of the graph $y = f(x)$ by a scale factor $q$ centred about the $y$-axis is represented by
  - $y = f\left(\frac{x}{q}\right)$
- The ***x*-coordinates change**
  - They are **divided** by $q$
- The ***y*-coordinates stay the same**
- The coordinates $(x, y)$ become $\left(\frac{x}{q}, y\right)$
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
  - $x = k$ becomes $x = \frac{k}{q}$

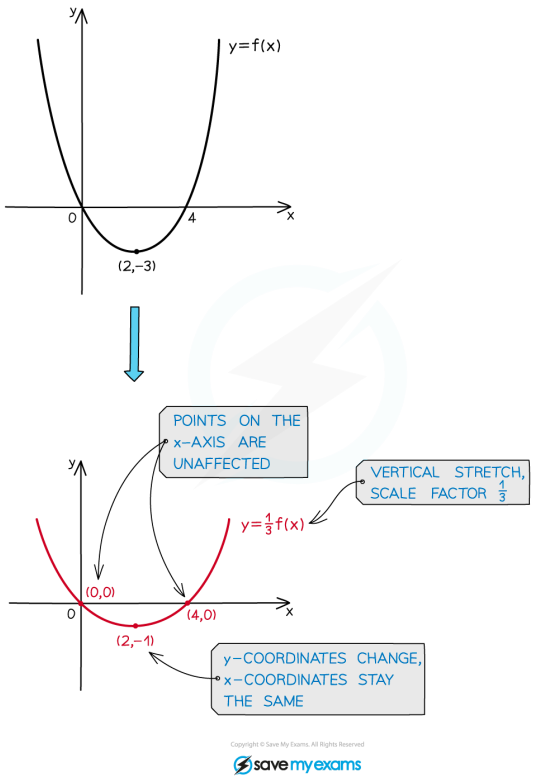
www.thrivingscholars.com




## What effects do vertical stretches have on the graphs and functions?

- A **vertical stretch** of the graph $y = f(x)$ by a scale factor $p$ centred about the $x$-axis is represented by
  - $\frac{y}{p} = f(x)$
  - This is often rearranged to $y = pf(x)$
- The **$x$-coordinates stay the same**
- The **$y$-coordinates change**
  - They are **multiplied** by $p$
- The coordinates $(x, y)$ become $(x, py)$
- **Horizontal** asymptotes **change**
  - $y = k$ becomes $y = pk$
- **Vertical** asymptotes **stay the same**

www.thrivingscholars.com



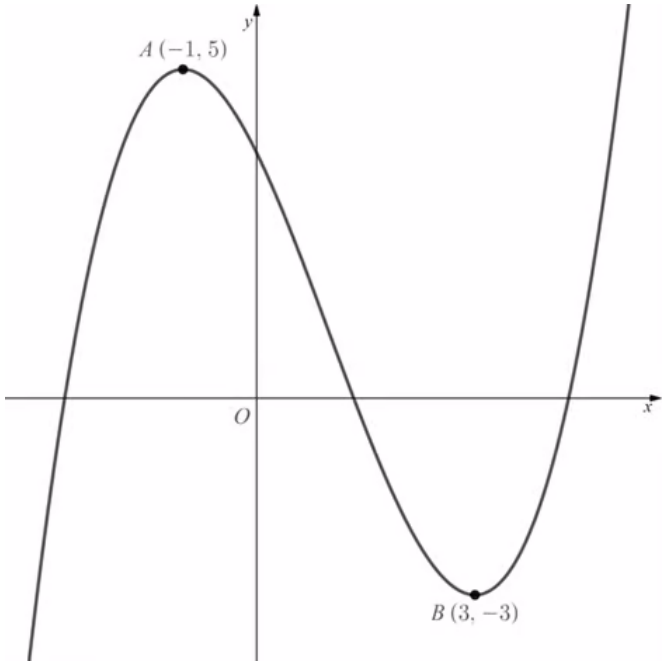

## Exam Tip

- To get full marks in an exam make sure you use correct mathematical terminology
  - For example: Stretch vertically by scale factor ½
  - Do not use the word "compress" in your exam

www.thrivingscholars.com

## Worked Example

The diagram below shows the graph of $y = f(x)$.




a)

Sketch the graph of $y = 2f(x)$.

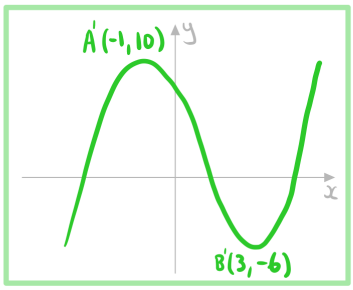
$y = kf(x)$ vertical stretch scale factor $k$

Stretch $y = f(x)$ vertically

scale factor 2

A becomes $(-1, 10)$

B becomes $(3, -6)$




b)

Sketch the graph of $y = f(2x)$.

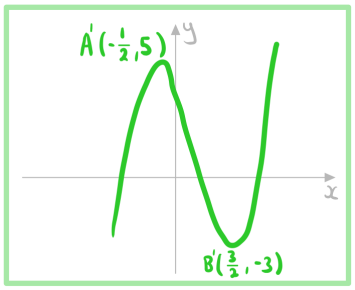
www.thrivingscholars.com

$y = f(kx)$ horizontal stretch scale factor $\frac{1}{k}$

Stretch $y = f(x)$ horizontally scale factor $\frac{1}{2}$

A becomes $(-\frac{1}{2}, 5)$

B becomes $(\frac{3}{2}, -3)$



www.thrivingscholars.com



# Composite Transformations of Graphs

## What transformations do I need to know?

- $y = f(x + k)$ is **horizontal translation** by vector $\begin{pmatrix} -k \\ 0 \end{pmatrix}$
  - If $k$ is **positive** then the graph moves **left**
  - If $k$ is **negative** then the graph moves **right**
- $y = f(x) + k$ is **vertical translation** by vector $\begin{pmatrix} 0 \\ k \end{pmatrix}$
  - If $k$ is **positive** then the graph moves **up**
  - If $k$ is **negative** then the graph moves **down**
- $y = f(kx)$ is a **horizontal stretch** by scale factor $\frac{1}{k}$ centred about the $y$-axis
  - If $\boldsymbol{k > 1}$ then the graph gets **closer** to the $y$-axis
  - If $\boldsymbol{0 < k < 1}$ then the graph gets **further** from the $y$-axis
- $y = kf(x)$ is a **vertical stretch** by scale factor $k$ centred about the $x$-axis
  - If $\boldsymbol{k > 1}$ then the graph gets **further** from the $x$-axis
  - If $\boldsymbol{0 < k < 1}$ then the graph gets **closer** to the $x$-axis
- $y = f(-x)$ is a **horizontal reflection** about the $y$-axis
  - A **horizontal reflection** can be viewed as a special case of a **horizontal stretch**
- $y = -f(x)$ is a **vertical reflection** about the $x$-axis
  - A **vertical reflection** can be viewed as a special case of a **vertical stretch**

## How do horizontal and vertical transformations affect each other?

- **Horizontal and vertical transformations** are **independent** of each other
  - The horizontal transformations involved will need to be applied in their correct order
  - The vertical transformations involved will need to be applied in their correct order
- Suppose there are **two horizontal** transformation **$H_1$ then $H_2$** and **two vertical** transformations **$V_1$ then $V_2$** then they can be applied in the following orders:
  - Horizontal then vertical:
    - $H_1H_2V_1V_2$
  - Vertical then horizontal:
    - $V_1V_2H_1H_2$
  - Mixed up (provided that **$H_1$ comes before $H_2$** and **$V_1$ comes before V2**):
    - $H_1V_1H_2V_2$
    - $H_1V_1V_2H_2$
    - $V_1H_1V_2H_2$
    - $V_1H_1H_2V_2$

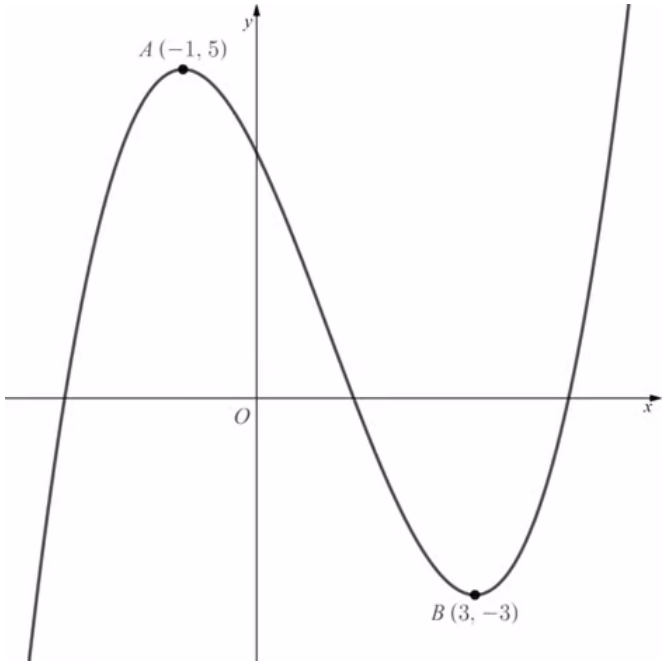
### Exam Tip

- In an exam you are more likely to get the correct solution if you deal with one transformation at a time and sketch the graph after each transformation

www.thrivingscholars.com

## Worked Example

The diagram below shows the graph of $y = f(x)$.




Sketch the graph of $y = \frac{1}{2} f\left(\frac{x}{2}\right)$.

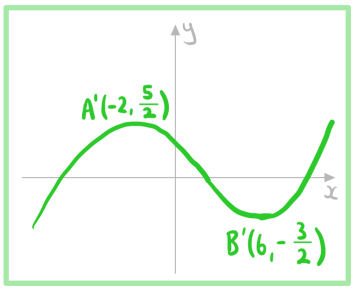
A vertical and horizontal transformation can be done in any order

$y = \frac{1}{2} f(x)$ : vertical stretch scale factor $\frac{1}{2}$

$y = f\left(\frac{x}{2}\right)$ : horizontal stretch scale factor 2

A becomes $\left(-2, \frac{5}{2}\right)$

B becomes $\left(6, -\frac{3}{2}\right)$



www.thrivingscholars.com

# Composite Vertical Transformations af(x)+b

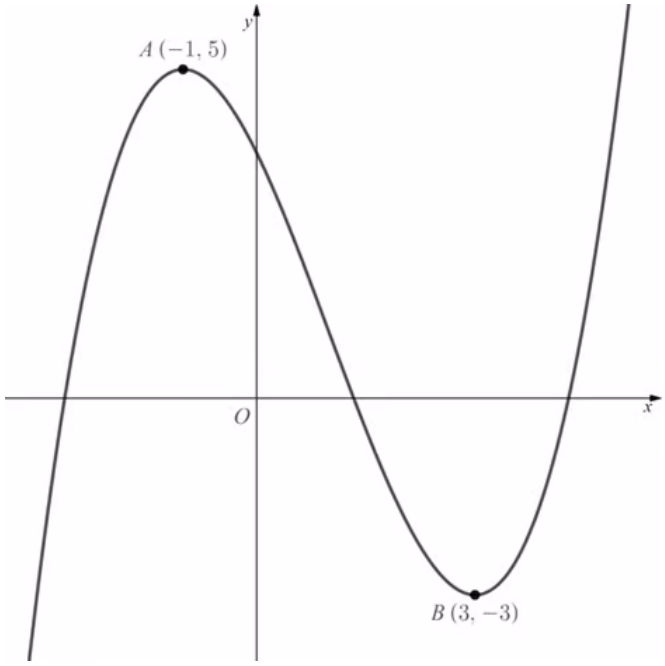
## How do I deal with multiple vertical transformations?

- **Order matters** when you have **more than one vertical transformations**
- If you are asked to find the equation then **build up the equation** by looking at the transformations in order
  - A **vertical stretch** by scale factor *a* followed by a **translation** of $\begin{pmatrix} 0 \\ b \end{pmatrix}$
    - Stretch: $y = af(x)$
    - Then translation: $y = [af(x)] + b$
    - Final equation: $y = af(x) + b$
  - A **translation** of $\begin{pmatrix} 0 \\ b \end{pmatrix}$ followed by a **vertical stretch** by scale factor *a*
    - Translation: $y = f(x) + b$
    - Then stretch: $y = a[f(x) + b]$
    - Final equation: $y = af(x) + ab$
- If you are asked to determine the **order**
  - The order of vertical transformations **follows the order of operations**
  - First write the equation in the form $y = af(x) + b$
    - **First stretch vertically** by scale factor *a*
    - If *a* is negative then the **reflection and stretch** can be **done in any order**
    - **Then translate** by $\begin{pmatrix} 0 \\ b \end{pmatrix}$

www.thrivingscholars.com

## Worked Example

The diagram below shows the graph of $y = f(x)$.




Sketch the graph of $y = 3f(x) - 2$.

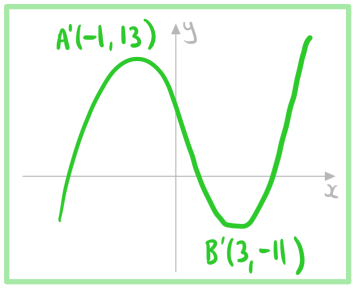
The order vertical transformations follows the order of operations

$y = 3f(x)$: Vertical stretch scale factor 3

$y = f(x) - 2$: Translate $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

A becomes $(-1, 13)$

B becomes $(3, -11)$



www.thrivingscholars.com



# Composite Horizontal Transformations f(ax+b)

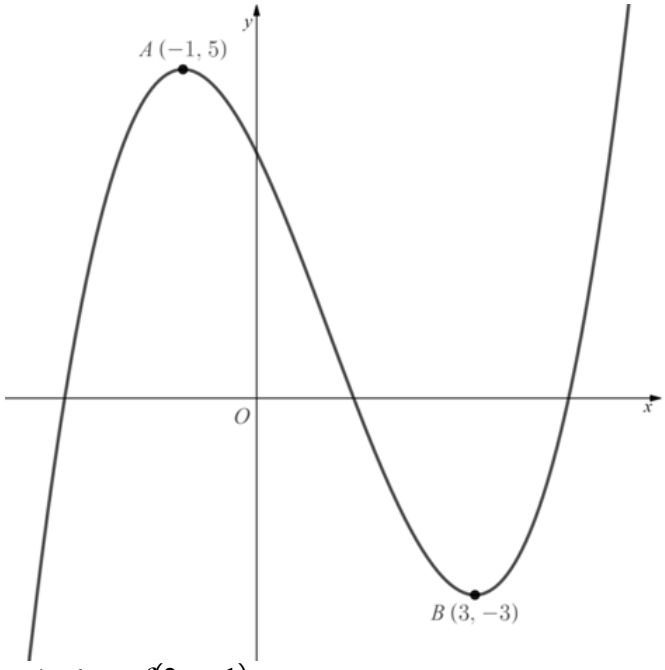
## How do I deal with multiple horizontal transformations?

- **Order matters** when you have **more than one horizontal transformations**
- If you are asked to find the equation then **build up the equation** by looking at the transformations in order
  - A **horizontal stretch** by scale factor $\frac{1}{a}$ followed by a **translation** of $\begin{pmatrix} -b \\ 0 \end{pmatrix}$
    - Stretch: $y = f(ax)$
    - Then translation: $y = f(a(x+b))$
    - Final equation: $y = f(ax+ab)$
  - A **translation** of $\begin{pmatrix} -b \\ 0 \end{pmatrix}$ followed by a **horizontal stretch** by scale factor $\frac{1}{a}$
    - Translation: $y = f(x+b)$
    - Then stretch: $y = f((ax)+b)$
    - Final equation: $y = f(ax+b)$
- If you are asked to determine the **order**
  - First write the equation in the form $y = f(ax+b)$
  - The order of horizontal transformations **is the reverse of the order of operations**
    - **First translate** by $\begin{pmatrix} -b \\ 0 \end{pmatrix}$
    - **Then stretch** by scale factor $\frac{1}{a}$
    - If *a* is negative then the **reflection and stretch** can be **done in any order**

www.thrivingscholars.com

## Worked Example

The diagram below shows the graph of $y = f(x)$.




Sketch the graph of $y = f(2x - 1)$.

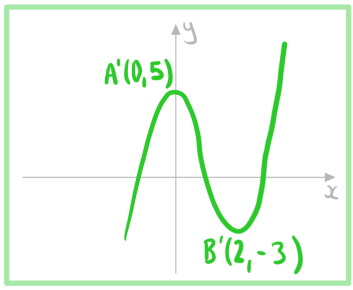
The order of horizontal transformations is the reverse of the order of operations

$y = f(x-1)$: Translate $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$y = f(2x)$: Horizontal stretch scale factor $\frac{1}{2}$

A becomes $(0, 5)$

B becomes $(2, -3)$



www.thrivingscholars.com

# 2.7 Polynomial Functions

## 2.7.1 Factor & Remainder Theorem

### Factor Theorem

### What is the factor theorem?

- The **factor theorem** is used to find the linear factors of **polynomial** equations
- This topic is closely tied to finding the **zeros** and **roots** of a **polynomial** function/equation
  - As a rule of thumb a **zero** refers to the polynomial function and a **root** refers to a polynomial equation
- For any **polynomial** function $P(x)$
  - $(x - k)$ is a **factor** of $P(x)$ if $\mathbf{P(k) = 0}$
  - $\mathbf{P(k) = 0}$ if $(x - k)$ is a **factor** of $P(x)$

### How do I use the factor theorem?

- Consider the polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ and $(x - k)$ is a **factor**
  - Then, due to the factor theorem $P(k) = a_n k^n + a_{n-1}k^{n-1} + \ldots + a_1 k + a_0 = 0$
  - $P(x) = (x - k) \times Q(x)$, where $Q(x)$ is a **polynomial** that is a factor of $P(x)$
  - Hence, $\frac{P(x)}{x - k} = Q(x)$, where $Q(x)$ is another factor of $P(x)$
- If the linear factor has a **coefficient of x** then you must first factorise out the coefficient
  - If the linear factor is $(ax - b) = a\left(x - \frac{b}{a}\right) \rightarrow P\left(\frac{b}{a}\right) = 0$

**Exam Tip**

- A common mistake in exams is using the incorrect sign for either the root or the factor
- If you are asked to find integer solutions to a polynomial then you only need to consider factors of the constant term

www.thrivingscholars.com

## Worked Example

Determine whether $(x-2)$ is a factor of the following polynomials:

a)

$f(x) = x^3 - 2x^2 - x + 2.$

Step 1: Determine k

Our linear function is $x - 2$

→ so $k = 2$

Step 2: Apply factor theorem

For $x - 2$ to be a factor of $f(x)$, $f(2)$ has to equal zero

$f(2) = (2)^3 - 2(2)^2 - (2) + 2$

$= 8 - 8 - 2 + 2$

$= 0$

$f(2) = 0$,
so $x - 2$ is a factor of $f(x)$

b)

$g(x) = 2x^3 + 3x^2 - x + 5.$

www.thrivingscholars.com



Step 1: Determine k

Our linear function is $x - 2$

$\rightarrow$ so $k = 2$

Step 2: Apply factor theorem

For $x - 2$ to be a factor of $g(x)$, $g(2)$ has to equal zero

$$g(2) = 2(2)^3 + 3(2)^2 - (2) + 5$$
$$= 16 - 12 - 2 + 5$$
$$= 7$$

$g(2) = 7$, so $x - 2$ is not a factor of $g(x)$

It is given that $(2x - 3)$ is a factor of $h(x) = 2x^3 - bx^2 + 7x - 6$.

c)
Find the value of $b$.

www.thrivingscholars.com



Step 1: Determine k

Our linear function is $2x-3$

$\rightarrow$ so $k = \frac{3}{2}$

Step 2: Apply factor theorem to find b

Since $2x-3$ is a factor of $h(x)$,

$h\left(\frac{3}{2}\right) = 0$

$$0 = 2\left(\frac{3}{2}\right)^3 - b\left(\frac{3}{2}\right)^2 + 7\left(\frac{3}{2}\right) - 6$$

$$= \frac{54}{8} - \frac{9}{4}b + \frac{21}{2} - 6$$

$$\boxed{b = 5}$$

www.thrivingscholars.com

# Remainder Theorem

## What is the remainder theorem?

- The **remainder theorem** is used to find the remainder when we divide a **polynomial** function by a linear function
- When any polynomial $P(x)$ is divided by any linear function $(x - k)$ the value of the remainder $R$ is given by $P(k) = R$
  - Note, when $P(k) = 0$ then $(x - k)$ is a factor of $P(x)$

## How do I use the remainder theorem?

- Consider the polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ and the linear function $(x - k)$
  - Then, due to the remainder theorem $P(k) = a_n k^n + a_{n-1}k^{n-1} + \ldots + a_1 k + a_0 = R$
  - $P(x) = (x - k) \times Q(x) + R$, where Q(x) is a **polynomial**
  - Hence, $\frac{P(x)}{x-k} = Q(x) + \frac{R}{x-k}$ , where $R$ is the remainder

- If the linear factor has a **coefficient of *x*** then you must first factorise out the coefficient
  - If the linear factor is $(ax - b) = a\left(x - \frac{b}{a}\right) \rightarrow P\left(\frac{b}{a}\right) = R$

www.thrivingscholars.com

## Worked Example

Let $f(x) = 2x^4 - 2x^3 - x^2 - 3x + 1$, find the remainder $R$ when $f(x)$ is divided by:

a)
$x - 3$.

Step 1: Determine k

Our linear function is $x - 3$

→ so $k = 3$

Step 2: Apply remainder theorem

$f(3) = R$

$f(3) = 2(3)^4 - 2(3)^3 - (3)^2 - 3(3) + 1$

$f(3) = 162 - 54 - 9 - 9 + 1$

$f(3) = 91$

**R = 91**

b)
$x + 2$.

Step 1: Determine k

Our linear function is $x + 2$

→ so $k = -2$

Step 2: Apply remainder theorem

$f(-2) = R$

$f(-2) = 2(-2)^4 - 2(-2)^3 - (-2)^2 - 3(-2) + 1$

$f(-2) = 32 + 16 - 4 + 6 + 1$

$f(-2) = 51$

**R = 51**

www.thrivingscholars.com

The remainder when $f(x)$ is divided by $(2x+k)$ is $\frac{893}{8}$.

c)
Given that $k>0$, find the value of $k$.

Step 1: Apply remainder theorem

$2x+k = 2\left(x+\frac{k}{2}\right)$ $\quad f\left(-\frac{k}{2}\right) = \frac{893}{8}$

$\frac{893}{8} = 2\left(-\frac{k}{2}\right)^4 - 2\left(-\frac{k}{2}\right)^3 - \left(-\frac{k}{2}\right)^2 - 3\left(-\frac{k}{2}\right) + 1$

Step 2: Solve for k using your GDC

$k = 5$

www.thrivingscholars.com



# Polynomial Division

## What is polynomial division?

- Polynomial division is the process of **dividing two polynomials**
  - This is usually only useful when the **degree of the denominator** is **less than or equal** to the **degree of the numerator**
- To do this we use an algorithm similar to that used for **division of integers**
- To divide the polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ by the polynomial $D(x) = b_k x^k + b_{k-1} x^{k-1} + \ldots + b_1 x + b_0$ where $k \leq n$
  - STEP 1
    **Divide** the **leading term of the polynomial** $P(x)$ by **the leading term of the divisor** $D(x)$
    $: \frac{a_n x^n}{b_b x^k} = q_m x^m$
  - STEP 2
    **Multiply the divisor** by this term: $D(x) \times q_m x^m$
  - STEP 3
    **Subtract this** from the **original polynomial** $P(x)$ to cancel out the leading term:
    $R(x) = P(x) - D(x) \times q_m x^m$
  - Repeat steps 1 – 3 using the new polynomial $R(x)$ in place of $P(x)$ until the subtraction results in an expression for $R(x)$ with degree less than the divisor
    - The quotient $Q(x)$ is the **sum of the terms** you multiplied the divisor by:
      $Q(x) = q_m x^m + q_{m-1} x^{m-1} + \ldots + q_1 x + q_0$
    - The remainder $R(x)$ is the polynomial after the final subtraction

## Division by linear functions

- If $P(x)$ has degree $n$ and is divided by a linear function $(ax + b)$ then
  - $\frac{P(x)}{ax+b} = Q(x) + \frac{R}{ax+b}$ where
    - $ax + b$ is the **divisor** (degree 1)
    - $Q(x)$ is the **quotient** (degree $n - 1$)
    - $R$ is the **remainder** (degree 0)
  - Note that $P(x) = Q(x) \times (ax + b) + R$

## Division by quadratic functions

- If $P(x)$ has degree $n$ and is divided by a quadratic function $(ax^2 + bx + c)$ then
  - $\frac{P(x)}{ax^2+bx+c} = Q(x) + \frac{ex+f}{ax^2 + bx + c}$ where
    - $ax^2 + bx + c$ is the **divisor** (degree 2)
    - $Q(x)$ is the **quotient** (degree $n - 2$)
    - $ex + f$ is the **remainder** (degree less than 2)

www.thrivingscholars.com



- The remainder will be **linear** (degree 1) if $e \neq 0$, and **constant** (degree 0) if $e = 0$
- Note that $P(x) = Q(x) \times (ax^2 + bx + c) + ex + f$

## Division by polynomials of degree $k \leq n$

- If $P(x)$ has degree $n$ and is divided by a polynomial $D(x)$ with degree $k \leq n$
  - $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$ where
    - $D(x)$ is the **divisor** (degree $k$)
    - $Q(x)$ is the **quotient** (degree $n - k$)
    - $R(x)$ is the **remainder** (degree less than $k$)
  - Note that $P(x) = Q(x) \times D(x) + R(x)$

## Are there other methods for dividing polynomials?

- **Synthetic division** is a faster and shorter way of setting out a division when dividing by a linear term of the form
  - To divide $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ by $(x - c)$:
    - Set $b_n = a_n$
    - Calculate $b_{n-1} = a_{n-1} + c \times b_n$
    - Continue this iterative process $b_{i-1} = a_{i-1} + c \times a_i$
    - The quotient is $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \ldots + b_2 x + b_1$ and the remainder is $r = b_0$
- You can also find quotients and remainders by **comparing coefficients**
  - Given a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$
  - And a divisor $D(x) = d_k x^k + d_{k-1} x^{k-1} + \ldots + d_1 x + d_0$
  - Write $Q(x) = q_{n-k} x^{n-k} + \ldots + q_1 x + q_0$ and $R(x) = r_{k-1} x^{k-1} + \ldots + r_1 x + r_0$
  - Write $P(x) = Q(x)D(x) + R(x)$
    - Expand the right-hand side
    - Equate the coefficients
    - Solve to find the unknowns $q$'s & $r$'s

### Exam Tip

- In an exam you can use whichever method to divide polynomials - just make sure your method is written clearly so that if you make a mistake you can still get a mark for your method!

www.thrivingscholars.com

## Worked Example

a)

Perform the division $\frac{x^4+11x^2-1}{x+3}$. Hence write $x^4+11x^2-1$ in the form $Q(x)\times(x+3)+R$.

Step 1: what do we multiply $x$ by to get $x^4$?

$$x+3 \;\overline{)\; x^4+0x^3+11x^2+0x-1}$$
$$\qquad\quad x^3$$

Note: $0x^3$ and $0x$ are used to keep like terms together.

Step 2: subtract $x^3(x+3) = x^4+3x^3$ from $x^4+0x^3$

$$\begin{array}{r} x^3 \\ x+3 \;\overline{)\; x^4+0x^3+11x^2+0x-1} \\ -(x^4+3x^3) \\ \hline -3x^3 \end{array}$$

www.thrivingscholars.com



Step 3: bring the $11x^2$ down and return to step 1.

$$
\begin{array}{r}
x^3 - 3x^2 + 20x - 60 \\
x+3 \;\overline{)\; x^4 + 0x^3 + 11x^2 + 0x - 1} \\
-(x^4 + 3x^3) \\
\hline
-3x^3 + 11x^2 \\
-(-3x^3 - 9x^2) \\
\hline
20x^2 + 0x \\
-(20x^2 + 60x) \\
\hline
-60x - 1 \\
-(-60x - 180) \\
\hline
179
\end{array}
$$

$$x^4 + 11x^2 - 1 = (x^3 - 3x^2 + 20x - 60)(x + 3) + 179$$

b)

Find the quotient and remainder for $\dfrac{x^4 + 4x^3 - x + 1}{x^2 - 2x}$. Hence write $x^4 + 4x^3 - x + 1$ in the form $Q(x) \times (x^2 - 2x) + R(x)$.

www.thrivingscholars.com



When dividing by quadratics use the same steps as above.

$$
\begin{array}{r}
x^2 + 6x + 12 \\
x^2 - 2x \;\overline{)\; x^4 + 4x^3 + 0x^2 - x + 1} \\
-\,(x^4 - 2x^3) \\
\hline
6x^3 + 0x^2 \\
-\,(6x^3 - 12x^2) \\
\hline
12x^2 - x \\
-\,(12x^2 - 24x) \\
\hline
23x + 1
\end{array}
$$

$$x^4 + 4x^3 - x + 1$$
$$= (x^2 + 6x + 12)(x^2 - 2x) + 23x + 1$$

www.thrivingscholars.com





# Sketching Polynomial Graphs

In exams you'll commonly be asked to sketch the graphs of different polynomial functions with and without the use of your GDC.

## What's the relationship between a polynomial's degree and its zeros?

- If a **real polynomial** $P(x)$ has **degree *n***, it will have ***n* zeros** which can be written in the form $a + bi$, where $a, b \in \mathbb{R}$
  - For example:
    - A quadratic will have 2 zeros
    - A cubic function will have 3 zeros
    - A quartic will have 4 zeros
  - Some of the zeros may be **repeated**
- Every **real polynomial** of **odd degree** has **at least one real zero**

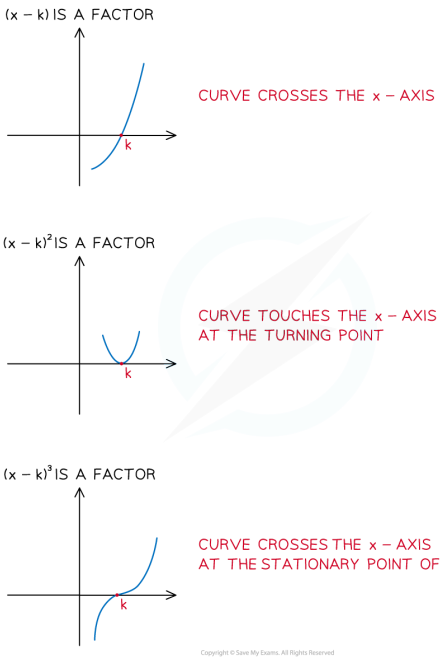
## How do I sketch the graph of a polynomial function without a GDC?

- Suppose $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ ... + a_1 x + a_0$ is a **real polynomial** with **degree *n***
- To sketch the graph of a polynomial you need to know three things:
  - The ***y*-intercept**
    - Find this by **substituting $x = 0$** to get **$y = a_0$**
  - The **roots**
    - You can find these by **factorising** or solving **$y = 0$**
  - The **shape**
    - This is determined by the **degree** ($n$) and the sign of the **leading coefficient** ($a_n$)

## How does the multiplicity of a real root affect the graph of the polynomial?

- The **multiplicity** of a root is the number of times it is **repeated** when the polynomial is factorised
  - If $x = k$ is a root with **multiplicity *m*** then $(x - k)^m$ is a **factor** of the polynomial
- The graph either **crosses** the $x$-axis or **touches** the $x$-axis at a **root $x = k$** where $k$ is a real number
  - If $x = k$ has **multiplicity 1** then the graph **crosses** the $x$-axis at $(k, 0)$
  - If $x = k$ has **multiplicity 2** then the graph has a **turning point** at $(k, 0)$ so **touches** the $x$-axis
    - If $x = k$ has **odd multiplicity** $m \geq 3$ then the graph has a **stationary point of inflection** at $(k, 0)$ so **crosses** the $x$-axis
    - If $x = k$ has **even multiplicity** $m \geq 4$ then the graph has a **turning point** at $(k, 0)$ so **touches** the $x$-axis

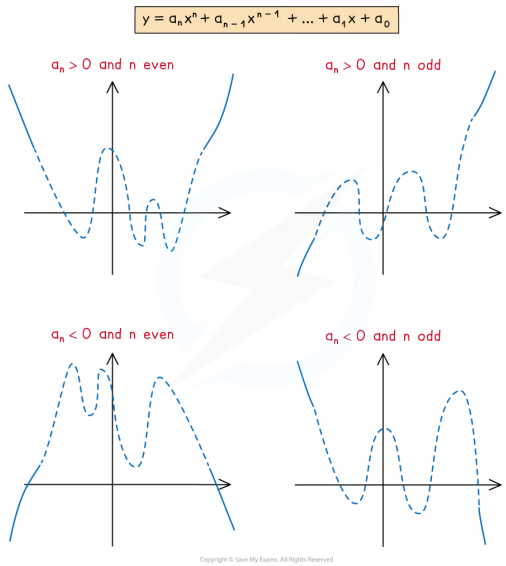
www.thrivingscholars.com




## How do I determine the shape of the graph of the polynomial?

- Consider what happens as ***x* tends to ± ∞**
  - If $a_n$ is **positive** and $n$ is **even** then the graph **approaches from the top left** and **tends to the top right**
    - $\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = +\infty$
  - If $a_n$ is **negative** and $n$ is **even** then the graph **approaches from the bottom left** and **tends to the bottom right**
    - $\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = +\infty$
  - If $a_n$ is **positive** and $n$ is **odd** then the graph **approaches from the bottom left** and **tends to the top right**
    - $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$
  - If $a_n$ is **negative** and $n$ is **odd** then the graph **approaches from the top left** and **tends to the bottom right**
    - $\lim_{x \to -\infty} f(x) = +\infty$ and $\lim_{x \to +\infty} f(x) = -\infty$
- Once you know the **shape**, the **real roots** and the **y-intercept** then you simply connect the points using a **smooth curve**
- There will be **at least one turning point** in-between each pair of roots
  - If the degree is $n$ then there is **at most $n - 1$ stationary points** (some will be **turning points**)
    - Every real polynomial of **even degree** has **at least one turning point**
    - Every real polynomial of **odd degree bigger than 1** has **at least one point of inflection**
  - If it is a calculator paper then you can use your GDC to find the coordinates of the turning points
  - You won't need to find their location without a GDC unless the question asks you to

www.thrivingscholars.com




Copyright © Save My Exams. All Rights Reserved

## Exam Tip

- If it is a calculator paper then you can use your GDC to find the coordinates of any turning points
- If it is the non-calculator paper then you will not be required to find the turning points when sketching unless specifically asked to

www.thrivingscholars.com

## Worked Example

a)

The function $f$ is defined by $f(x)=(x+1)(2x-1)(x-2)^2$. Sketch the graph of $y=f(x)$.

Find the y-intercept

$x=0$ : $y=(1)(-1)(-2)^2=-4$

Find the roots and determine if graphs crosses or touches the x-axis

$(x+1)(2x-1)(x-2)^2$

$(-1,0)$ $(\frac{1}{2},0)$ $(2,0)$

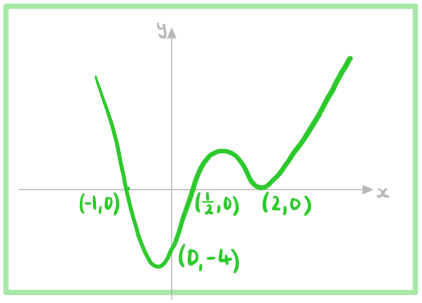
cross cross touch

Determine the shape by looking at the leading term

Leading term is $(x)(2x)(x)^2=2x^4$

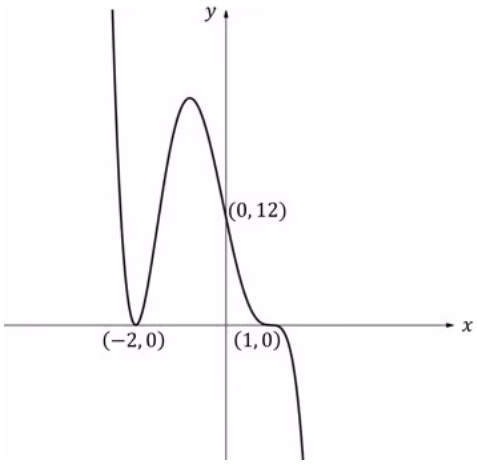
As $x\to-\infty$ $y\to+\infty$

As $x\to+\infty$ $y\to+\infty$




b)

The graph below shows a polynomial function. Find a possible equation of the polynomial.



www.thrivingscholars.com



Touches at $(-2,0)$ $(x+2)^2$ is a factor

Point of inflection at $(1,0)$ $(x-1)^3$ is a factor

Write in the form of : $y = a(x+2)^2(x-1)^3$

Use the y-intercept to find a

$12 = a(2)^2(-1)^3 \Rightarrow -4a = 12 \quad \therefore a = -3$

$$y = -3(x+2)^2(x-1)^3$$

www.thrivingscholars.com

# Solving Polynomial Equations

## What is "The Fundamental Theorem of Algebra"?

- Every **real polynomial** with degree *n* can be factorised into ***n* complex linear factors**
  - Some of which may be **repeated**
  - This means the polynomial will have *n* zeros (some may be repeats)
- Every **real polynomial** can be expressed as a product of **real linear factors** and **real irreducible quadratic factors**
  - An irreducible quadratic is where it **does not have real roots**
    - The **discriminant** will be negative: $b^2 - 4ac < 0$
- If $a + bi$ ($b \neq 0$) is a **zero** of a **real polynomial** then its **complex conjugate** $a - bi$ is also a **zero**
- Every **real polynomial** of **odd degree** will have **at least one real zero**

## How do I solve polynomial equations?

- Suppose you have an equation $P(x) = 0$ where $P(x)$ is a **real polynomial of degree *n***
  - $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ ... + a_1 x + a_0$
- You may be given one zero or you might have to find a zero $x = k$ by substituting values into $P(x)$ until it equals 0
- If you know a **root** then you know a **factor**
  - If you know ***x* = *k* is a root** then **(*x* – *k*) is a factor**
  - If you know ***x* = *a* + *b*i is a root** then you know a **quadratic factor (*x* – (*a* + *b*i))( *x* – (*a* – *b*i))**
    - Which can be written as $((x - a) - bi)((x - a) + bi)$ and **expanded quickly using difference of two squares**
- You can then **divide** $P(x)$ by this factor to get **another factor**
  - For example: dividing a cubic by a linear factor will give you a quadratic factor
- You then may be able to factorise this new factor

### Exam Tip

- If a polynomial has three or less terms check whether a substitution can turn it into a quadratic
  - For example: $x^6 + 3x^3 + 2$ can be written as $(x^3)^2 + 3(x^3) + 2$

www.thrivingscholars.com

## Worked Example

Given that $x = \frac{1}{2}$ is a zero of the polynomial defined by $f(x) = 2x^3 - 3x^2 + 5x - 2$, find all three zeros of $f$.

$x = \frac{1}{2}$ is a root $\therefore (2x-1)$ is a factor

Find the quadratic factor $(2x^3 - 3x^2 + 5x - 2) = (2x-1)(ax^2 + bx + c)$

Compare coefficients : $2x^3 = 2ax^3 \quad \therefore a = 1$

$-2 = -c \quad \therefore c = 2$

$5x = 2cx - bx \Rightarrow 5 = 4 - b \quad \therefore b = -1$

Solve the quadratic : $x^2 - x + 2 = 0$

Formula booklet

| | |
|---|---|
| Solutions of a quadratic equation | $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{-7}}{2} = \frac{1}{2} \pm \frac{\sqrt{7}}{2}i$$

Roots : $\frac{1}{2}, \frac{1}{2} + \frac{\sqrt{7}}{2}i, \frac{1}{2} - \frac{\sqrt{7}}{2}i$

www.thrivingscholars.com



# Sum & Product of Roots

## How do I find the sum & product of roots of polynomials?

- Suppose $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ ... + a_1 x + a_0$ is a **polynomial** of **degree** $n$ with $n$ roots $\alpha_1, \alpha_2, ..., \alpha_n$
  - The polynomial is written as $\sum_{r=0}^{n} a_r x^r = 0,\ a_n \neq 0$ in the **formula booklet**
  - $a_n$ is the coefficient of the **leading term**
  - $a_{n-1}$ is the coefficient of the **$x^{n-1}$ term**
    - Be careful: this could be equal to zero
  - $a_0$ is the **constant term**
    - Be careful: this could be equal to zero
- In factorised form: $P(x) = a_n(x - \alpha_1)(x - \alpha_2)...(x - \alpha_n)$
  - Comparing coefficients of the **$x^{n-1}$ term** and the **constant term** gives
    - $a_{n-1} = a_n(-\alpha_1 - \alpha_2 - ... - \alpha_n)$
    - $a_0 = a_n(-\alpha_1) \times (-\alpha_2) \times ... \times (-\alpha_n)$
- The **sum** of the roots is given by:
  - $\alpha_1 + \alpha_2 + \ ... + \alpha_n = -\dfrac{a_{n-1}}{a_n}$
- The **product** of the roots is given by:
  - $\alpha_1 \times \alpha_2 \times \ ... \times \alpha_n = \dfrac{(-1)^n a_0}{a_n}$
    - both of these formulae are in your **formula booklet**

## How can I find unknowns if I am given the sum and/or product of the roots of a polynomial?

- If you know a complex root of a real polynomial then its **complex conjugate** is **another root**
- Form **two equations** using the roots
  - One using the **sum of the roots formula**
  - One using the **product of the roots formula**
- **Solve** for any unknowns

### Exam Tip

- Examiners might trick you by not having an $x^{n-1}$ term or a constant term
- To make sure you do not get tricked you can write out the full polynomial using 0 as a coefficient where needed
  - For example: Write $x^4 + 2x^2 - 5x$ as $x^4 + 0x^3 + 2x^2 - 5x + 0$

www.thrivingscholars.com

## Worked Example

$2-3\mathrm{i}$, $\frac{5}{3}\mathrm{i}$ and $\alpha$ are three roots of the equation

$18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k = 0$.

a)

Use the sum of all the roots to find the value of $\alpha$.

It is a real polynomial so if $a+bi$ is a root then $a-bi$ is also a root

Roots: $2-3i$, $2+3i$, $\frac{5}{3}i$, $-\frac{5}{3}i$, $\alpha$

Formula booklet

| Sum & product of the roots of polynomial equations of the form $\sum_{r=0}^{n} a_r x^r = 0$ | Sum is $\frac{-a_{n-1}}{a_n}$ |
|---|---|

$18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k$

$a_n = 18$ $a_{n-1} = -9$

$(2-3i) + (2+3i) + \left(\frac{5}{3}i\right) + \left(-\frac{5}{3}i\right) + \alpha = \frac{-(-9)}{18}$

$4 + \alpha = \frac{1}{2}$

$\alpha = -\frac{7}{2}$

b)

Use the product of all the roots to find the value of $k$.

Formula booklet

| Sum & product of the roots of polynomial equations of the form $\sum_{r=0}^{n} a_r x^r = 0$ | product is $\frac{(-1)^n a_0}{a_n}$ |
|---|---|

$18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k$

$a_n = 18$ $n = 5$ $a_0 = k$

$(2-3i)(2+3i)\left(\frac{5}{3}i\right)\left(-\frac{5}{3}i\right)\left(-\frac{7}{2}\right) = \frac{(-1)^5 k}{18}$

$(13)\left(\frac{25}{9}\right)\left(-\frac{7}{2}\right) = \frac{-k}{18}$

$-\frac{2275}{18} = -\frac{k}{18}$

$k = 2275$

www.thrivingscholars.com

# 2.8 Inequalities

## 2.8.1 Solving Inequalities Graphically

### Solving Inequalities Graphically

### How can I solve inequalities graphically?

- Consider the inequality **$f(x) \leq g(x)$**, where $f(x)$ and $g(x)$ are functions of $x$
  - if we **move $g(x)$ to the LHS** we get
    - **$f(x) - g(x) \leq 0$**
- Solve **$f(x) - g(x) = 0$ to** find the **zeros** of $f(x) - g(x)$
  - These correspond to the $x$-coordinates of the points of intersection of the graphs **$y = f(x)$** and **$y = g(x)$**
- To solve the inequality we can use a **graph**
  - **Graph $y = f(x) - g(x)$** and label its zeros
  - Hence find the intervals of $x$ that satisfy the inequality $f(x) - g(x) \leq 0$
    - These are the **intervals which satisfies the original inequality** $f(x) \leq g(x)$
  - This method is particularly useful when finding the intersections between the functions is difficult due to needing large $x$ and $y$ windows on your GDC

### Be careful when rearranging inequalities!

- Remember to **flip the sign** of the inequality when you **multiply or divide** both sides by a **negative** number
  - e. **$1 < 2$** → [times both sides by $(-1)$] → **$-1 > -2$** (sign flips)
- **Never multiply or divide** by a **variable** as this could be **positive or negative**
  - You can only multiply by a term if you are certain it is always positive (or always negative)
    - Such as $x^2$, $|x|$, $e^x$
- Some **functions reverse the inequality**
  - Taking reciprocals of positive values
    - $0 < x < y \Rightarrow \frac{1}{x} > \frac{1}{y}$
  - Taking logarithms when the base is $0 < a < 1$
    - $0 < x < y \Rightarrow \log_a(x) > \log_a(y)$
- The **safest way** to rearrange is simply to add & subtract to move all the terms onto one side

www.thrivingscholars.com

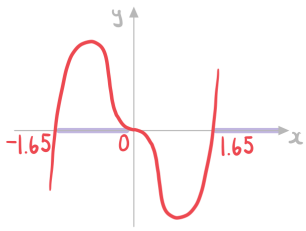
## Worked Example

Use a GDC to solve the inequality $2x^3 < x^5 - 2x$.

Rearrange to get one side as zero

$x^5 - 2x^3 - 2x > 0$

On GDC sketch $y = x^5 - 2x^3 - 2x$ and find zeros




Identify the sections where the graph is above the $x$-axis

$-1.65 < x < 0$ or $x > 1.65$

www.thrivingscholars.com



# Polynomial Inequalities

## How do I solve polynomial inequalities?

- **STEP 1**: **Rearrange the inequality** so that **one of the sides is equal to zero**
  - For example: $P(x) \leq 0$
- **STEP 2**: Find the **roots** of the polynomial
  - You can do this by factorising or using GDC to solve $P(x) = 0$
- **STEP 3**: Choose one of the following methods:
- **Graph method**
  - Sketch a graph of the polynomial (with or without a GDC)
  - Choose the intervals for $x$ corresponding to the sections of the graph that satisfy the inequality
    - For example: for $P(x) \leq 0$ you would want the sections below the $x$-axis
- **Sign table method**
  - If you are unsure how to sketch a polynomial graph then this method is best
  - **Split the real numbers** into the possible **intervals** using the roots
    - If the roots are $a$ and $b$ then the intervals would be $x<a$, $a<x<b$, $x>b$
  - **Test a value** from each interval using the inequality
    - Choose a value within an interval and substitute into $P(x)$ to determine if it is positive or negative
  - Alternatively if the polynomial is factorised you can **determine the sign of each factor** in each interval
    - An odd number of negative factors in an interval will mean the polynomial is negative on that interval
  - If the value satisfies the inequality then that interval is part of the solution

### Exam Tip

- In exams most solutions will be intervals but some could be a single point
  - For example: Solution to $(x-3)^2 \leq 0$ is $x = 3$

www.thrivingscholars.com

## Worked Example

Solve the inequality $x^3 + 2x^2 > x + 2$ using an algebraic method.

Rearrange $x^3 + 2x^2 - x - 2 > 0$

Let $P(x) = x^3 + 2x^2 - x - 2$

Find a factor $P(1) = 0 \quad \therefore (x-1)$ is a factor

Factorise $(x-1)(x^2+3x+2) > 0$ Compare coefficients or use division

$(x-1)(x+1)(x+2) > 0$

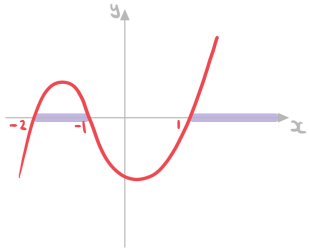
Find the roots $1, -1, -2$

Construct a sign table

| For $x < -2$: | For $-2 < x < -1$: | For $-1 < x < 1$: | For $x > 1$: |
|---|---|---|---|
| $(x+2) < 0$ | $(x+2) > 0$ | $(x+2) > 0$ | $(x+2) > 0$ |
| $(x+1) < 0$ | $(x+1) < 0$ | $(x+1) > 0$ | $(x+1) > 0$ |
| $(x-1) < 0$ | $(x-1) < 0$ | $(x-1) < 0$ | $(x-1) > 0$ |
| $\therefore P(x) < 0$ | $\therefore P(x) > 0$ | $\therefore P(x) < 0$ | $\therefore P(x) > 0$ |




Or sketch




Choose the regions that satisfy the inequality

$-2 < x < -1 \quad$ or $\quad x > 1$

www.thrivingscholars.com

# 2.9 Further Functions & Graphs

## 2.9.1 Modulus Functions

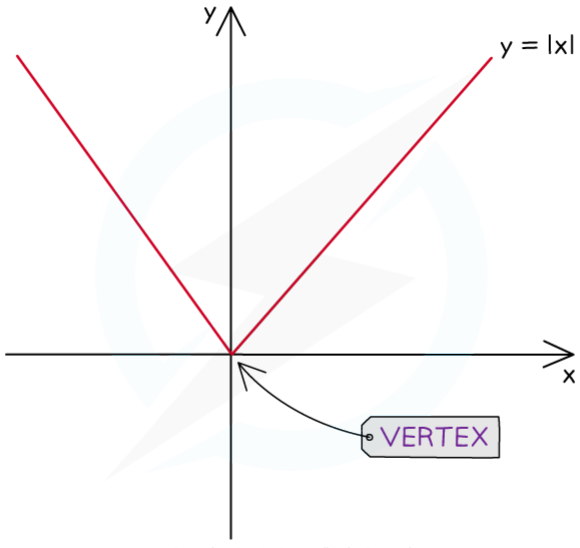
### Modulus Functions & Graphs

### What is the modulus function?

- The **modulus function** is defined by $f(x) = |x|$
  - $|x| = \sqrt{x^2}$
  - Equivalently it can be defined $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$
- Its **domain** is the set of **all real values**
- Its **range** is the set of **all real non-negative values**
- The modulus function gives the **distance** between 0 and $x$
  - This is also called the **absolute value** of $x$

### What are the key features of the modulus graph: $y = |x|$?

- The graph has a ***y*-intercept** at (0, 0)
- The graph has **one root** at (0, 0)
- The graph has a **vertex** at (0, 0)
- The graph is **symmetrical** about the ***y*-axis**
- At the **origin**
  - The function is **continuous**
  - The function is **not differentiable**




Copyright © Save My Exams. All Rights Reserved

### What are the key features of the modulus graph: $y = a|x + p| + q$?

- Every **modulus graph** which is formed by **linear transformations** can be written in this form using key features of the modulus function
  - $|ax| = |a||x|$

www.thrivingscholars.com



    - For example: $|2x+1| = 2\left|x+\frac{1}{2}\right|$
  - $|p-x| = |x-p|$
    - For example: $|4-x| = |x-4|$
- The graph has a ***y*-intercept** when $x = 0$
- The graph can have 0, 1 or 2 **roots**
  - If *a* and *q* have the **same sign** then there will be **0 roots**
  - If $q = 0$ then there will be **1 root** at $(-p, 0)$
  - If *a* and *q* have **different signs** then there will be **2 roots** at $\left(-p \pm \frac{q}{a}, 0\right)$
- The graph has a **vertex** at $(-p, q)$
- The graph is **symmetrical** about the line $x = -p$
- The value of *a* determines the **shape** and the **steepness** of the graph
  - If *a* is **positive** the graph looks like ∨
  - If *a* is **negative** the graph looks like ∧
  - The **larger** the value of $|a|$ the **steeper** the lines
- At the **vertex**
  - The function is **continuous**
  - The function is **not differentiable**

www.thrivingscholars.com



# Modulus Transformations

## How do I sketch the graph of the modulus of a function: $y = |f(x)|$?

- **STEP 1**: Keep the parts of the graph of $y = f(x)$ that are **on or above the $x$-axis**
- **STEP 2**: Any parts of the **graph below the x-axis** get **reflected** in the $x$-axis anything

## How do I sketch the graph of a function of a modulus: $y = f(|x|)$?

- **STEP 1**: Keep the graph of $y = f(x)$ **only for $x \geq 0$**
- **STEP 2**: **Reflect** this in the **$y$-axis**

## What is the difference between $y = |f(x)|$ and $y = f(|x|)$?

- The graph of $y = |f(x)|$ **never goes below the $x$-axis**
  - It does not have to have any lines of symmetry
- The graph of $y = f(|x|)$ is **always symmetrical about the $y$-axis**
  - It can go below the $y$-axis

## When multiple transformations are involved how do I determine the order?

- The transformations **outside the function** follow the **same order** as the **order of operations**
  - $y = |af(x) + b|$
    - Deal with the $a$ then the $b$ then the modulus
  - $y = a|f(x)| + b$
    - Deal with the modulus then the $a$ then the $b$
- The transformations **inside the function** are in the **reverse order** to the **order of operations**
  - $y = f(|ax + b|)$
    - Deal with the modulus then the $b$ then the $a$
  - $y = f(a|x| + b)$
    - Deal with the $b$ then the $a$ then the modulus

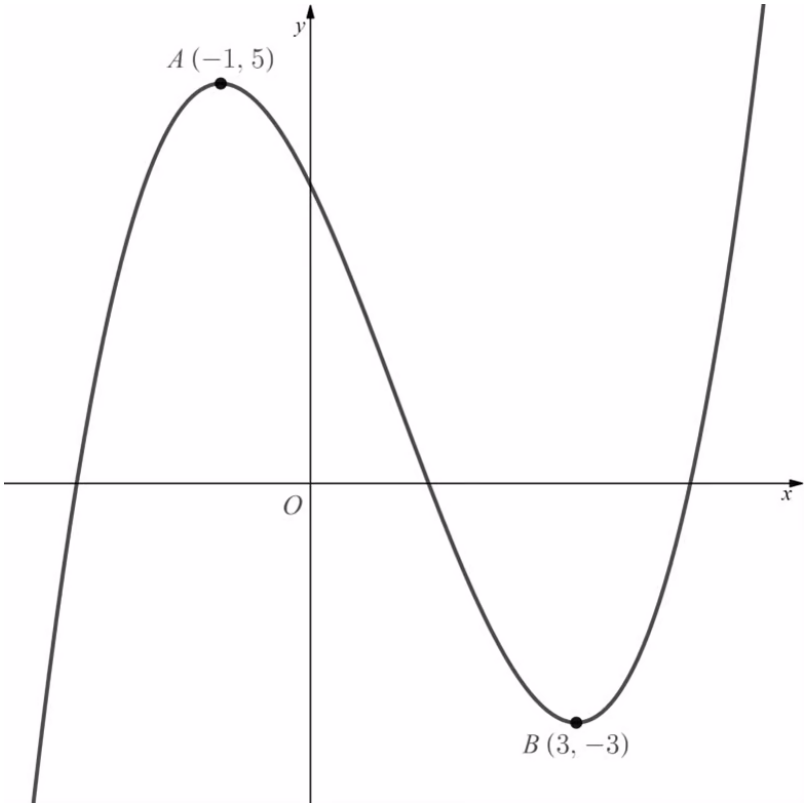
### Exam Tip

- When sketching one of these transformations in an exam question make sure that the graphs do not look smooth at the points where the original graph have been reflected
  - For $y = |f(x)|$ the graph should look "sharp" at the points where it has been reflected on the $x$-axis
  - For $y = f(|x|)$ the graph should look "sharp" at the point where it has been reflected on the $y$-axis

www.thrivingscholars.com

## Worked Example

The diagram below shows the graph of $y = f(x)$.



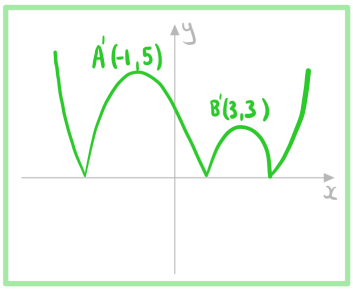

(a) Sketch the graph of $y = |f(x)|$.

If the graph is on or above the x-axis then it stays the same

If the graph is below the x-axis the it is reflected in the x-axis

A stays the same (-1,5)

B becomes (3, 3)




(b) Sketch the graph of $y = f(|x|)$.

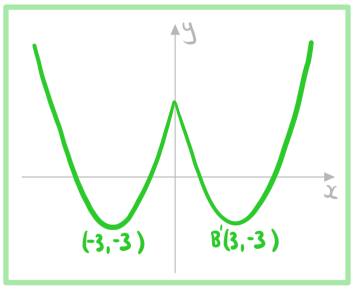
www.thrivingscholars.com

keep the graph for $x \geq 0$

Reflect this in the y-axis

A disappears

B stays the same (3, -3)



www.thrivingscholars.com





# Modulus Equations

## How do I find the modulus of a function?

- The **modulus of a function *f*(*x*)** is
  - $|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$ or
  - $|f(x)| = \sqrt{[f(x)]^2}$

## How do I solve modulus equations graphically?

- To solve $|f(x)| = g(x)$ graphically
  - Draw $y = |f(x)|$ and $y = g(x)$ into your GDC
  - Find the $x$-coordinates of the **points of intersection**

## How do I solve modulus equations analytically?

- To solve $|f(x)| = g(x)$ analytically
  - Form **two equations**
    - $f(x) = g(x)$
    - $f(x) = -g(x)$
  - Solve both equations
  - **Check solutions** work in the original equation
    - For example: $x - 2 = 2x - 3$ has solution $x = 1$
    - But $|(1) - 2| = 1$ and $2(1) - 3 = -1$
    - So $x = 1$ is not a solution to $|x - 2| = 2x - 3$

www.thrivingscholars.com

## Worked Example

Solve for $x$:

a)

$$\left|\frac{2x+3}{2-x}\right| = 5$$

**Analytically**

Split into two equations

$$\frac{2x+3}{2-x} = \pm 5$$

Solve individually

$\frac{2x+3}{2-x} = 5$

$2x+3 = 10-5x$

$7x = 7$

$x = 1$

$\frac{2x+3}{2-x} = -5$

$2x+3 = 5x-10$

$13 = 3x$

$x = \frac{13}{3}$

Check:

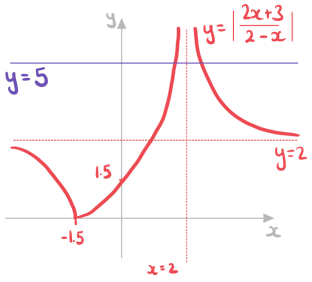
$\left|\frac{2(1)+3}{2-(1)}\right| = 5$ ✓

$\left|\frac{2(\frac{13}{3})+3}{2-(\frac{13}{3})}\right| = 5$ ✓

$x = 1$ or $x = \frac{13}{3}$

**Graphically**

Sketch the two graphs




Find the points of intersection

(1, 5) (4.33, 5)

Choose the $x$-coordinates

$x = 1$ or $x = 4.33$ (3sf)

b)

$$|3x-1| = 5x-11.$$

**Analytically**

Split into two equations

$3x-1 = \pm(5x-11)$

Solve individually

$3x-1 = 5x-11$

$10 = 2x$

$x = 5$

$3x-1 = 11-5x$

$8x = 12$

$x = 1.5$

Check:

$|3(5)-1| = 14$

$5(5)-11 = 14$ ✓

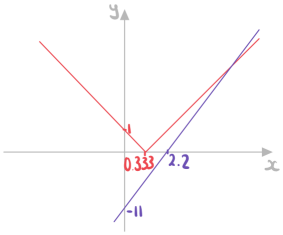
$|3(1.5)-1| = 3.5$

$5(1.5)-11 = -3.5$ ✗

$x = 5$

**Graphically**

Sketch the two graphs




Find the points of intersection

(5, 14)

Choose the $x$-coordinates

$x = 5$

www.thrivingscholars.com

# Modulus Inequalities

## How do I solve modulus inequalities analytically?

- To solve **any** modulus inequality
  - First solve the corresponding modulus equation
    - Remembering to **check whether solutions are valid**
  - Then use a graphical method or a sign table to find the intervals that satisfy the inequality
- Another method is to solve **two pairs of inequalities**
  - For $|f(x)| < g(x)$ solve:
    - $f(x) < g(x)$ when $f(x) \geq 0$
    - $f(x) > -g(x)$ when $f(x) \leq 0$
  - For $|f(x)| > g(x)$ solve:
    - $f(x) > g(x)$ when $f(x) \geq 0$
    - $f(x) < -g(x)$ when $f(x) \leq 0$

### Exam Tip

- If a question on this appears on a calculator paper then use the same ideas as solving other inequalities
  - Sketch the graphs and find the intersections

www.thrivingscholars.com

## Worked Example

Solve the following inequalities for $x$.

a)

$|2x-1|<4$

Solve for $2x-1 \geq 0$

For $x \geq \frac{1}{2}$: $2x-1<4 \Rightarrow x<\frac{5}{2}$ $\therefore \frac{1}{2} \leq x<\frac{5}{2}$

Solve for $2x-1 \leq 0$

For $x \leq \frac{1}{2}$: $2x-1>-4 \Rightarrow x>-\frac{3}{2}$ $\therefore -\frac{3}{2}<x \leq \frac{1}{2}$

Combine inequalities $\boxed{-\frac{3}{2}<x<\frac{5}{2}}$

b)

$|x+1|<|2x+3|$

Solve the corresponding equation

$|x+1|=|2x+3| \Rightarrow x+1=\pm(2x+3)$

Solve

$x+1=2x+3$

$x=-2$

$x+1=-2x-3$

$x=-\frac{4}{3}$

Check

$|(-2)+1|=1$

$|2(-2)+3|=1$ ✓

$|(-\frac{4}{3})+1|=\frac{1}{3}$

$|2(-\frac{4}{3})+3|=\frac{1}{3}$ ✓

Use a sign table

| Check $x=-3$ | Check $x=-1.5$ | Check $x=0$ |
|---|---|---|
| $\lvert(-3)+1\rvert < \lvert2(-3)+3\rvert$ | $\lvert(-1.5)+1\rvert < \lvert2(-1.5)+3\rvert$ | $\lvert0+1\rvert < \lvert2(0)+3\rvert$ |
| $2 < 3$ | $0.5 < 0$ | $1 < 3$ |
| True ✓ | False ✗ | True ✓ |




Write solution $\boxed{x<-2 \text{ or } x>-\frac{4}{3}}$

www.thrivingscholars.com



# Reciprocal Transformations

## What effects do reciprocal transformations have on the graphs?

- The ***x*-coordinates stay the same**
- The ***y*-coordinates change**
  - Their values become their **reciprocals**
- The coordinates $(x, y)$ become $\left(x, \frac{1}{y}\right)$ where $y \neq 0$
  - If $y = 0$ then a vertical asymptote goes through the original coordinate
  - Points that lie on the line $y = 1$ or the line $y = -1$ **stay the same**

## How do I sketch the graph of the reciprocal of a function: $y = 1/f(x)$?

- Sketch the **reciprocal transformation** by considering the **different features** of the original graph
- Consider key points on the original graph
  - If $(x_1, y_1)$ is a point on $y = f(x)$ where $y_1 \neq 0$
    - $\left(x_1, \frac{1}{y_1}\right)$ is a point on $y = \frac{1}{f(x)}$
    - If $|y_1| < 1$ then the point gets **further away from the *x*-axis**
    - If $|y_1| > 1$ then the point gets **closer to the *x*-axis**
  - If $y = f(x)$ has a ***y*-intercept** at $(0, c)$ where $c \neq 0$
    - The reciprocal graph $y = \frac{1}{f(x)}$ has a ***y*-intercept** at $\left(0, \frac{1}{c}\right)$
  - If $y = f(x)$ has a **root** at $(a, 0)$
    - The reciprocal graph $y = \frac{1}{f(x)}$ has a **vertical asymptote** at $x = a$
  - If $y = f(x)$ has a **vertical asymptote** at $x = a$
    - The reciprocal graph $y = \frac{1}{f(x)}$ has a **discontinuity** at $(a, 0)$
    - The **discontinuity** will look like a **root**
  - If $y = f(x)$ has a **local maximum** at $(x_1, y_1)$ where $y_1 \neq 0$
    - The reciprocal graph $y = \frac{1}{f(x)}$ has a **local minimum** at $\left(x_1, \frac{1}{y_1}\right)$
  - If $y = f(x)$ has a **local minimum** at $(x_1, y_1)$ where $y_1 \neq 0$

www.thrivingscholars.com



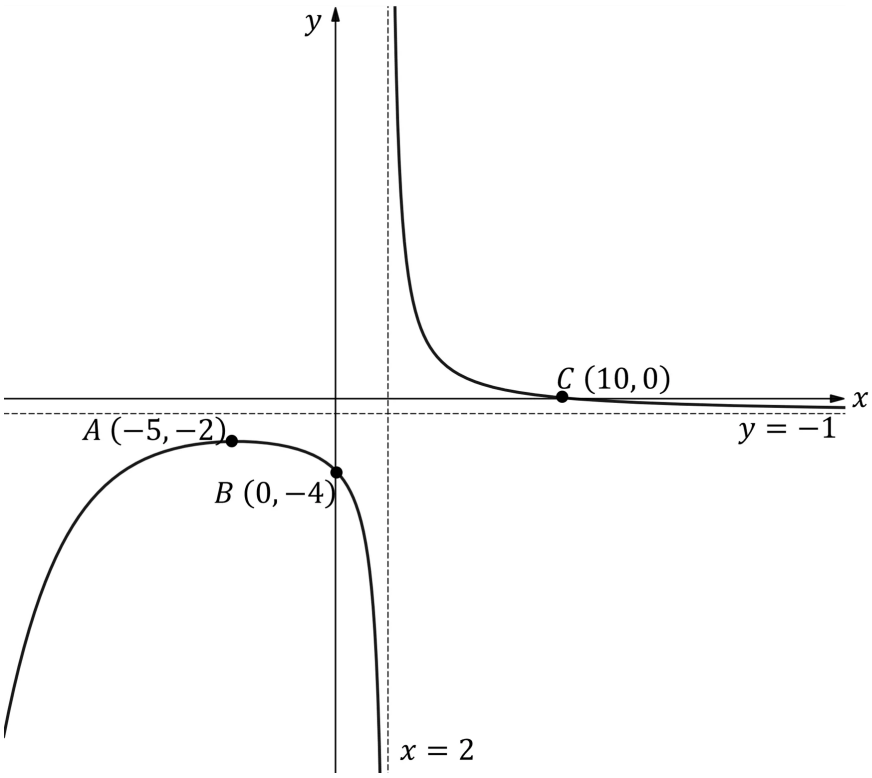
- The reciprocal graph $y = \frac{1}{f(x)}$ has a **local maximum** at $\left(x_1, \frac{1}{y_1}\right)$

- Consider key regions on the original graph
  - If $y = f(x)$ is **positive** then $y = \frac{1}{f(x)}$ is **positive**
    - If $y = f(x)$ is **negative** then $y = \frac{1}{f(x)}$ is **negative**
  - If $y = f(x)$ is **increasing** then $y = \frac{1}{f(x)}$ is **decreasing**
    - If $y = f(x)$ is **decreasing** then $y = \frac{1}{f(x)}$ is **increasing**
  - If $y = f(x)$ has a **horizontal asymptote** at $y = k$
    - $y = \frac{1}{f(x)}$ has a **horizontal asymptote** at $y = \frac{1}{k}$ if $\boldsymbol{k \neq 0}$
    - $y = \frac{1}{f(x)}$ **tends to ± ∞** if $\boldsymbol{k = 0}$
  - If $y = f(x)$ **tends to ± ∞** as $x$ tends to $+\infty$ or $-\infty$
    - $y = \frac{1}{f(x)}$ has a **horizontal asymptote** at $y = 0$

www.thrivingscholars.com

## Worked Example

The diagram below shows the graph of $y = f(x)$ which has a local maximum at the point $A$.




Sketch the graph of $y = \frac{1}{f(x)}$.

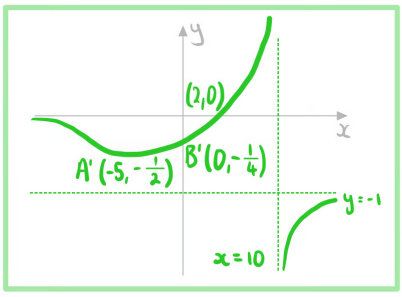
A becomes local minimum $(-5, -\frac{1}{2})$

Vertical asymptote becomes root $(2, 0)$

B becomes $(0, -\frac{1}{4})$

C becomes vertical asymptote $x = 10$

Horizontal asymptote $y = -1$ remains



www.thrivingscholars.com

# Square Transformations

## What effects do square transformations have on the graphs?

- The effects are **similar to** the transformation $y = |f(x)|$
  - The parts **below the $x$-axis are reflected**
  - The **vertical distance** between a point and the $x$-axis is **squared**
    - This has the effect of **smoothing the curve** at the $x$-axis
- $y = [f(x)]^2$ is **never below the $x$-axis**
- The **$x$-coordinates stay the same**
- The **$y$-coordinates change**
  - Their values are **squared**
- The coordinates $(x, y)$ become $(x, y^2)$
  - Points that lie on the **$x$-axis** or the line **$y = 1$ stay the same**

## How do I sketch the graph of the square of a function: $y = [f(x)]^2$?

- Sketch the **square transformation** by considering the **different features** of the original graph
- Consider key points on the original graph
  - If $(x_1, y_1)$ is a point on $y = f(x)$
    - $(x_1, y_1^2)$ is a point on $y = [f(x)]^2$
    - If $|y_1| < 1$ then the point gets **closer to the $x$-axis**
    - If $|y_1| > 1$ then the point gets **further away from the $x$-axis**
  - If $y = f(x)$ has a **$y$-intercept** at $(0, c)$
    - The square graph $y = [f(x)]^2$ has a **$y$-intercept** at $(0, c^2)$
  - If $y = f(x)$ has a **root** at $(a, 0)$
    - The square graph $y = [f(x)]^2$ has a **root** and **turning point** at $(a, 0)$
  - If $y = f(x)$ has a **vertical asymptote** at $x = a$
    - The square graph $y = [f(x)]^2$ has a **vertical asymptote** at $x = a$
  - If $y = f(x)$ has a **local maximum** at $(x_1, y_1)$
    - The square graph $y = [f(x)]^2$ has a **local maximum** at $(x_1, y_1^2)$ if $\mathbf{y_1 > 0}$
    - The square graph $y = [f(x)]^2$ has a **local minimum** at $(x_1, y_1^2)$ if $\mathbf{y_1 \leq 0}$
  - If $y = f(x)$ has a **local minimum** at $(x_1, y_1)$
    - The square graph $y = [f(x)]^2$ has a **local minimum** at $(x_1, y_1^2)$ if $\mathbf{y_1 \geq 0}$
    - The square graph $y = [f(x)]^2$ has a **local maximum** at $(x_1, y_1^2)$ if $\mathbf{y_1 < 0}$

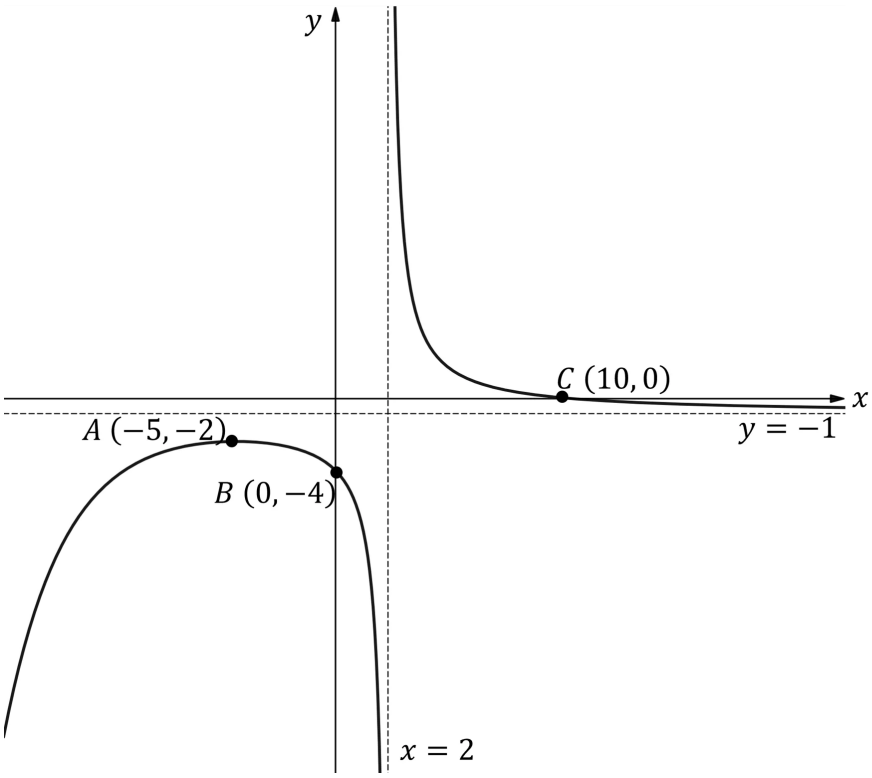
### Exam Tip

- In an exam question when sketching $y = [f(x)]^2$ make it clear that the points where the new graph touches the $x$-axis are smooth
  - This will make it clear to the examiner that you understand the difference between the roots of the graphs $y = |f(x)|$ and $y = [f(x)]^2$

www.thrivingscholars.com

## Worked Example

The diagram below shows the graph of $y = f(x)$ which has a local maximum at the point $A$.




Sketch the graph of $y = [f(x)]^2$.

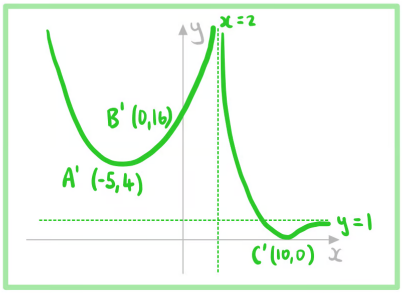
A becomes local minimum (-5, 4)

Vertical asymptote $x = 2$ remains

B becomes (0, 16)

C becomes local minimum

Horizontal asymptote becomes $y = 1$



www.thrivingscholars.com

www.thrivingscholars.com

www.thrivingscholars.com

www.thrivingscholars.com

www.thrivingscholars.com

www.thrivingscholars.com

www.thrivingscholars.com

# IB Maths DP

## 3. Geometry & Trigonometry

**CONTENTS**


3.1 Geometry Toolkit
- 3.1.1 Coordinate Geometry
- 3.1.2 Radian Measure
- 3.1.3 Arcs & Sectors

3.2 Geometry of 3D Shapes
- 3.2.1 3D Coordinate Geometry
- 3.2.2 Volume & Surface Area

3.3 Trigonometry Toolkit
- 3.3.1 Pythagoras & Right-Angled Trigonometry
- 3.3.2 Non Right-Angled Trigonometry
- 3.3.3 Applications of Trigonometry & Pythagoras

3.4 Trigonometry
- 3.4.1 The Unit Circle
- 3.4.2 Exact Values

3.5 Trigonometric Functions & Graphs
- 3.5.1 Graphs of Trigonometric Functions
- 3.5.2 Transformations of Trigonometric Functions
- 3.5.3 Modelling with Trigonometric Functions

3.6 Trigonometric Equations & Identities
- 3.6.1 Simple Identities
- 3.6.2 Compound Angle Formulae
- 3.6.3 Double Angle Formulae
- 3.6.4 Relationship Between Trigonometric Ratios
- 3.6.5 Linear Trigonometric Equations
- 3.6.6 Quadratic Trigonometric Equations

3.7 Inverse & Reciprocal Trig Functions
- 3.7.1 Reciprocal Trig Functions
- 3.7.2 Inverse Trig Functions

3.8 Further Trigonometry
- 3.8.1 Trigonometric Proof
- 3.8.2 Strategy for Trigonometric Equations

3.9 Vector Properties

www.thrivingscholars.com

3.9.1 Introduction to Vectors
3.9.2 Position & Displacement Vectors
3.9.3 Magnitude of a Vector
3.9.4 The Scalar Product
3.9.5 Geometric Proof with Vectors
3.10 Vector Equations of Lines
3.10.1 Vector Equations of Lines
3.10.2 Applications to Kinematics
3.10.3 Pairs of Lines in 3D
3.10.4 The Vector Product
3.10.5 Shortest Distances with Lines
3.11 Vector Planes
3.11.1 Vector Equations of Planes
3.11.2 Intersections of Lines & Planes
3.11.3 Angles Between Lines & Planes
3.11.4 Shortest Distances with Planes

www.thrivingscholars.com



# 3.1 Geometry Toolkit

## 3.1.1 Coordinate Geometry

### Basic Coordinate Geometry

### What are cartesian coordinates?

- **Cartesian** coordinates are basically the *x*-*y* coordinate system
  - They allow us to label where things are in a two-dimensional plane
- In the 2D cartesian system, the horizontal axis is labelled *x* and the vertical axis is labelled *y*

### What can we do with coordinates?

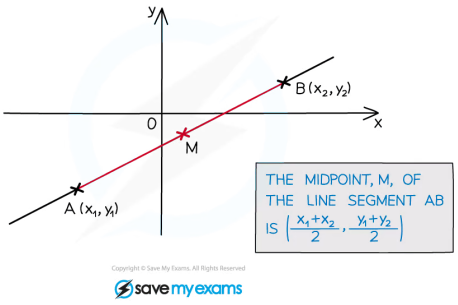
- If we have two points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ then we should be able to find
  - The **midpoint** of the two points
  - The **distance** between the two points
  - The **gradient** of the line between them

### How do I find the midpoint of two points?

- The midpoint is the **average (middle) point**
  - It can be found by finding the middle of the *x*-coordinates and the middle of the *y*-coordinates
- The coordinates of the midpoint will be

$$\left( \frac{x_1 + x_2}{2} , \frac{y_1 + y_2}{2} \right)$$

  - This is given in the formula booklet under the prior learning section at the beginning




Copyright © Save My Exams. All Rights Reserved
save my exams

### How do I find the distance between two points?

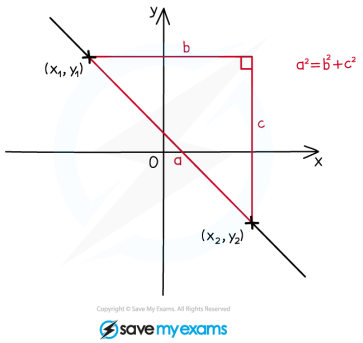
- The distance between two points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ can be found using the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

www.thrivingscholars.com

- This is given in the formula booklet in the *prior learning* section at the beginning

- Pythagoras' Theorem $a^2 = b^2 + c^2$ is used to find the length of a line between two coordinates
- If the coordinates are labelled A and B then the line segment between them is written with the notation [AB]




## How do I find the gradient of the line between two points?

- The gradient of a line between two points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ can be found using the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

  - This is given in the formula booklet under section 2.1 Gradient formula
  - This is usually known as $m = \frac{\text{rise}}{\text{run}}$

www.thrivingscholars.com

## Worked Example

Point $A$ has coordinates $(3, -4)$ and point $B$ has coordinates $(-5, 2)$.

i)

Calculate the distance of the line segment $AB$.

$A: (3, -4)$ $\quad B: (-5, 2)$

$x_1$ $\quad y_1$ $\quad x_2$ $\quad y_2$

Formula for distance between two points:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Sub coordinates for A and B into the formula:

$$d = \sqrt{(3 - (-5))^2 + (-4 - 2)^2}$$

$$= \sqrt{8^2 + (-6)^2} = \sqrt{100}$$

$$d = 10 \text{ units}$$

ii)

Find the gradient of the line connecting points $A$ and $B$.

$A: (3, -4)$ $\quad B: (-5, 2)$

$x_1$ $\quad y_1$ $\quad x_2$ $\quad y_2$

Formula for gradient of a line segment:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Sub coordinates for A and B into the formula:

$$m = \frac{2 - -4}{-5 - 3} = \frac{6}{-8} = -\frac{3}{4}$$

$$m = -\frac{3}{4}$$

iii)

Find the midpoint of $[AB]$.

www.thrivingscholars.com



$A:(3, -4)$ $\quad B:(-5, 2)$

$x_1$ $\quad y_1$ $\quad x_2$ $\quad y_2$

Formula for the midpoint of two coordinates:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Sub values in:

$$\text{Midpoint} = \left(\frac{3 + (-5)}{2}, \frac{-4 + 2}{2}\right) = (-1, -1)$$

**Midpoint = $(-1, -1)$**

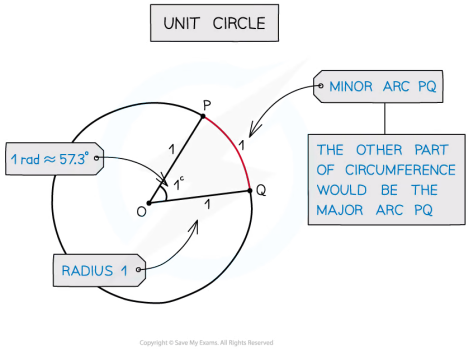
www.thrivingscholars.com



# Radian Measure

## What are radians?

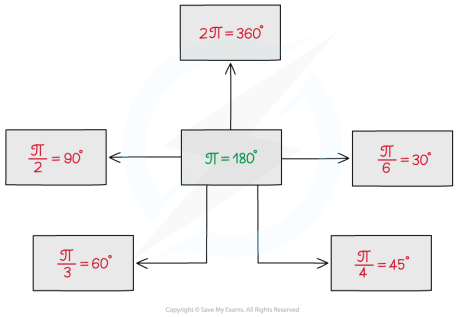
- Radians are an alternative to degrees for measuring angles
- 1 radian is the angle in a **sector** of radius 1 and arc length 1
  - A circle with radius 1 is called a **unit circle**
- Radians are normally quoted in terms of π
  - 2π radians = 360°
  - π radians = 180°
- The symbol for radians is $^{c}$ but it is more usual to see **rad**
  - Often, when π is involved, no symbol is given as it is obvious it is in radians
  - Whilst it is okay to omit the symbol for radians, you should never omit the symbol for degrees
- In the exam you should use radians unless otherwise indicated




## How do I convert between radians and degrees?

- Use $\pi^{c} = 180°$ to convert between radians and degrees
  - To convert from radians to degrees multiply by $\frac{180}{\pi}$
  - To convert from degrees to radians multiply by $\frac{\pi}{180}$
- Some of the common conversions are:
  - $2\pi^{c} = 360°$
  - $\pi^{c} = 180°$
  - $\frac{\pi}{2}^{c} = 90°$
  - $\frac{\pi}{3}^{c} = 60°$
  - $\frac{\pi}{4}^{c} = 45°$
  - $\frac{\pi}{6}^{c} = 30°$
- It is a good idea to remember some of these and use them to work out other conversions
- Your GDC will be able to work with both radians and degrees

www.thrivingscholars.com




## Exam Tip

- Sometimes an exam question will specify whether you should be using degrees or radians and sometimes it will not, if it doesn't it is expected that you will work in radians
- If the question involves π then working in radians is useful as there will likely be opportunities where you can cancel out π
- Make sure that your calculator is in the correct mode for the type of angle you are working with

www.thrivingscholars.com

## Worked Example

i)

Convert 43.8° to radians.

$43.8^\circ \rightarrow \div 180^\circ \rightarrow \frac{73}{300} \rightarrow \times \pi^c \rightarrow \frac{73\pi}{300^c}$ $(\pi^c = 180^\circ)$

$43.8^\circ = 0.764^c$ (3 s.f.)

ii) Convert $\frac{5\pi}{4}$ to degrees.

$\frac{5\pi}{4} \rightarrow \div \pi^c \rightarrow \frac{5}{4} \rightarrow \times 180^\circ \rightarrow 225^\circ$ $(\pi^c = 180^\circ)$

$\frac{5\pi}{4} = 225^\circ$

www.thrivingscholars.com



# Length of an Arc

## What is an arc?

- An arc is a part of the **circumference** of a circle
  - It is easiest to think of it as the crust of a single slice of pizza
- The length of an arc depends of the size of the angle at the centre of the circle
- If the angle at the centre is **less than 180°** then the arc is known as a **minor arc**
  - This could be considered as the crust of a single slice of pizza
- If the angle at the centre is **more than 180°** then the arc is known as a **major arc**
  - This could be considered as the crust of the remaining pizza after a slice has been taken away

## How do I find the length of an arc?

- The length of an arc is simply a fraction of the circumference of a circle
  - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the length, $l$, of an arc is

$$l = \frac{\theta}{360} \times 2\pi r$$

  - Where $\theta$ is the angle measured in degrees
  - $r$ is the radius
  - This is **in the formula booklet for radian measure only**
    - Remember 2π radians = 360°

### Exam Tip

- Make sure that you read the question carefully to determine if you need to calculate the arc length of a sector, the perimeter or something else that incorporates the arc length!

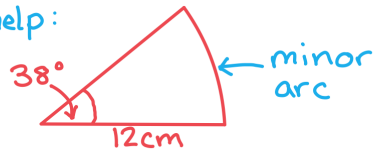
www.thrivingscholars.com

## Worked Example

A circular pizza has had a slice cut from it, the angle of the slice that was cut was 38 °. The radius of the pizza is 12 cm. Find

i)
the length of the outside crust of the slice of pizza (the minor arc),

A diagram will help:




Formula for the length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

$$l = \frac{38}{360} \times 2\pi (12)$$

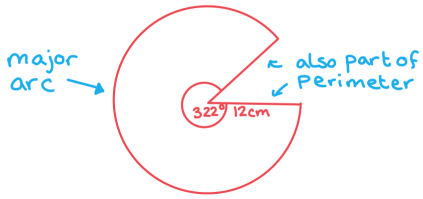
$$= \frac{38\pi}{15} = 7.9587\ldots \text{ cm}$$

**Length of crust = 7.96 cm (3s.f)**

ii)
the perimeter of the remaining pizza.

www.thrivingscholars.com

A diagram will help:




formula for the Length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

$$l = \frac{322}{360} \times 2\pi (12)$$

$$= \frac{322\pi}{15}$$ ← Length of major arc

Find perimeter:

$$P = \text{major arc} + \text{radius} + \text{radius}$$

$$= \frac{322\pi}{15} + 12 + 12 = 91.4395\ldots \text{ cm}$$

**Perimeter = 91.4 cm (3s.f)**

www.thrivingscholars.com

# Area of a Sector

## What is a sector?

- A sector is a part of a circle enclosed by two radii (radiuses) and an arc
  - It is easier to think of this as the shape of a single slice of pizza
- The area of a sector depends of the size of the angle at the centre of the sector
- If the angle at the centre is **less than 180°** then the sector is known as a **minor sector**
  - This could be considered as the shape of a single slice of pizza
- If the angle at the centre is **more than 180°** then the sector is known as a **major sector**
  - This could be considered as the shape of the remaining pizza after a slice has been taken away

## How do I find the area of a sector?

- The area of a sector is simply a fraction of the area of the whole circle
  - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the area, $A$, of a sector is

$$A = \frac{\theta}{360} \times \pi r^2$$

  - Where $\theta$ is the angle measured in degrees
  - $r$ is the radius
  - This is **in the formula booklet for radian measure only**
    - Remember $2\pi$ radians = 360°

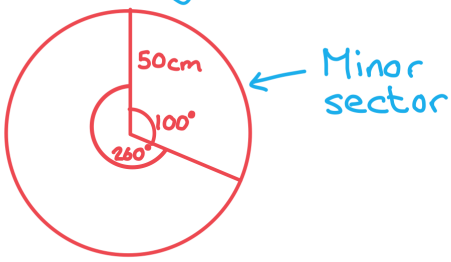
www.thrivingscholars.com

## Worked Example

Jamie has divided a circle of radius 50 cm into two sectors; a minor sector of angle 100° and a major sector of angle 260°. He is going to paint the minor sector blue and the major sector yellow. Find

i)
the area Jamie will paint blue,

Start with a diagram:




Formula for the area of a sector:

$$A = \frac{\theta}{360^\circ} \times \pi r^2$$

Substitute: $A = \frac{100}{360} \times \pi \times 50^2$

$$= \frac{6250}{9}\pi$$

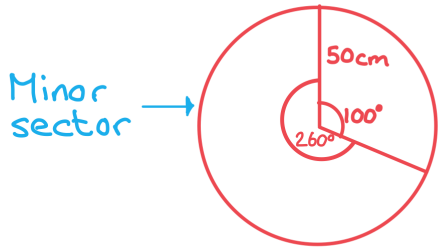
$$= 2181.66\ldots \text{ cm}^2$$

**Blue area = 2180 cm² (3sf)**

ii)
the area Jamie will paint yellow.

www.thrivingscholars.com

Start with a diagram:




Formula for the area of a sector:

$$A = \frac{\theta}{360^\circ} \times \pi r^2$$

Substitute: $A = \frac{260}{360} \times \pi \times 50^2$

$$= \frac{16250}{9}\pi$$

$$= 5672.32\ldots \text{cm}^2$$

**Yellow area = 5670 $\text{cm}^2$ (3sf)**

www.thrivingscholars.com

# Arcs & Sectors Using Radians

## How do I use radians to find the length of an arc?

- As the radian measure for a **full turn** is $2\pi$, the fraction of the circle becomes $\frac{\theta}{2\pi}$
- Working in radians, the formula for the length of an arc will become

$$l = \frac{\theta}{2\pi} \times 2\pi r$$

- Simplifying, the formula for the length, $l$, of an arc is

$$l = r\theta$$

  - $\theta$ is the angle measured in **radians**
  - $r$ is the radius
  - This is **given in the formula booklet,** you do not need to remember it

## How do I use radians to find the area of a sector?

- As the radian measure for a **full turn** is $2\pi$, the fraction of the circle becomes $\frac{\theta}{2\pi}$
- Working in radians, the formula for the area of a sector will become

$$A = \frac{\theta}{2\pi} \times \pi r^2$$

- Simplifying, the formula for the area, $A$, of a sector is

$$A = \frac{1}{2} r^2 \theta$$

  - $\theta$ is the angle measured in **radians**
  - $r$ is the radius
  - This is **given in the formula booklet,** you do not need to remember it

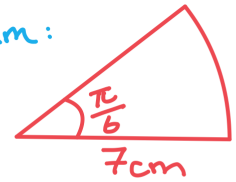
www.thrivingscholars.com

## Worked Example

A slice of cake forms a sector of a circle with an arc of $\frac{\pi}{6}$ radians and radius of 7 cm.

Find the area of the surface of the slice of cake and its perimeter.

Draw a diagram:




Area of a sector: $A = \frac{1}{2}r^2\theta$

Substitute: $r = 7$, $\theta = \frac{\pi}{6}$

$$A = \frac{1}{2}(7)^2\left(\frac{\pi}{6}\right) = \frac{49\pi}{12}$$

**Area = 12.8 cm² (3 s.f.)**

Perimeter = arc Length + 2(radius)

Length of an arc: $l = r\theta$

$$P = 7\left(\frac{\pi}{6}\right) + 2(7)$$

**Perimeter = 17.7 cm (3 s.f.)**

www.thrivingscholars.com





# 3D Coordinate Geometry

## How does the 3D coordinate system work?

- In three-dimensional space we can label where any object is using the $x$-$y$-$z$ coordinate system
- In the 3D cartesian system, the $x$- and $y$- axes usually represent lateral space (length and width) and the $z$-axis represents vertical height

## What can we do with 3D coordinates?

- If we have two points with coordinates $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ then we should be able to find:
  - The **midpoint** of the two points
  - The **distance** between the two points
- If the coordinates are labelled A and B then the line segment between them is written with the notation [AB]

## How do I find the midpoint of two points in 3D?

- The midpoint is the **average (middle) point**
  - It can be found by finding the middle of the $x$-coordinates and the middle of the $y$-coordinates
- The coordinates of the midpoint will be

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

  - This is given in the formula booklet, you do not need to remember it

## How do I find the distance between two points in 3D?

- The distance between two points with coordinates $((x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ can be found using the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

  - This is given in the formula booklet, you do not need to remember it

www.thrivingscholars.com

## Worked Example

The points $A$ and $B$ have coordinates $(-2, 1, 5)$ and $(4, -3, 2)$ respectively.

i)

Calculate the distance of the line segment $AB$.

Formula for the distance of a line segment:

in formula booklet

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

$A: (-2, 1, 5)$ $\quad$ $B: (4, -3, 2)$

$x_1$ $y_1$ $z_1$ $\quad$ $x_2$ $y_2$ $z_2$

Substitute:

$$d = \sqrt{(-2 - 4)^2 + (1 - (-3))^2 + (5 - 2)^2}$$

$$= \sqrt{(-6)^2 + 4^2 + 3^2}$$

$$= \sqrt{36 + 16 + 9}$$

$$= \sqrt{61}$$

$$d = 7.81 \text{ units (3 sf)}$$

ii)

Find the midpoint of $[AB]$.

www.thrivingscholars.com

Formula for the midpoint of a line segment:

$$MP = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

in formula booklet

$A: (-2, 1, 5)$ $\quad$ $B: (4, -3, 2)$

$x_1$ $y_1$ $z_1$ $\quad$ $x_2$ $y_2$ $z_2$

Substitute:

$$MP = \left( \frac{-2+4}{2}, \frac{1+(-3)}{2}, \frac{5+2}{2} \right)$$

$$= \left( \frac{2}{2}, -\frac{2}{2}, \frac{7}{2} \right)$$

$$MP = (1, -1, 3.5)$$

www.thrivingscholars.com



# Volume of 3D Shapes

## What is volume?

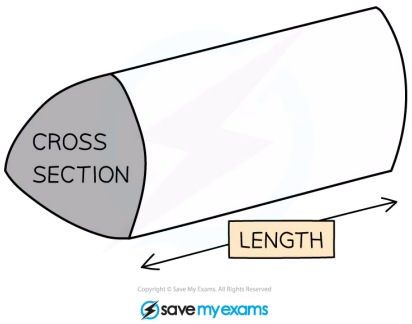
- The volume of a 3D shape is a measure of how much 3D space it takes up
  - A 3D shape is also called a **solid**
- You need to be able to calculate the volume of a number of common shapes

## How do I find the volume of cuboids, prisms and cylinders?

- A prism is a 3-D shape that has two identical **base** shapes connected by parallel **edges**
  - A prism has the same base shape all the way through
  - A **prism** takes its name from its base
- To find the **volume** of any prism use the formula:

$$\text{Volume of a prism} = Ah$$

  - Where ***A*** is the area of the cross section and ***h*** is the base height
    - *h* could also be the length of the prism, depending on how it is oriented
  - This is in the formula booklet in the **prior learning** section at the beginning
  - The base could be any shape so as long as you know its area and length you can calculate the volume of any prism




Copyright © Save My Exams. All Rights Reserved
save my exams

- Note two special cases:
  - To find the volume of a cuboid use the formula:

$$\text{Volume of a cuboid} = \text{length} \times \text{width} \times \text{height}$$
$$V = lwh$$

  - The volume of a **cylinder** can be found in the same way as a prism using the formula:

$$\text{Volume of a cylinder} = \pi r^2 h$$

  - where $r$ is the radius, $h$ is the height (or length, depending on the orientation
  - Note that a cylinder is technically not a prism as its base is not a polygon, however the method for finding its volume is the same
- Both of these are **in the formula booklet** in the **prior learning** section

www.thrivingscholars.com

## How do I find the volume of pyramids and cones?

- In a **right-pyramid** the apex (the joining point of the triangular faces) is vertically above the centre of the base
  - The base can be any shape but is usually a square, rectangle or triangle
- To calculate the volume of a **right-pyramid** use the formula

$$V = \frac{1}{3}Ah$$

  - Where $A$ is the area of the base, $h$ is the height
  - Note that the height must be **vertical to the base**

- **A right cone** is a circular-based pyramid with the vertical height joining the apex to the centre of the circular base
- To calculate the volume of a **right-cone** use the formula

$$V = \frac{1}{3}\pi r^2 h$$

  - Where $r$ is the radius, $h$ is the height

- These formulae are both given in the formula booklet

## How do I find the volume of a sphere?

- To calculate the volume of a **sphere** use the formula

$$V = \frac{4}{3}\pi r^3$$

  - Where $r$ is the radius
    - the line segment from the centre of the sphere to the surface
  - This formula is given in the formula booklet

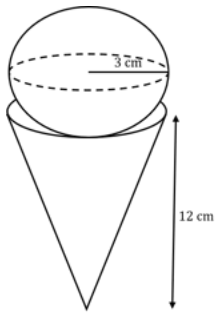
### Exam Tip

- Remember to make use of the formula booklet in the exam as all the volume formulae you need will be here
  - Formulae for basic 3D objects (cuboid, cylinder and prism) are in the **prior learning** section
  - Formulae for other 3D objects (pyramid, cone and sphere) are in the **Topic 3: Geometry** section

www.thrivingscholars.com

## Worked Example

A dessert can be modelled as a right-cone of radius 3 cm and height 12 cm and a scoop of ice-cream in the shape of a sphere of radius 3 cm. Find the total volume of the ice-cream and cone.




Volume of a sphere: $V = \frac{4}{3}\pi r^3$ (In formula booklet)

Substitute: $r = 3 \Rightarrow V = \frac{4}{3}\pi \times 3^3$

$= 36\pi$

Volume of a right cone: $V = \frac{1}{3}\pi r^2 h$ (In formula booklet)

Substitute: $r = 3,\ h = 12 \Rightarrow V = \frac{1}{3}\pi(3)^2(12)$

$= 36\pi$

Total Volume $= 72\pi\ cm^3$

**Total Volume = 226 $cm^3$ (3sf)**

www.thrivingscholars.com

# Surface Area of 3D Shapes

## What is surface area?

- The surface area of a 3D shape is the sum of the areas of all the **faces** that make up a shape
  - A **face** is one of the flat or curved surfaces that make up a 3D shape
  - It often helps to consider a 3D shape in the form of its 2D net

## How do I find the surface area of cuboids, pyramids and prisms?

- Any prisms and pyramids that have polygons as their bases have only flat faces
  - The surface area is simply found by adding up the areas of these flat faces
  - Drawing a 2D net will help to see which faces the 3D shape is made up of

## How do I find the surface area of cylinders, cones and spheres?

- Cones, cylinders and spheres all have curved faces so it is not always as easy to see their shape
  - The net of a **cylinder** is made up of two identical circles and a rectangle
  - The rectangle is the curved surface area and is harder to identify
  - The length of the rectangle is the same as the circumference of the circle
  - The area of the **curved surface area** is

$$A = 2\pi rh$$

    - where $r$ is the radius, $h$ is the height
  - This is given in the formula book in the prior learning section
  - The area of the **total surface area of a cylinder** is

$$A = 2\pi rh + 2\pi r^2$$

  - This is **not** given in the formula book, however it is easy to put together as both the area of a circle and the area of the curved surface area are given

- The net of a **cone** consists of the circular base along with the curved surface area
  - The area of the **curved surface area** is

$$A = \pi rl$$

    - Where $r$ is the radius and $l$ is the **slant height**
  - This is **given in the formula book**
    - Be careful not to confuse the slant height, $l$, with the vertical height, $h$
    - Note that $r$, $h$ and $l$ will create a **right-triangle** with $l$ as the hypotenuse
  - The area of the **total surface area of a cone** is

$$A = \pi rl + \pi r^2$$

  - This is **not** given in the formula book, however it is easy to put together as both the area of a circle and the area of the curved surface area are given

- To find the surface area of a **sphere** use the formula

$$A = 4\pi r^2$$

www.thrivingscholars.com

        - where $r$ is the radius (line segment from the centre to the surface)
    - This is given in the formula booklet, you do not have to remember it

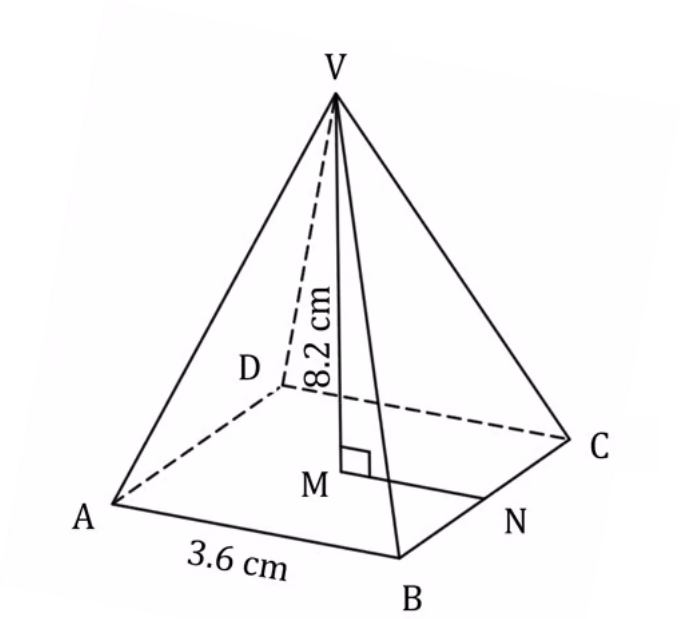
### Exam Tip

- Remember to make use of the formula booklet in the exam as all the area formulae you need will be here
    - Formulae for basic 2D shapes (parallelogram, triangle, trapezoid, circle, curved surface of a cylinder) are in the **prior learning** section
    - Formulae for other 2D shapes (curved surface area of a cone and surface area of a sphere ) are in the **Topic 3: Geometry** section

www.thrivingscholars.com

## Worked Example

In the diagram below $ABCD$ is the square base of a right pyramid with vertex $V$. The centre of the base is $M$. The sides of the square base are 3.6 cm and the vertical height is 8.2 cm.

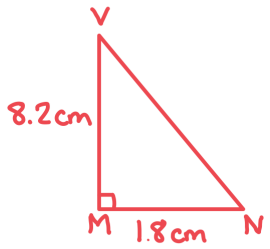



i)

Use the Pythagorean Theorem to find the distance VN.

www.thrivingscholars.com

Sketch the triangle MNV:




M is the midpoint
so MN = 3.6 ÷ 2

By the Pythagorean Theorem:

$$VN^2 = \sqrt{VM^2 + MN^2}$$

$$= \sqrt{8.2^2 + 1.8^2}$$

$$= \sqrt{70.48}$$

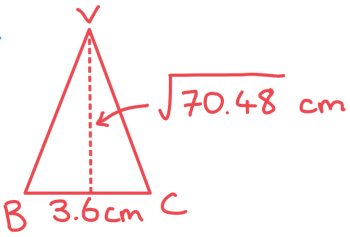
**VN = 8.40 cm (3sf)**

ii)
Calculate the area of the triangle ABV.

www.thrivingscholars.com

Area $\Delta ABV$ = area $\Delta BCV$

Sketch $\Delta BCV$:




Area of a triangle $= \frac{1}{2}bh$

Substitute $b = 3.6$, $h = \sqrt{70.48}$

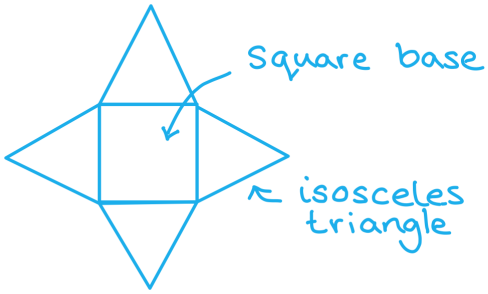
$$\text{Area} = \frac{1}{2}(3.6)(\sqrt{70.48})$$
$$= 15.111\ldots \text{ cm}^2$$

**Area $\Delta ABV$ = 15.1 cm²**

iii)

Find the surface area of the right pyramid.

Considering the net may help:




$$\text{Surface area} = \text{area square} + 4(\text{area triangle})$$
$$SA = 3.6^2 + 4(15.111\ldots)$$
$$= 73.405\ldots \text{ cm}^2$$

**SA = 73.4 cm² (3sf)**

www.thrivingscholars.com

www.thrivingscholars.com



# 3.3 Trigonometry Toolkit

## 3.3.1 Pythagoras & Right-Angled Trigonometry

www.thrivingscholars.com

# Pythagoras

## What is the Pythagorean theorem?

- Pythagoras' theorem is a formula that works for **right-angled triangles** only
- It states that for any right-angled triangle, the **square of the hypotenuse is equal to the sum of the squares of the two shorter sides**
  - The **hypotenuse** is the **longest side** in a right-angled triangle
    - It will always be **opposite** the right angle
  - If we label the hypotenuse *c*, and label the other two sides *a* and *b*, then Pythagoras' theorem tells us that

$$a^2 + b^2 = c^2$$

- The formula for Pythagoras' theorem is assumed prior knowledge and is **not in the formula booklet**
  - You will need to remember it

## How can we use Pythagoras' theorem?

- If you know two sides of any right-angled triangle you can use Pythagoras' theorem to find the length of the third side
  - Substitute the values you have into the formula and either solve or rearrange
- To find the length of the **hypotenuse** you can use:

$$c = \sqrt{a^2 + b^2}$$

- To find the length of **one of the other sides** you can use:

$$a = \sqrt{c^2 - b^2} \text{ or } b = \sqrt{c^2 - a^2}$$

- Note that when finding the **hypotenuse** you should **add** inside the square root and when finding **one of the other sides** you should **subtract** inside the square root
- Always **check** your answer carefully to make sure that the hypotenuse is the longest side
- Note that Pythagoras' theorem questions will rarely be standalone questions and will often be 'hidden' in other geometry questions

## What is the converse of the Pythagorean theorem?

- The converse of the Pythagorean theorem states that if $a^2 + b^2 = c^2$ is true then the triangle must be a right-angled triangle
  - This is a very useful way of determining whether a triangle is right-angled
- If a diagram in a question does not clearly show that something is right-angled, you may need to use Pythagoras' theorem to check

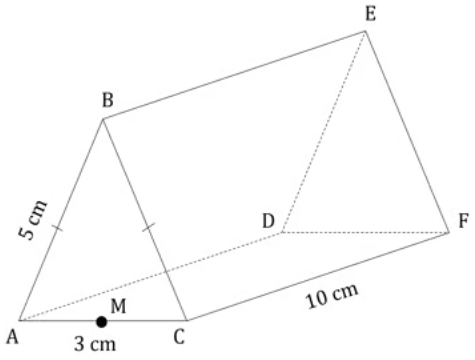
### Exam Tip

- Pythagoras' theorem pops up in lots of exam questions so bear it in mind whenever you see a right-angled triangle in an exam question!

www.thrivingscholars.com

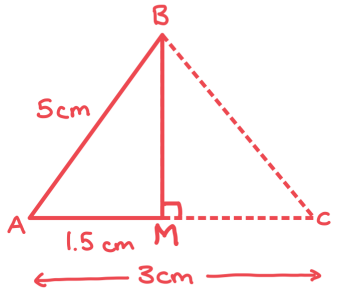
## Worked Example

ABCDEF is a chocolate bar in the shape of a triangular prism. The end of the chocolate bar is an isosceles triangle where AC = 3 cm and AB = BC = 5 cm. M is the midpoint of AC. This information is shown in the diagram below.




Calculate the length BM.

Sketch the triangle ABM:




By the Pythagorean Theorem:

$$BM^2 = \sqrt{AB^2 - AM^2}$$

shorter side — hypotenuse — shorter side

$$= \sqrt{5^2 - 1.5^2}$$

$$= \sqrt{22.75}$$

**BM = 4.77 cm (3sf)**

www.thrivingscholars.com

# Right-Angled Trigonometry

## What is Trigonometry?

- Trigonometry is the mathematics of angles in triangles
- It looks at the relationship between side lengths and angles of triangles
- It comes from the Greek words *trigonon* meaning 'triangle' and *metron* meaning 'measure'

## What are Sin, Cos and Tan?

- The three trigonometric functions Sine, Cosine and Tangent come from ratios of side lengths in right-angled triangles
- To see how the ratios work you must first label the sides of a right-angled triangle in relation to a chosen angle
  - The **hypotenuse, *H*,** is the **longest side** in a right-angled triangle
    - It will always be **opposite** the right angle
  - If we label one of the other angles $\theta$, the side opposite $\theta$ will be labelled **opposite**, ***O,*** and the side next to $\theta$ will be labelled **adjacent**, ***A***
- The functions Sine, Cosine and Tangent are the ratios of the lengths of these sides as follows

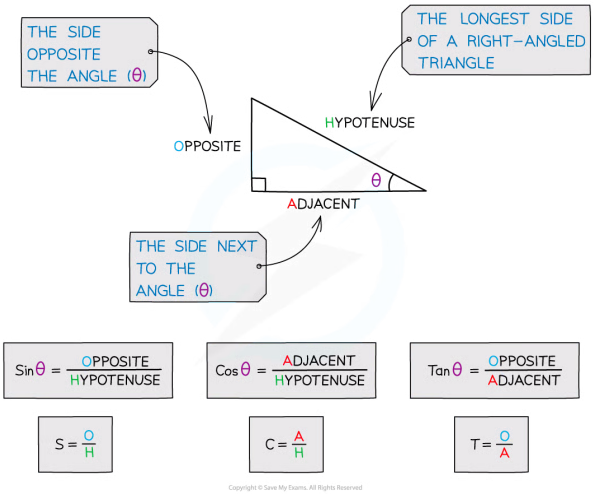
$$\text{Sin}\ \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{O}{H}$$

$$\text{Cos}\ \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{A}}{\text{H}}$$

$$\text{Tan}\ \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{O}}{\text{A}}$$

  - These are **not in the formula book**, you must remember them
- The mnemonic **SOHCAHTOA** is often used as a way of remembering which ratio is which
  - **S**in is **O**pposite over **H**ypotenuse
  - **C**os is **A**djacent over **H**ypotenuse
  - **T**an is **O**pposite over **A**djacent

www.thrivingscholars.com




## How can we use SOHCAHTOA to find missing lengths?

- If you know the length of one of the sides of any right-angled triangle and one of the angles you can use SOHCAHTOA to find the length of the other sides
  - Always start by **labelling the sides** of the triangle with H, O and A
  - Choose the correct ratio by looking only at the values that you have and that you want
    - For example if you know the angle and the side opposite it (O) and you want to find the hypotenuse (H) you should use the sine ratio
  - Substitute the values into the ratio
  - Use your calculator to find the solution

## How can we use SOHCAHTOA to find missing angles?

- If you know two sides of any right-angled triangle you can use SOHCAHTOA to find the size of one of the angles
- Missing angles are found using the **inverse functions**:

$$\theta = \text{Sin}^{-1}\frac{O}{H} \quad , \quad \theta = \text{Cos}^{-1}\frac{\text{A}}{\text{H}} \quad , \quad \theta = \text{Tan}^{-1}\frac{\text{O}}{\text{A}}$$

- After choosing the correct ratio and substituting the values use the inverse trigonometric functions on your calculator to find the correct answer

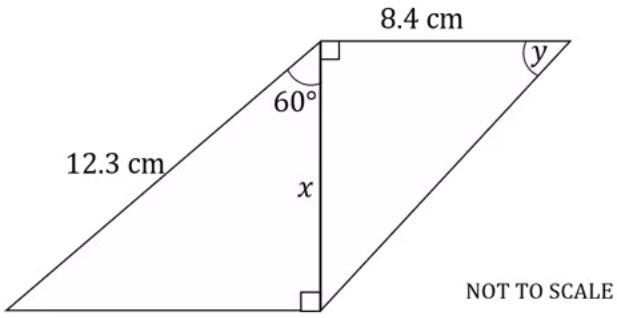
### Exam Tip

- You need to remember the sides involved in the different trig ratios as they are not given to you in the exam

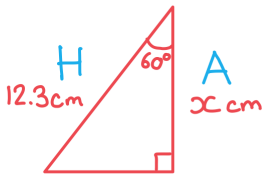
www.thrivingscholars.com

## Worked Example

Find the values of $x$ and $y$ in the following diagram. Give your answers to 3 significant figures.




Start by labelling the sides of the triangle:




SOHCAHTOA

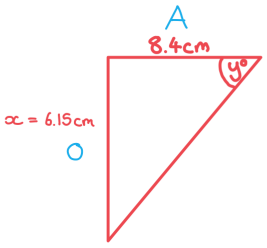
We know H and we want to find A so we need to use

$\cos\theta = \frac{A}{H}$

$\cos 60° = \frac{x}{12.3}$

$x = 12.3\cos 60°$

$x = 6.15\text{cm}$




SOHCAHTOA

$\tan y° = \frac{O}{A}$

$\tan y° = \frac{6.15}{8.4}$

$y° = \tan^{-1}\left(\frac{6.15}{8.4}\right)$

$y° = 36.2°$ (3 s.f.)

www.thrivingscholars.com

# 3D Problems

## How does Pythagoras work in 3D?

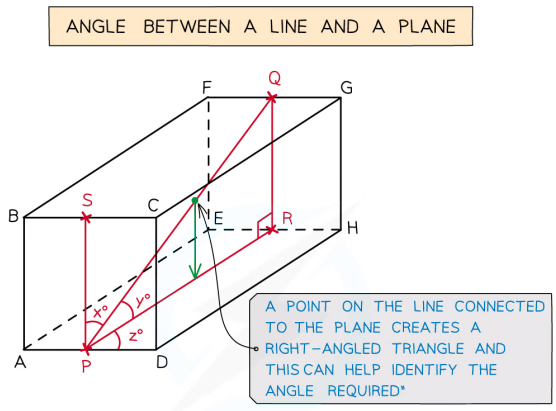
- 3D shapes can often be broken down into several 2D shapes
- With Pythagoras' Theorem you will be specifically looking for right-angled triangles
  - The right-angled triangles you need will have two known sides and one unknown side
  - Look for perpendicular lines to help you spot right-angled triangles
- There is a 3D version of the Pythagorean theorem formula:

$$d^2 = x^2 + y^2 + z^2$$

  - However it is usually easier to see a problem by breaking it down into two or more 2D problems

## How does SOHCAHTOA work in 3D?

- Again look for a combination of right-angled triangles that would lead to the missing angle or side
- The angle you are working with can be awkward in 3D
  - The angle between a line and a plane is not always obvious
  - If unsure put a point on the line and draw a new line to the plane
    - This should create a right-angled triangle



Copyright © Save My Exams. All Rights Reserved

www.thrivingscholars.com

### Exam Tip

- Annotate diagrams that are given to you with values that you have calculated
- It can be useful to make additional sketches of parts of any diagrams that are given to you, especially if there are multiple lengths/angles that you are asked to find
- If you are not given a diagram, sketch a nice, big, clear one!

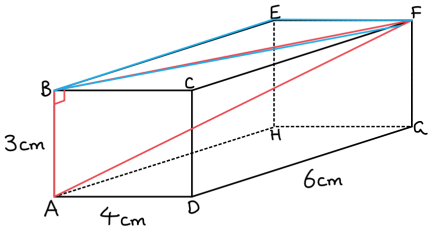
www.thrivingscholars.com

## Worked Example

A pencil is being put into a cuboid shaped box which has dimensions 3 cm by 4 cm by 6 cm. Find:

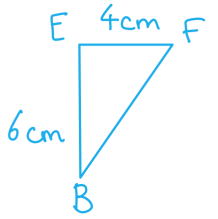
a)
the length of the longest pencil that could fit inside the box,

Draw a diagram:




The longest pencil could fit on any of the diagonals, e.g. AF.

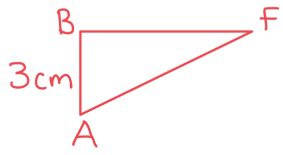
To find AF we must first find BF:




$BF^2 = 4^2 + 6^2$

$BF^2 = 16 + 36$

$BF^2 = 52$ ← Can leave as $BF^2$ for now.




$AF^2 = 3^2 + BF^2$

$= 9 + 52$

$AF^2 = 61$

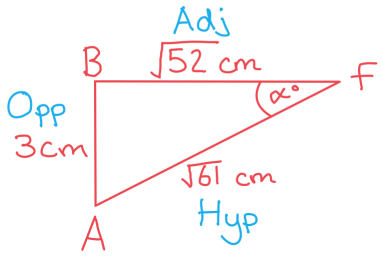
$AF = \sqrt{61} = 7.8102...$

**7.81 cm (3 s.f.)**

b)
the angle that the pencil would make with the top of the box.

www.thrivingscholars.com

Find $A\hat{F}B$:




All three sides are known so can use any of the trig ratios.

SOH CAH TOA

↑ Choose

$$\tan\alpha = \frac{\text{opp}}{\text{adj}}$$

$$\tan\alpha = \frac{3}{\sqrt{52}}$$

$$\alpha = \tan^{-1}\left(\frac{3}{\sqrt{52}}\right)$$

$$= 22.588...$$

$A\hat{F}B = 22.6°$ (3 s.f.)

www.thrivingscholars.com



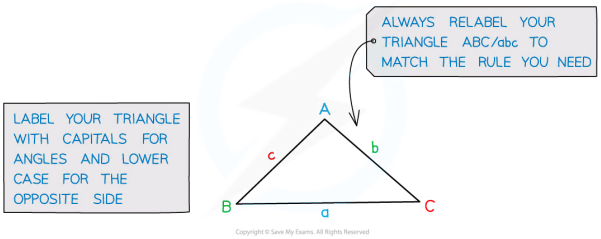
# Sine Rule

## What is the sine rule?

- The sine rule allows us to find missing side lengths or angles in **non-right-angled triangles**
- It states that for any triangle with angles $A$, $B$ and $C$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

  - Where
    - $a$ is the side **opposite** angle $A$
    - $b$ is the side **opposite** angle $B$
    - $c$ is the side **opposite** angle $C$
- This formula **is in the formula booklet**, you do not need to remember it
- Sin 90° = 1 so if one of the angles is 90° this becomes SOH from **SOHCAHTOA**




## How can we use the sine rule to find missing side lengths or angles?

- The sine rule can be used when you have any opposite pairs of sides and angles
- Always **start by labelling your triangle** with the angles and sides
  - Remember the sides with the lower-case letters are **opposite** the angles with the equivalent upper-case letters
- Use the formula in the formula booklet to find the **length of a side**
- To find a missing angle you can rearrange the formula and use the form

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

  - This is **not in the formula booklet** but can easily be found by rearranging the one given

- Substitute the values you have into the formula and solve

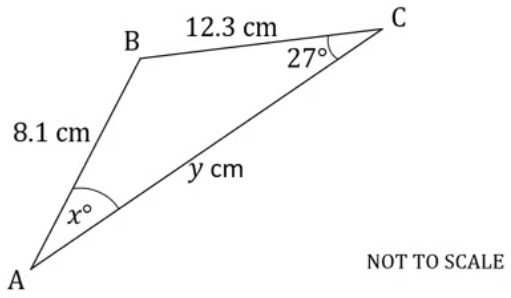
### Exam Tip

- If you're using a calculator make sure that it is in the correct mode (degrees/radians)
- Remember to give your answers as exact values if you are asked too

www.thrivingscholars.com

## Worked Example

The following diagram shows triangle ABC. $AB = 8.1$ cm, $AC = 12.3$ cm, $B\widehat{C}A = 27°$.

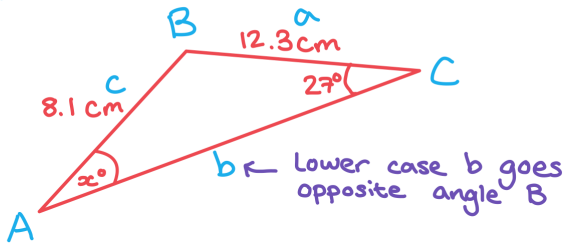



Use the sine rule to calculate the value of:

i)

$x$,

Sketch the diagram and label the sides:




Using the sine rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ ← We are looking for an angle so this version is easier.

$$\frac{\sin x}{12.3} = \frac{\sin 27}{8.1}$$

$$\sin x = \frac{12.3 \sin 27}{8.1}$$

$$x = \sin^{-1}\left(\frac{12.3 \sin 27}{8.1}\right)$$

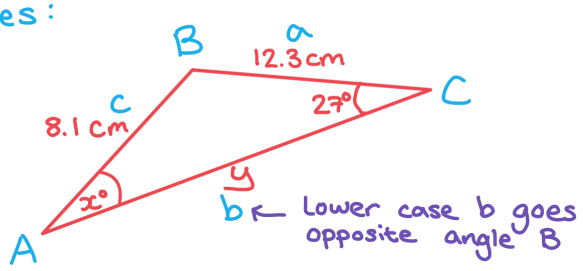
$$x = 43.6° \text{ (3s.f.)}$$

ii)

www.thrivingscholars.com

$y$.

Sketch the diagram and label the sides:




Find $A\hat{B}C$: $180 - (27 + 43.582...)$

$A\hat{B}C = 109.417...$

Using the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ ← We are looking for a side so this version is easier.

$$\frac{y}{\sin(109.417...)} = \frac{8.1}{\sin 27}$$

$$y = \frac{8.1\sin(109.417...)}{\sin 27}$$

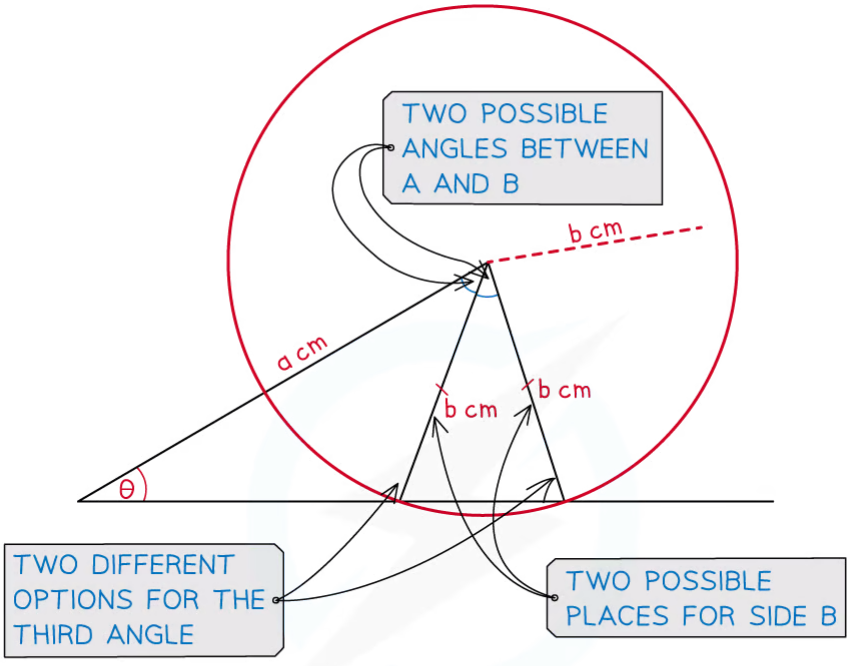
$y = 16.8\text{cm}$ (3 s.f.)

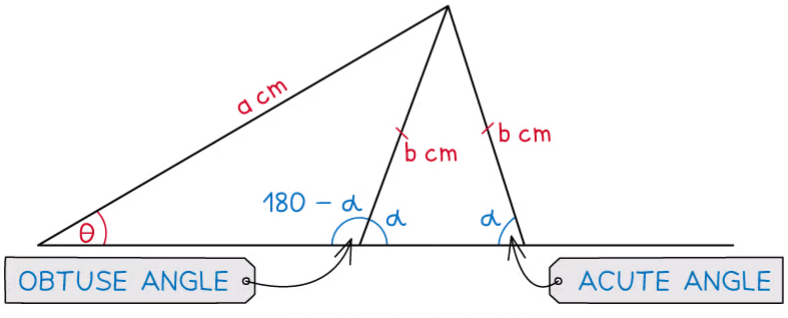
www.thrivingscholars.com

# Ambiguous Sine Rule

## What is the ambiguous case of the sine rule?

- If the sine rule is used in a triangle **given two sides and an angle which is not the angle between them** there **may** be more than one possible triangle which could be drawn
- The side **opposite** the given angle could be in two possible positions
- This will create two possible values for each of the missing angles and two possible lengths for the missing side
- The two angles found **opposite** the given side (not the ambiguous side) will **add up to 180°**
  - In IB the question will usually tell you whether the angle you are looking for is **acute** or **obtuse**
  - The sine rule will always give you the acute option but you can **subtract from 180°** to find the obtuse angle
  - Sometimes the obtuse angle will not be valid
    - It could cause the sum of the three interior angles of the triangle to exceed 180°




Copyright © Save My Exams. All Rights Reserved

www.thrivingscholars.com

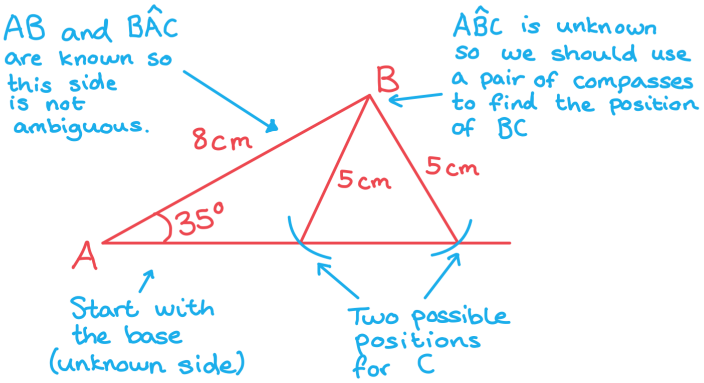
## Exam Tip

- Make sure that you are clear which of the two answers is the one that is required and make sure that you communicate this clearly to the examiner by writing it on the answer line!

## Worked Example

Given triangle $ABC$, $AB = 8$ cm, $BC = 5$ cm, $B\widehat{A}C = 35°$. Find the two possible options for $A\widehat{C}B$, giving both answers to 1 decimal place.

There are two ways triangle ABC can be drawn:




Find $A\hat{C}B$: $\frac{\sin 35°}{5} = \frac{\sin C}{8}$

$C = \sin^{-1}\left(\frac{8 \sin 35°}{5}\right)$

$= 66.59...$

$A\hat{C}B = 66.6°$ or $113.4°$ (1dp)

www.thrivingscholars.com

# Cosine Rule

## What is the cosine rule?

- The cosine rule allows us to find missing side lengths or angles in **non-right-angled triangles**
- It states that for any triangle

$$c^2 = a^2 + b^2 - 2ab\cos C \; ; \; \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

  - Where
    - c is the side **opposite** angle C
    - *a* and *b* are the other two sides
- Both of these formulae **are in the formula booklet**, you do not need to remember them
  - The first version is used to find a missing side
  - The second version is a rearrangement of this and can be used to find a missing angle
- Cos 90° = 0 so if C = 90° this becomes **Pythagoras' Theorem**

## How can we use the cosine rule to find missing side lengths or angles?

- The cosine rule can be used when you have two sides and the angle between them or all three sides
- Always **start by labelling your triangle** with the angles and sides
  - Remember the sides with the lower-case letters are **opposite** the angles with the equivalent upper-case letters
- Use the formula $c^2 = a^2 + b^2 - 2ab\cos C$ to find an unknown side
- Use the formula $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ to find an unknown angle
  - C is the angle **between** sides *a* and *b*
- Substitute the values you have into the formula and solve

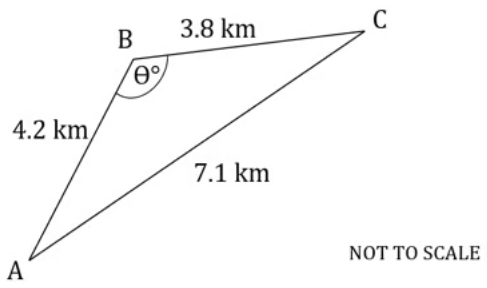
### Exam Tip

- If you're using a calculator make sure that it is in the correct mode (degrees/radians)
- Remember to give your answers as exact values if you are asked too

www.thrivingscholars.com

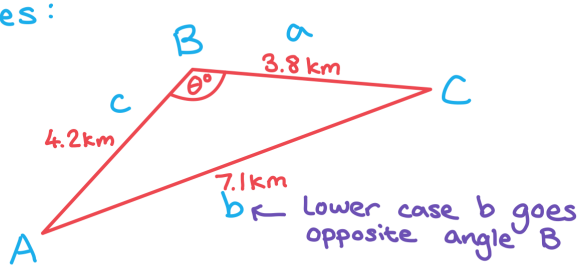
## Worked Example

The following diagram shows triangle $ABC$. $AB = 4.2$ km, $BC = 3.8$ km, $AC = 7.1$ km.




Calculate the value of $A\hat{B}C$.

Sketch the diagram and label the sides:




← Lower case b goes opposite angle B

Using the cosine rule:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

← We are looking for an angle so this version is easier.

$$\cos\theta = \frac{4.2^2 + 3.8^2 - 7.1^2}{2(4.2)(3.8)}$$

$$\theta = \cos^{-1}\left(\frac{4.2^2 + 3.8^2 - 7.1^2}{2(4.2)(3.8)}\right)$$

$$\theta = 128^\circ \text{ (3 s.f.)}$$

www.thrivingscholars.com



# Area of a Triangle

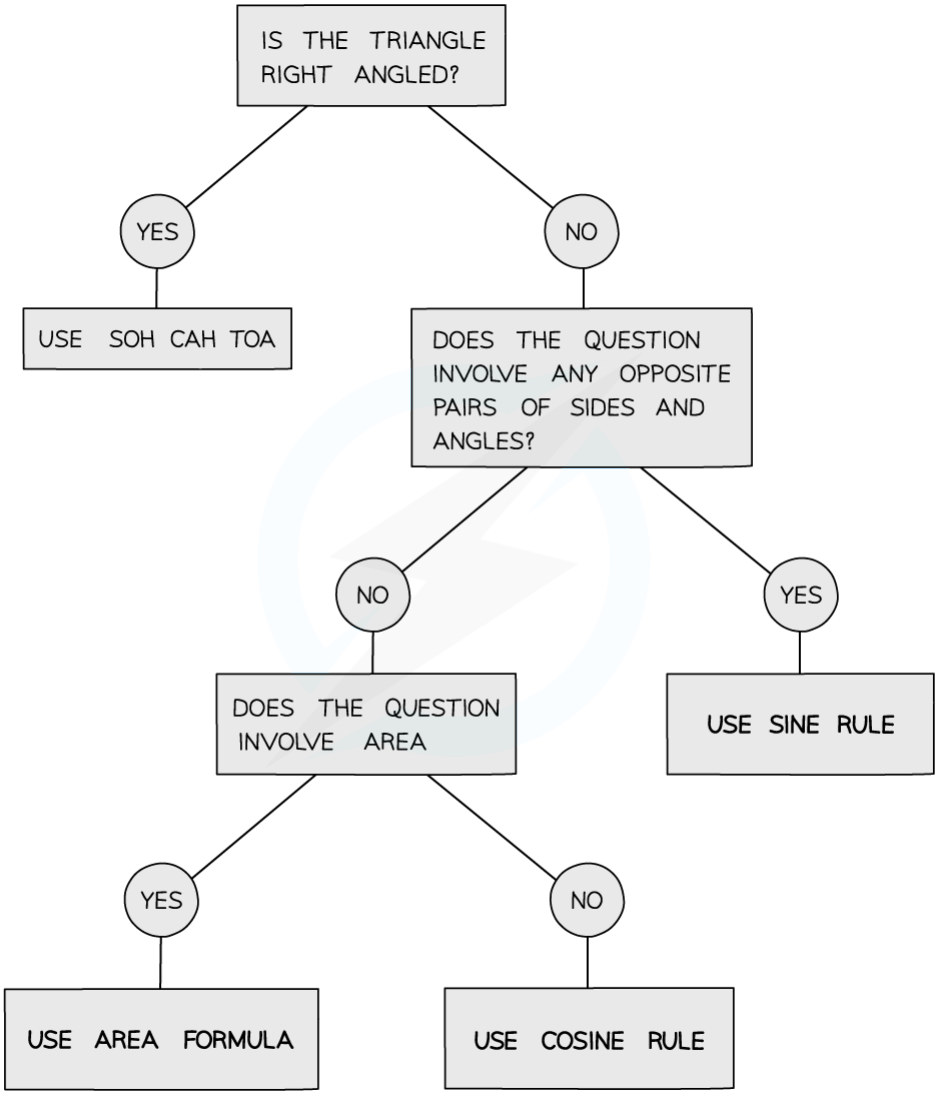
## How do I find the area of a non-right triangle?

- The area of **any triangle** can be found using the formula

$$A = \frac{1}{2}ab\sin C$$

  - Where $C$ is the angle between sides $a$ and $b$
  - This formula **is in the formula booklet**, you do not need to remember it
- Be careful to label your triangle correctly so that $C$ is always the angle **between** the two sides
- Sin 90° = 1 so if C = 90° this becomes Area = ½ × base × height

www.thrivingscholars.com




Copyright © Save My Exams. All Rights Reserved

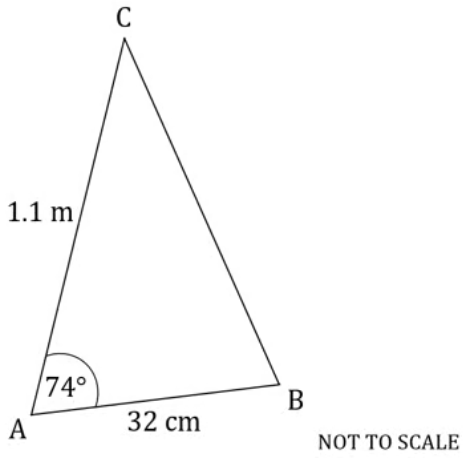
## Exam Tip

- If you're using a calculator make sure that it is in the correct mode (degrees/radians)
- Remember to give your answers as exact values if you are asked too

www.thrivingscholars.com

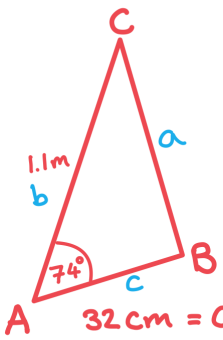
## Worked Example

The following diagram shows triangle $ABC$. $AB = 32$ cm, $AC = 1.1$ m, $B\widehat{A}C = 74°$.




Calculate the area of triangle .

Label the sides of the triangle:




32 cm = 0.32 m

↖ change all units to be the same

Area of a triangle: $A = \frac{1}{2}ab\sin C$

$A = \frac{1}{2}(1.1)(0.32)\sin 74°$

$A = 0.169\ m^2$

www.thrivingscholars.com



# Bearings

## What are bearings?

- **Bearings** are a way of describing and using **directions** as **angles**
- They are specifically defined for use in navigation because they give a precise **location** and/or **direction**

## How are bearings defined?

- There are **three rules** which must be followed every time a bearing is defined
  - They are **measured** from the **North** direction
    - An arrow showing the North line should be included on the diagram
  - They are **measured clockwise**
  - The angle is always written in **3 figures**
    - If the angle is less than 100° the first digit will be a zero

## What are bearings used for?

- Bearings questions will normally involve the use of Pythagoras or trigonometry to find missing distances (lengths) and directions (angles) within navigation questions
  - You should always **draw a diagram**
- There may be a scale given or you may need to consider using a scale
  - However normally in IB you will be using triangle calculations to find the distances
- Some questions may also involve the use of angle facts to find the missing directions
- To answer a question involving **drawing bearings** the following steps may help:
  - STEP 1: Draw a diagram adding in any points and distances you have been given
  - STEP 2: Draw a North line (arrow pointing vertically up) at the point you wish to measure the bearing **from**
    - If you are given the bearing **from A to B** draw the North line at **A**
  - STEP 3: Measure the angle of the bearing given **from the North line** in the **clockwise direction**
  - STEP 4: Draw a line and add the point B at the given distance
- You will likely then need to use trigonometry to calculate the shortest distance or another given distance

### Exam Tip

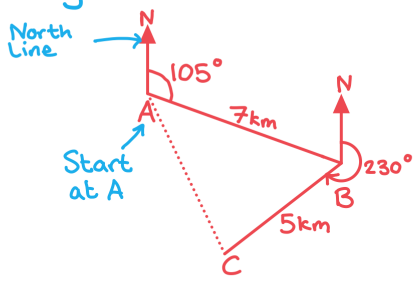
- **Always** draw a big, clear diagram and annotate it, be especially careful to label the angles in the correct places!

www.thrivingscholars.com

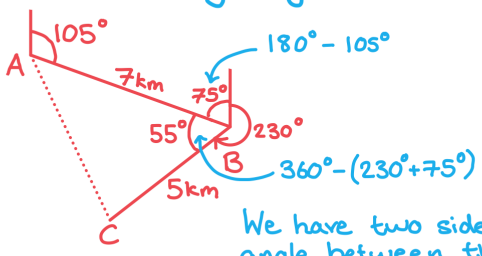
## Worked Example

The point B is 7 km from A on a bearing of 105°. The distance from B to C is 5 km and the bearing from B to C is 230°. Find the distance from A to C.

Always start with a diagram:




Fill in the angles you can on the diagram




We have two sides and the angle between them so we can use the cosine rule for the third side

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$AC^2 = 7^2 + 5^2 - 2(7)(5)\cos(55°)$$

$$= 33.849...$$

$$AC = 5.82 \text{ km (3 s.f.)}$$

www.thrivingscholars.com

# Elevation & Depression

## What are the angles of elevation and depression?

- If a person looks at an **object** that is not on the same horizontal line as their eye-level they will be looking at either an angle of **elevation** or **depression**
  - If a person looks **up** at an object their line of sight will be at an **angle of elevation** with the horizontal
  - If a person looks **down** at an object their line of sight will be at an **angle of depression** with the horizontal
- Angles of elevation and depression are measured **from the horizontal**
- **Right-angled trigonometry** can be used to find an angle of elevation or depression or a missing distance
- Tan is often used in real-life scenarios with angles of elevation and depression
  - For example if we know the distance we are standing from a tree and the angle of elevation of the top of the tree we can use Tan to find its height
  - Or if we are looking at a boat at to sea and we know our height above sea level and the angle of depression we can find how far away the boat is

### Exam Tip

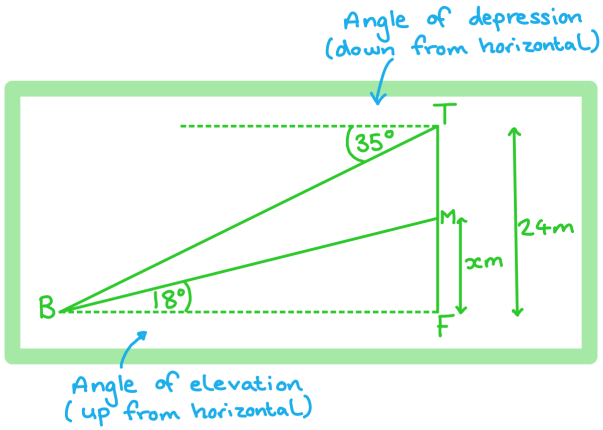
- It may be useful to draw more than one diagram if the triangles that you are interested in overlap one another

www.thrivingscholars.com

## Worked Example

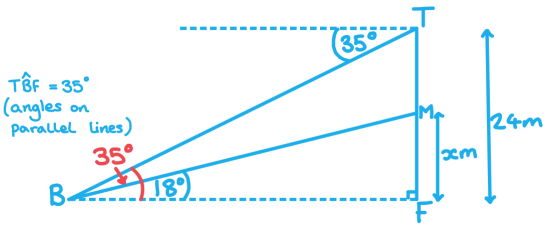
A cliff is perpendicular to the sea and the top of the cliff stands 24 m above the level of the sea. The angle of depression from the cliff to a boat at sea is 35°. At a point $x$ m up the cliff is a flag marker and the angle of elevation from the boat to the flag marker is 18°.

a)
Draw and label a diagram to show the top of the cliff, T, the foot of the cliff, F, the flag marker, M, and the boat, B, labelling all the angles and distances given above.

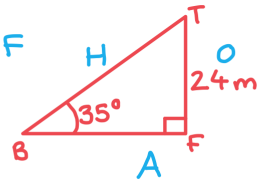



b)
Find the distance from the boat to the foot of the cliff.

www.thrivingscholars.com




Consider triangle TBF




SOHCAHTOA

we have opposite and adjacent so use Tan

$$\tan 35^\circ = \frac{24}{BF}$$

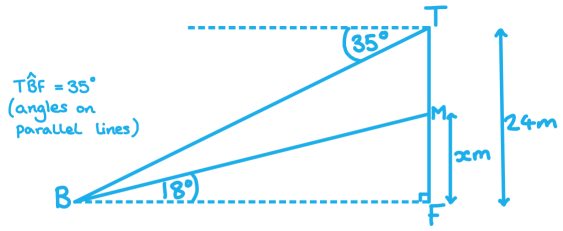
$$BF = \frac{24}{\tan 35^\circ}$$

**BF = 34.3m (3s.f.)**

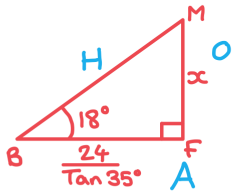
c)

Find the value of $x$.

www.thrivingscholars.com




Consider triangle FBM




SOHCAHTOA

we have opposite and adjacent so use Tan

$$\tan 18° = \frac{x}{\left(\frac{24}{\tan 35°}\right)}$$

$$x = \tan 18° \times \left(\frac{24}{\tan 35°}\right)$$

$$= 11.136...$$

$$x = 11.1\text{ m (3s.f.)}$$

www.thrivingscholars.com

# Constructing Diagrams

## What diagrams will I need to construct?

- In IB you will be expected to construct diagrams based on information given
- The information will include **compass directions, bearings, angles**
  - Look out for the **plane** the diagram should be drawn in
  - It will either be **horizontal** (something occurring at sea or on the ground)
  - Or it will be **vertical** (Including height)
- Work through the statements given in the instructions systematically

## What do I need to know?

- Your diagrams will be sketches, they do not need to be accurate or to scale
  - However the more accurate your diagram is the easier it is to work with
- Read the full set of instructions once before beginning to draw the diagram so you have a rough idea of where each object is
- Make sure you know your **compass directions**
  - **Due east** means on a **bearing of 090°**
    - Draw the line directly to the right
  - **Due south** means on a **bearing of 180°**
    - Draw the line vertically downwards
  - **Due west** means on a **bearing of 270°**
    - Draw the line directly to the left
  - **Due north** means on a **bearing of 360° (or 000°)**
    - Draw the line vertically upwards
- Using the above bearings for compass directions will help you to estimate angles for other bearings on your diagram

### Exam Tip

- Draw your diagrams in pencil so that you can easily erase any errors

www.thrivingscholars.com

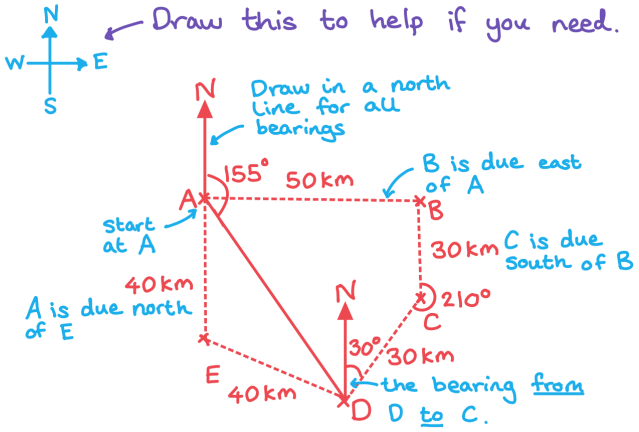
## Worked Example

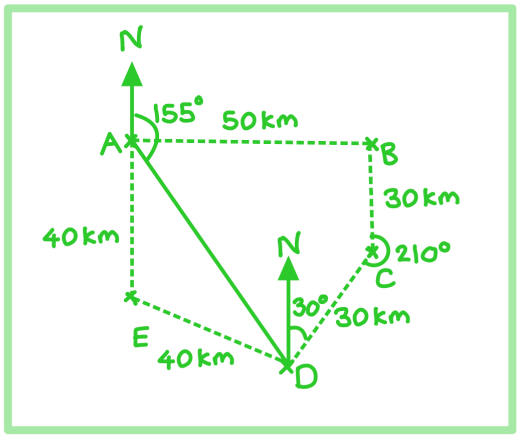
A city at B is due east of a city at A and A is due north of a city at E. A city at C is due south of B.

The bearing from A to D is 155° and the bearing from D to C is 30°.

The distance AB = 50 km, the distances BC = CD = 30 km and the distances DE = AE = 40 km.

Draw and label a diagram to show the cities A, B, C, D and E and clearly mark the bearings and distances given.



www.thrivingscholars.com

# 3.4 Trigonometry

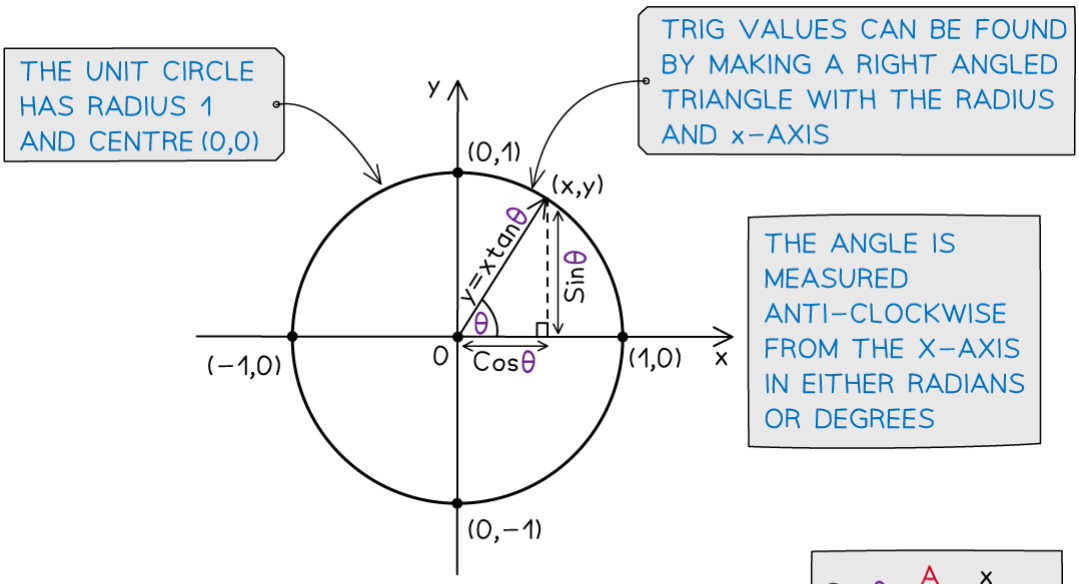
## 3.4.1 The Unit Circle

### Defining Sin, Cos and Tan

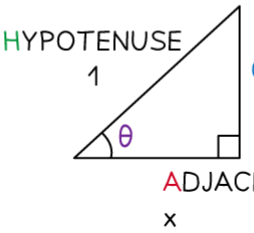
### What is the unit circle?

- The unit circle is a circle with radius 1 and centre (0, 0)
- Angles are always measured from the positive x-axis and turn:
  - **anticlockwise** for **positive** angles
  - **clockwise** for **negative** angles
- It can be used to calculate trig values as a coordinate point (x, y) on the circle
  - Trig values can be found by making a right triangle with the radius as the hypotenuse
  - Where $\theta$ is the angle measured anticlockwise from the positive $x$-axis
  - The $x$-axis will always be adjacent to the angle, $\theta$
- SOHCAHTOA can be used to find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ easily
- As the radius is 1 unit
  - the ***x* coordinate** gives the value of **$\cos\theta$**
  - the ***y* coordinate** gives the value of **$\sin\theta$**
- As the origin is one of the end points - dividing the y coordinate by the x coordinate gives the gradient
  - the **gradient** of the line gives the value of **$\tan\theta$**
- It allows us to calculate sin, cos and tan for angles greater than 90° ($\frac{\pi}{2}$ rad)

www.thrivingscholars.com




ANY POINT (x,y) ON THE UNIT CIRCLE CAN BE FOUND USING $(\cos\theta, \sin\theta)$



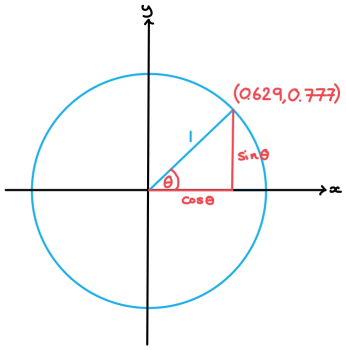

$$\cos\theta = \frac{A}{H} = \frac{x}{1} = x$$

$$\sin\theta = \frac{O}{H} = \frac{y}{1} = y$$

$$\tan\theta = \frac{O}{A} = \frac{y}{x}$$

## Worked Example

The coordinates of a point on a unit circle, to 3 significant figures, are (0.629, 0.777). Find $\theta°$ to the nearest degree.




We know $(x,y) = (\cos\theta, \sin\theta)$

So,

$\cos\theta = 0.629$

$\sin\theta = 0.777$

Using either ratio:

$\theta = \cos^{-1}(0.629)$

$= 51.023\ldots$

$\theta = 51°$ (nearest degree)

www.thrivingscholars.com

# Using The Unit Circle

## What are the properties of the unit circle?

- The unit circle can be split into four **quadrants** at every 90° ($\frac{\pi}{2}$ rad)
  - The first quadrant is for angles between 0 and 90°
    - **A**ll three of Sin$\theta$, Cos$\theta$ and Tan$\theta$ are positive in this quadrant
  - The second quadrant is for angles between 90° and 180° ($\frac{\pi}{2}$ rad and $\pi$ rad)
    - **S**in$\theta$ is positive in this quadrant
  - The third quadrant is for angles between 180° and 270° ($\pi$ rad and $\frac{3\pi}{2}$)
    - **T**an$\theta$ is positive in this quadrant
  - The fourth quadrant is for angles between 270° and 360° ($\frac{3\pi}{2}$ rad and $2\pi$)
    - **C**os$\theta$ is positive in this quadrant
  - Starting from the **fourth** quadrant (on the bottom right) and working anti-clockwise the positive trig functions spell out **CAST**
    - This is why it is often thought of as the **CAST** diagram
    - You may have your own way of remembering this
    - A popular one starting from the first quadrant is **All Students Take Calculus**
  - To help picture this better try sketching all three trig graphs on one set of axes and look at which graphs are positive in each 90° section

## How is the unit circle used to find secondary solutions?

- Trigonometric functions have more than one input to each output
  - For example $\sin 30° = \sin 150° = 0.5$
  - This means that trigonometric equations have more than one solution
  - For example both 30° and 150° satisfy the equation $\sin x = 0.5$
- The unit circle can be used to find all solutions to trigonometric equations in a given interval
  - Your calculator will only give you the first solution to a problem such as $x = \sin^{-1}(0.5)$
    - This solution is called the **primary value**
  - However, due to the **periodic** nature of the trig functions there could be an infinite number of solutions
    - Further solutions are called the **secondary values**
  - This is why you will be given a **domain** in which your solutions should be found
    - This could either be in degrees or in radians
    - If you see $\pi$ or some multiple of $\pi$ then you must work in radians
- The following steps may help you use the unit circle to find **secondary values**

STEP 1: Draw the angle into the first quadrant using the $x$ or $y$ coordinate to help you

- If you are working with $\sin x = k$, draw the line from the origin to the circumference of the circle at the point where the **y coordinate** is $k$
- If you are working with $\cos x = k$, draw the line from the origin to the circumference of the circle at the point where the **x coordinate** is $k$
- If you are working with $\tan x = k$, draw the line from the origin to the circumference of the circle such that the gradient of the line is $k$
  - This will give you the angle which should be measured from the **positive x-axis…**

www.thrivingscholars.com



    - … anticlockwise for a positive angle
    - … clockwise for a negative angle

STEP 2: Draw the radius in the other quadrant which has the same...

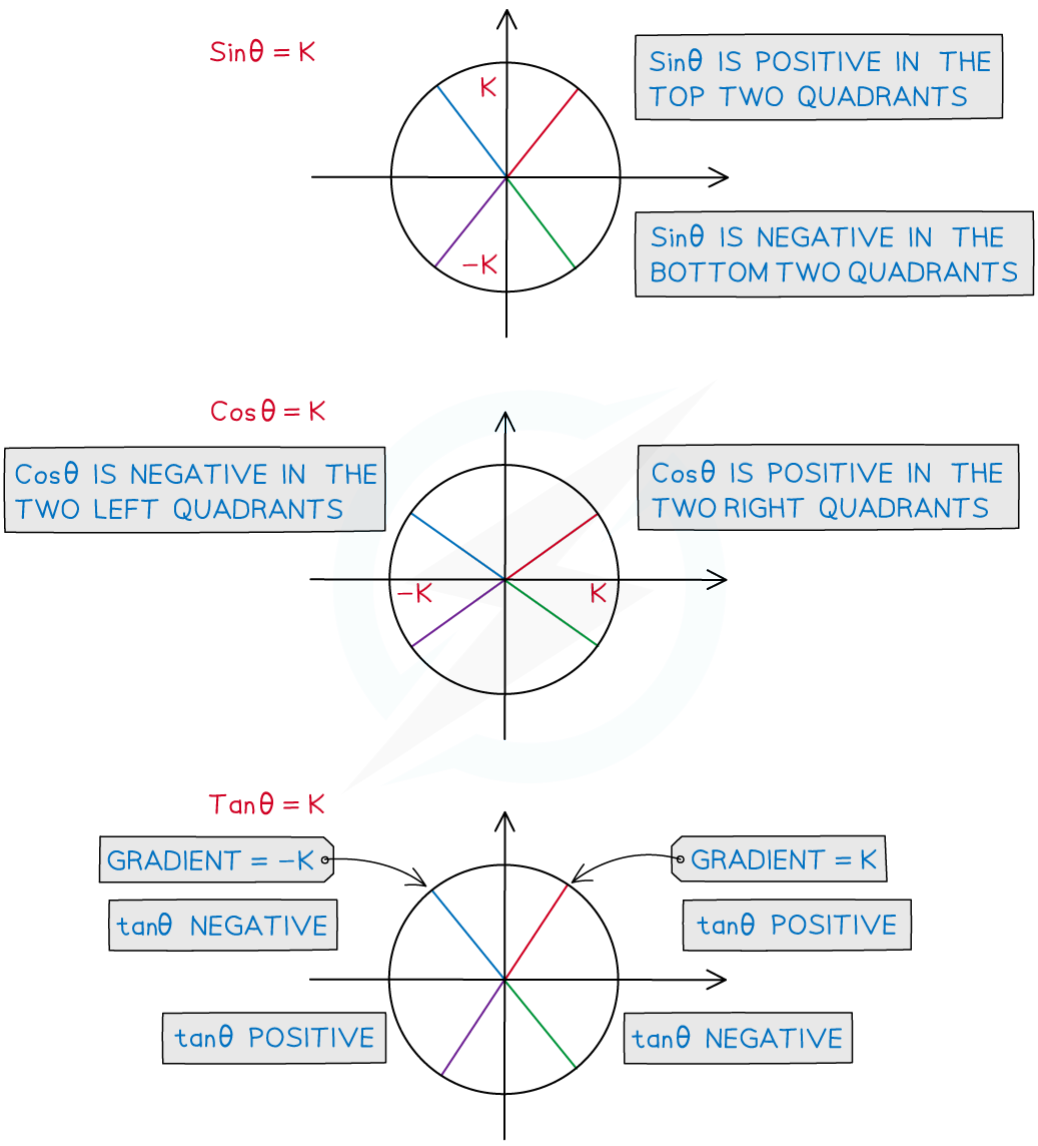
- ... $x$-coordinate if solving $\cos x = k$
    - This will be the quadrant which is vertical to the original quadrant
- ... $y$-coordinate if solving $\sin x = k$
    - This will be the quadrant which is horizontal to the original quadrant
- ... gradient if solving $\tan x = k$
    - This will be the quadrant diagonal to the original quadrant

STEP 3: Work out the size of the second angle, measuring from the positive x-axis

- … anticlockwise for a positive angle
- … clockwise for a negative angle
    - You should look at the given range of values to decide whether you need the negative or positive angle

STEP 4: Add or subtract either 360° or $2\pi$ radians to both values until you have all solutions in the required range

www.thrivingscholars.com




Copyright © Save My Exams. All Rights Reserved

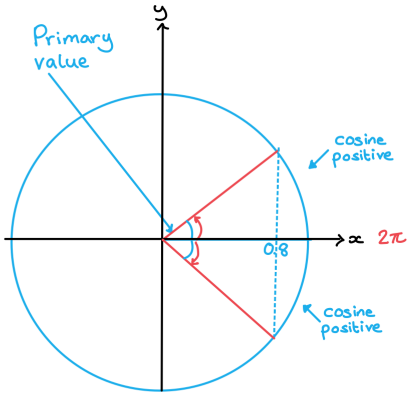
## Exam Tip

- Being able to sketch out the unit circle and remembering CAST can help you to find all solutions to a problem in an exam question

www.thrivingscholars.com

## Worked Example

Given that one solution of $\cos\theta = 0.8$ is $\theta = 0.6435$ radians correct to 4 decimal places, find all other solutions in the range $-2\pi \leq \theta \leq 2\pi$. Give your answers correct to 3 significant figures.




Cosine is positive in the first and fourth quadrants so draw the angle from the horizontal axis in both quadrants.

Primary value $= 0.6435$

Using diagram, secondary value $= -0.6435$

Therefore all values are: $0.6435 \pm 2\pi n$

and $-0.6435 \pm 2\pi n$

Within given domain: $-2\pi \leq \theta \leq 2\pi$

$$\theta = -5.64^c,\ -0.644^c,\ 0.644^c,\ 5.64^c$$

www.thrivingscholars.com



# Trigonometry Exact Values

## What are exact values in trigonometry?

- For certain angles the values of sin θ, cos θ and tan θ can be written **exactly**
  - This means using fractions and surds
  - You should be familiar with these values and be able to derive the values using geometry
- You are expected to know the exact values of sin, cos and tan for angles of 0°, 30°, 45°, 60°, 90°, 180° and their multiples
  - In **radians** this is $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ and their multiples
- The exact values you are expected to know are here:

| DEGREES | 0° | 30° | 45° | 60° | 90° | 180° | 360° |
|---|---|---|---|---|---|---|---|
| RADIANS | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $2\pi$ |
| sin | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 | 0 | 0 |
| cos | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 | −1 | 1 |
| tan | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | UNDEFINED | 0 | 0 |

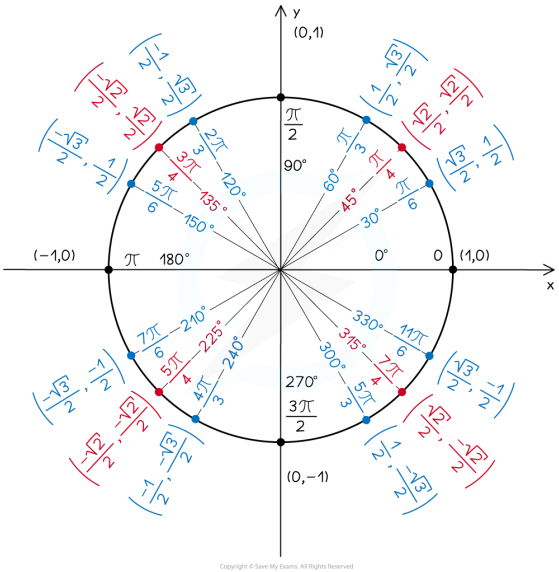
Copyright © Save My Exams. All Rights Reserved

## How do I find the exact values of other angles?

- The exact values for sin and cos can be seen on the **unit circle** as the *y* and *x* coordinates respectively
  - If using the coordinates on the unit circle to memorise the exact values, remember that cos comes **before** sin
- The **unit circle** can also be used to find exact values of other angles using symmetry
- If you know the exact value for an angle in the first quadrant you can draw the same angle from the x-axis in any other quadrant to find other angles
- Remember that the angles are **measured anticlockwise** from the positive x-axis
- For example if you know that the exact value for is 0.5
  - draw the angle 30° from the horizontal in the three other quadrants
  - measuring from the positive x-axis you have the angles of 150°, 210° and 330°
    - sin is positive in the second quadrant so sin150° = 0.5
    - sin is negative in the third quadrant so sin210° = - 0.5

www.thrivingscholars.com

    - sin is negative in the fourth quadrant so sin330° = - 0.5
- It is also possible to find the **negative** angles by measuring **clockwise** from the positive x-axis
    - draw the angle 30° from the horizontal in the three other quadrants
    - measuring **clockwise** from the positive x-axis you have the angles of -30°, -150°, -210° and -330°
        - sin is negative in the fourth quadrant so sin(-30°) = - 0.5
        - sin is negative in the third quadrant so sin(-150°) = - 0.5
        - sin is positive in the second quadrant so sin(-210°) = 0.5
        - sin is positive in the fourth quadrant so sin(-330°) = 0.5

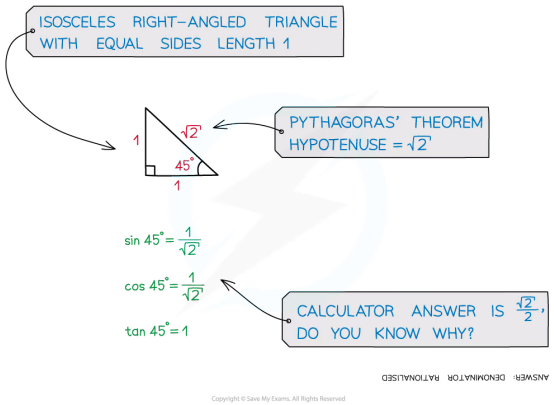



## How are exact values in trigonometry derived?

- There are two special **right-triangles** that can be used to derive all of the exact values you need to know
- Consider a **right-triangle** with a hypotenuse of 2 units and a shorter side length of 1 unit
    - Using Pythagoras' theorem the third side will be $\sqrt{3}$
    - The angles will be $\frac{\pi}{2}$ radians (90°), $\frac{\pi}{3}$ radians (60°) and $\frac{\pi}{6}$ radians (30°)
    - Using SOHCAHTOA gives...
        - Sin $\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ Sin $\frac{\pi}{6} = \frac{1}{2}$
        - Cos $\frac{\pi}{3} = \frac{1}{2}$ Cos $\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
        - Tan $\frac{\pi}{3} = \sqrt{3}$ Tan $\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
- Consider an **isosceles triangle** with two equal side lengths (the opposite and adjacent) of 1 unit
    - Using Pythagoras' theorem it will have a hypotenuse of $\sqrt{2}$
    - The two equal angles will be $\frac{\pi}{4}$ radians (45°)

www.thrivingscholars.com

- Using SOHCAHTOA gives...
  - Sin $\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
  - Cos $\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
  - Tan $\frac{\pi}{4} = 1$




Copyright © Save My Exams. All Rights Reserved

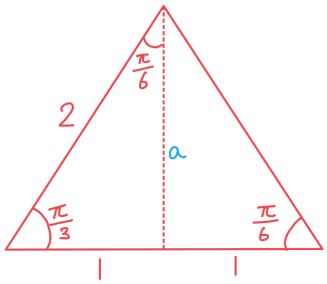
## Exam Tip

- You will be expected to be comfortable using exact trig values for certain angles but it can be easy to muddle them up if you just try to remember them from a list, sketch the triangles and trig graphs on your paper so that you can use them as many times as you need to during the exam!
  - sketch the triangles for the key angles $45° / \frac{\pi}{4}, 30° / \frac{\pi}{6}, 60° / \frac{\pi}{3}$
  - sketch the trig graphs for the key angles $0°, 90° / \frac{\pi}{2}, 180° / \pi, 270° / \frac{3\pi}{2}, 360° / 2\pi$

www.thrivingscholars.com

## Worked Example

Using an equilateral triangle of side length 2 units, derive the exact values for the sine, cosine and tangent of $\frac{\pi}{6}$ and $\frac{\pi}{3}$.




Use Pythagoras' Theorem to find a:

$a^2 = \sqrt{2^2 - 1^2}$

$= \sqrt{3}$

Using SOHCAH TOA:

| | |
|---|---|
| $\sin(\frac{\pi}{6}) = \frac{1}{2}$ | $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ |
| $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$ | $\cos(\frac{\pi}{3}) = \frac{1}{2}$ |
| $\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}$ | $\tan(\frac{\pi}{3}) = \frac{\sqrt{3}}{1} = \sqrt{3}$ |

www.thrivingscholars.com



# 3.5 Trigonometric Functions & Graphs

## 3.5.1 Graphs of Trigonometric Functions

### Graphs of Trigonometric Functions

### What are the graphs of trigonometric functions?

- The trigonometric functions sin, cos and tan all have special **periodic graphs**
- You'll need to know their properties and how to sketch them for a given domain in either **degrees** or **radians**
- Sketching the trigonometric graphs can help to
  - Solve trigonometric equations and find all solutions
  - Understand transformations of trigonometric functions

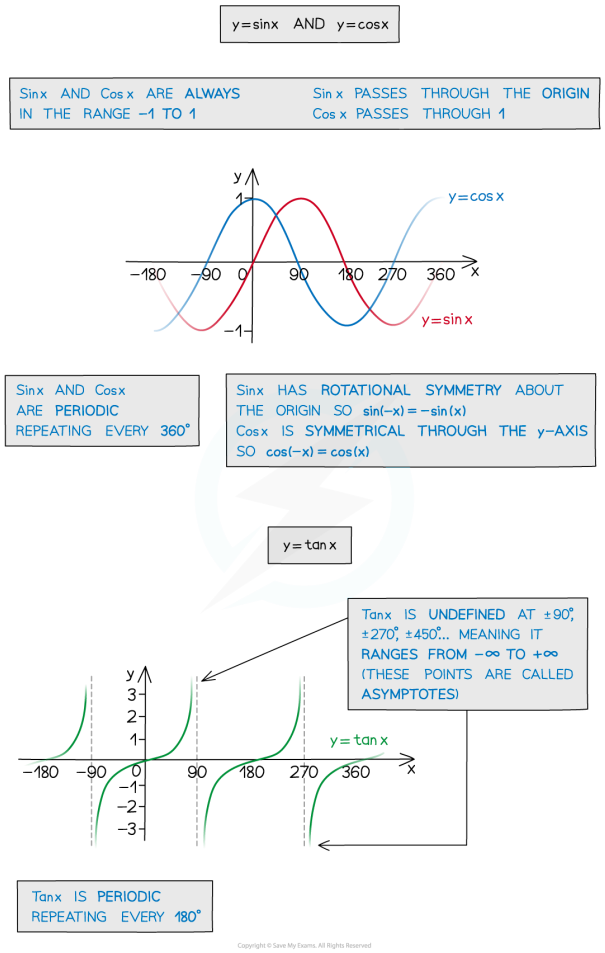
### What are the properties of the graphs of sin x and cos x?

- The graphs of sin x and cos x are both **periodic**
  - They **repeat every 360°** ($2\pi$ radians)
  - The angle will always be on the x-axis
    - Either in degrees or radians
- The graphs of sin x and cos x are always in the **range $-1 \leq y \leq 1$**
  - **Domain:** $\{x \mid x \in \mathbb{R}\}$
  - **Range:** $\{y \mid -1 \leq y \leq 1\}$
  - The graphs of sin x and cos x are identical however one is a **translation** of the other
    - sin x passes through the origin
    - cos x passes through $(0, 1)$
- The **amplitude** of the graphs of sin x and cos x is 1

### What are the properties of the graph of tan x?

- The graph of tan x is **periodic**
  - It **repeats every 180°** ($\pi$ radians)
  - The angle will always be on the x-axis
    - Either in degrees or radians
- The graph of tan x is **undefined** at the points $\pm 90°, \pm 270°$ etc
  - There are **asymptotes** at these points on the graph
  - In radians this is at the points $\pm \frac{\pi}{2}, \pm \frac{3\pi}{2}$ etc
- The range of the graph of tan x is
  - **Domain:** $\left\{x \mid x \neq \frac{\pi}{2} + k\pi, \ k \in \mathbb{R}\right\}$
  - **Range:** $\{y \mid y \in \mathbb{R}\}$

www.thrivingscholars.com




## How do I sketch trigonometric graphs?

- You may need to sketch a trigonometric graph so you will need to remember the key features of each one
- The following steps may help you sketch a trigonometric graph
  - STEP 1: Check whether you should be working in degrees or radians
    - You should check the domain given for this
    - If you see π in the given domain then you should work in radians
  - STEP 2: Label the x-axis in multiples of 90°
    - This will be multiples of $\frac{\pi}{2}$ if you are working in radians
    - Make sure you cover the whole domain on the x-axis
  - STEP 3: Label the y-axis
    - The range for the y-axis will be $-1 \leq y \leq 1$ for sin or cos
    - For tan you will not need any specific points on the y-axis
  - STEP 4: Draw the graph
    - Knowing exact values will help with this, such as remembering that sin(0) = 0 and cos(0) = 1
    - Mark the important points on the axis first
    - If you are drawing the graph of tan x put the asymptotes in first

www.thrivingscholars.com

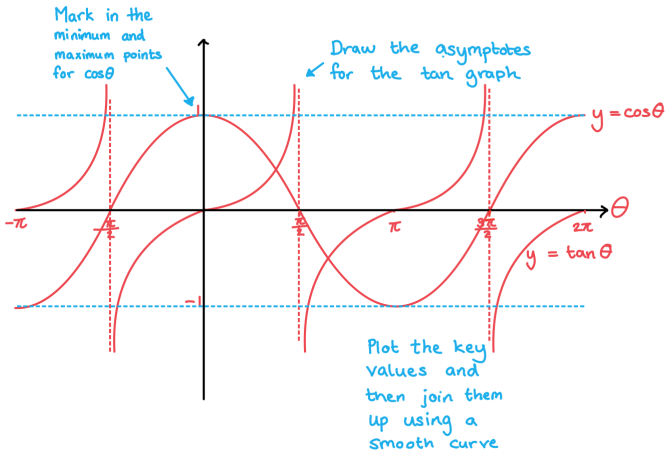
- If you are drawing sin x or cos x mark in where the maximum and minimum points will be
- Try to keep the symmetry and rotational symmetry as you sketch, as this will help when using the graph to find solutions

## Exam Tip

- Sketch all three trig graphs on your exam paper so you can refer to them as many times as you need to!

## Worked Example

Sketch the graphs of $y = \cos\theta$ and $y = \tan\theta$ on the same set of axes in the interval $-\pi \leq \theta \leq 2\pi$. Clearly mark the key features of both graphs.



www.thrivingscholars.com

# Using Trigonometric Graphs

## How can I use a trigonometric graph to find extra solutions?

- Your calculator will only give you the first solution to a problem such as $\sin^{-1}(0.5)$
  - This solution is called the **primary value**
- However, due to the **periodic** nature of the trig functions there could be an infinite number of solutions
  - Further solutions are called the **secondary values**
- This is why you will be given a **domain** (interval) in which your solutions should be found
  - This could either be in degrees or in radians
    - If you see π or some multiple of π then you must work in radians
- The following steps will help you use the **trigonometric graphs** to find **secondary values**
  - STEP 1: Sketch the graph for the given function and interval
    - Check whether you should be working in degrees or radians and label the axes with the key values
  - STEP 2: Draw a horizontal line going through the y-axis at the point you are trying to find the values for
    - For example if you are looking for the solutions to $\sin^{-1}(-0.5)$ then draw the horizontal line going through the y-axis at -0.5
    - The number of times this line cuts the graph is the number of solutions within the given interval
  - STEP 3: Find the primary value and mark it on the graph
    - This will either be an exact value and you should know it
    - Or you will be able to use your calculator to find it
  - STEP 4: Use the symmetry of the graph to find all the solutions in the interval by adding or subtracting from the key values on the graph

## What patterns can be seen from the graphs of trigonometric functions?

- The graph of sin x has rotational symmetry about the origin
  - So $\sin(-x) = -\sin(x)$
  - $\sin(x) = \sin(180° - x)$ or $\sin(\pi - x)$
- The graph of cos x has reflectional symmetry about the y-axis
  - So $\cos(-x) = \cos(x)$
  - $\cos(x) = \cos(360° - x)$ or $\cos(2\pi - x)$
- The graph of tan x repeats every 180° (π radians)
  - So $\tan(x) = \tan(x \pm 180°)$ or $\tan(x \pm \pi)$
- The graphs of sin x and cos x repeat every 360° (2π radians)
  - So $\sin(x) = \sin(x \pm 360°)$ or $\sin(x \pm 2\pi)$
  - $\cos(x) = \cos(x \pm 360°)$ or $\cos(x \pm 2\pi)$

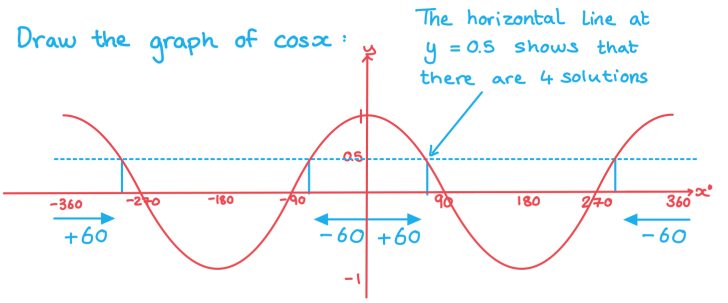
### Exam Tip

- Take care to always check what the **interval** for the angle is that the question is focused on

www.thrivingscholars.com

## Worked Example

One solution to cos x = 0.5 is 60°. Find all the other solutions in the range -360° ≤ x ≤ 360°.




Solutions are : $60°, 360° - 60°, -60°, -360° + 60°$

$$-60°, \quad -300°, \quad 60°, \quad 300°$$

www.thrivingscholars.com





# Transformations of Trigonometric Functions

## What transformations of trigonometric functions do I need to know?

- As with other graphs of functions, trigonometric graphs can be transformed through **translations, stretches** and **reflections**
- **Translations** can be either horizontal (parallel to the x-axis) or vertical (parallel to the y-axis)
  - For the function **y = sin (x)**
    - A **vertical** translation of ***a* units** in the **positive direction** (up) is denoted by **y = sin (x) + *a***
    - A **vertical** translation of ***a* units** in the **negative direction** (down) is denoted by **y = sin (x) – *a***
    - A **horizontal** translation in the **positive direction** (right) is denoted by **y = sin (x – *a*)**
    - A **horizontal** translation in the **negative direction** (left) is denoted by **y = sin (x + *a*)**
- **Stretches** can be either horizontal (parallel to the x-axis) or vertical (parallel to the y-axis)
  - For the function **y = sin (x)**
    - A **vertical** stretch of a **factor *a* units** is denoted by **y = *a* sin (x)**
    - A **horizontal** stretch of a **factor *a* units** is denoted by **y = sin ($\frac{x}{a}$)**
- **Reflections** can be either across the x-axis or across the y-axis
  - For the function **y = sin (x)**
    - A **reflection** across the x-axis is denoted by **y = – sin (x)**
    - A **reflection** across the y-axis is denoted by **y = sin (–x)**

## What combined transformations are there?

- **Stretches** in the horizontal and vertical direction are often combined
- The functions ***a* sin(*bx*)** and ***a* cos(*bx*)** have the following properties:
  - The **amplitude** of the graph is $|a|$
  - The **period** of the graph is $\frac{360}{b}°$ (or $\frac{2\pi}{b}$ rad)
- **Translations** in both directions could also be combined with the stretches
- The functions ***a* sin(*b*(*x* – c )) + *d*** and ***a* cos(*b*(*x* – c )) + *d*** have the following properties:
  - The **amplitude** of the graph is $|a|$
  - The **period** of the graph is $\frac{360}{b}°$ (or $\frac{2\pi}{b}$)
  - The translation in the horizontal direction is c
  - The translation in the vertical direction is *d*
    - *d* represents the **principal axis** (the line that the function fluctuates about)
- The function ***a* tan(*b*(*x* – c )) + *d*** has the following properties:
  - The **amplitude** of the graph does not exist
  - The **period** of the graph is $\frac{180}{b}°$ (or $\frac{2\pi}{b}$)
  - The translation in the horizontal direction is c
  - The translation in the vertical direction (principal axis) is *d*

## How do I sketch transformations of trigonometric functions?

- Sketch the graph of the original function first

www.thrivingscholars.com



- Carry out each transformation separately
  - The **order** in which you carry out the transformations is important
  - Given the form $y = a\sin(b(x - c)) + d$ carry out any **stretches** first, **translations** next and **reflections** last
    - If the function is written in the form $y = a\sin(bx - bc) + d$ factorise out the coefficient of $x$ before carrying out any transformations
  - Use a very light pencil to mark where the graph has moved for each transformation
- It is a good idea to mark in the principal axis the lines corresponding to the maximum and minimum points first
  - The **principal axis** will be the line $y = d$
  - The **maximum points** will be on the line $y = d + a$
  - The **minimum points** will be on the line $y = d - a$
- Sketch in the new transformed graph
- Check it is correct by looking at some key points from the **exact values**



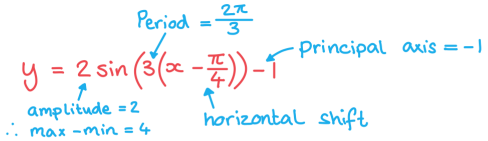
## Exam Tip

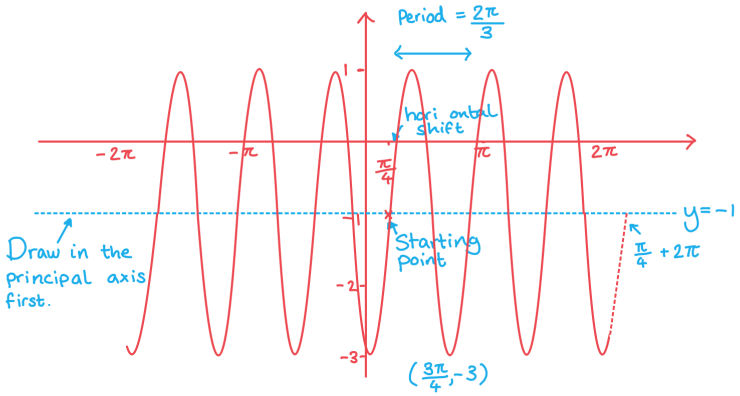
- Be sure to apply transformations in the correct order – applying them in the wrong order can produce an incorrect transformation
- When you sketch a transformed graph, indicate the new coordinates of any points that are marked on the original graph
- Try to indicate the coordinates of points where the transformed graph intersects the coordinate axes (although if you don't have the equation of the original function this may not be possible)
- If the graph has asymptotes, don't forget to sketch the asymptotes of the transformed graph as well

www.thrivingscholars.com

## Worked Example

Sketch the graph of $y = 2\sin\left(3\left(x - \frac{\pi}{4}\right)\right) - 1$ for the interval $-2\pi \le x \le 2\pi$. State the amplitude, period and principal axis of the function.




amplitude : 2
period : $\frac{2\pi}{3}$
principal axis : $y = -1$

www.thrivingscholars.com





# Modelling with Trigonometric Functions

## What can be modelled with trigonometric functions?

- Anything that oscillates (fluctuates periodically) can be modelled using a trigonometric function
  - Normally some transformation of the sine or cosine function
- Examples include:
  - $D(t)$ is the depth of water at a shore $t$ hours after midnight
  - $T(d)$ is the temperature of a city $d$ days after the 1st January
  - $H(t)$ is vertical height above ground of a person $t$ seconds after entering a Ferris wheel
- Notice that the x-axis will not always contain an angle
  - In the examples above time or number of days would be on the x-axis
  - Depth of the water, temperature or vertical height would be on the y-axis

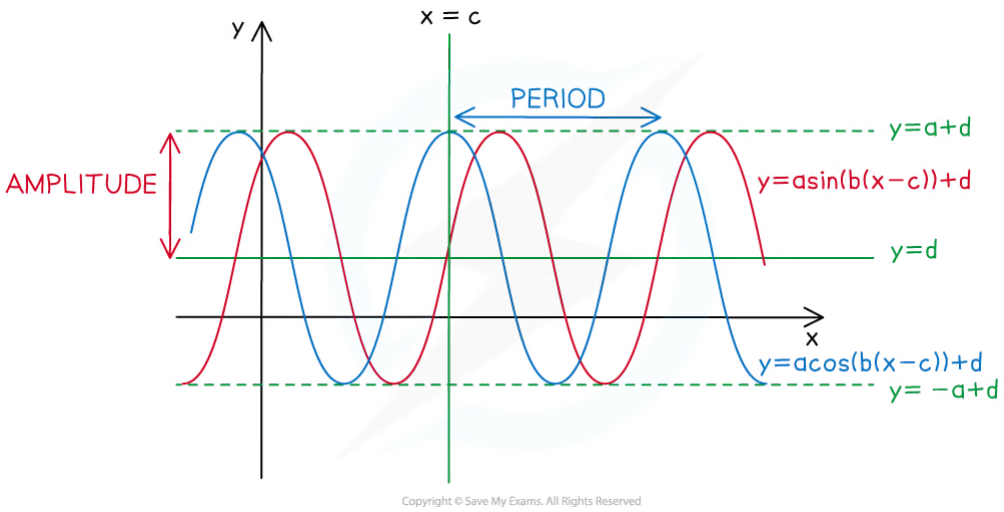
## What are the parameters of trigonometric models?

- A **trigonometric model** could be of the form
  - $f(x) = a\sin(b(x-c)) + d$
  - $f(x) = a\cos(b(x-c)) + d$
  - $f(x) = a\tan(b(x-c)) + d$
- The $a$ represents the **amplitude** of the function
  - The bigger the value of $a$ the bigger the **range** of values of the function
  - For the function $a\tan(b(x-c)) + d$ the amplitude is undefined
- The $b$ determines the **period** of the function
  - Period $= \frac{360°}{b} = \frac{2\pi}{b}$
  - The bigger the value of $b$ the quicker the function repeats a cycle
- The $c$ represents the **horizontal shift**
- The $d$ represents the **vertical shift**
  - This is the **principal axis**

## What are possible limitations of a trigonometric model?

- The amplitude is the same for each cycle
  - In real-life this might not be the case
  - The function might get closer to the value of $d$ over time
- The period is the same for each cycle
  - In real-life this might not be the case
  - The time to complete a cycle might change over time

www.thrivingscholars.com



Copyright © Save My Exams. All Rights Reserved

## Exam Tip

- The variable in these questions is often **t** for time.
- Read the question carefully to make sure you know what you are being asked to solve.

www.thrivingscholars.com

## Worked Example

The water depth, $D$, in metres, at a port can be modelled by the function

$$D(t) = 3\sin(15°(t-2)) + 12, \qquad 0 \leq t < 24$$

where $t$ is the elapsed time, in hours, since midnight.

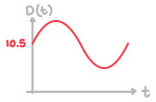
a)
Write down the depth of the water at midnight.

Substitute $t = 0$ for midnight:

$D(0) = 3\sin(15(0-2)) + 12$

$= 3\sin(-30) + 12$

$= 3(-\frac{1}{2}) + 12 \quad = 10.5$




$D = 10.5$ m

b)
Find the minimum water depth and the number of hours after midnight that this depth occurs.

www.thrivingscholars.com

$D(t) = 3\sin(15(t-2)) + 12$

3: amplitude; 12: principal axis

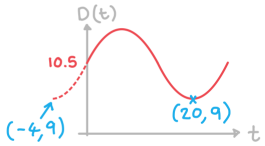
Principal axis is at $y = 12$

amplitude is 3 minimum $= 12 - 3 = 9$

Let $D(t) = 9$

$$3\sin(15(t-2)) + 12 = 9$$
$$3\sin(15(t-2)) = -3$$
$$\sin(15(t-2)) = -1$$
$$15(t-2) = -90$$
$$t = -4 + 24n$$

cycle repeats every 24 hours




**Minimum = 9m**
**20 hours after midnight**

c)
Calculate how long the water depth is at least 13.5 m each day.

www.thrivingscholars.com

Let $D(t) = 13.5$

$$3\sin(15(t-2)) + 12 = 13.5$$

$$3\sin(15(t-2)) = 1.5$$

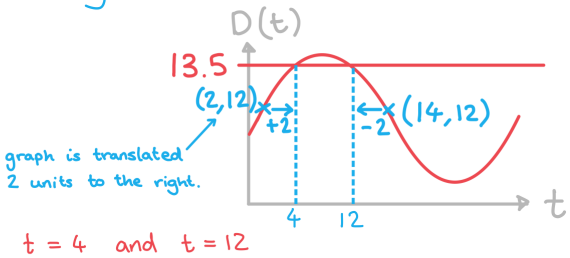
$$\sin(15(t-2)) = 0.5$$

$$15(t-2) = 30$$

$$t = 4 + 24n$$

cycle repeats every 24 hours

Use symmetry and properties of the graph to find secondary value of $t$:




$t = 4$ and $t = 12$

Find the difference between the times.

$12 - 4 = 8$

**8 hours**

www.thrivingscholars.com



# 3.6 Trigonometric Equations & Identities

## 3.6.1 Simple Identities

### Simple Identities

### What is a trigonometric identity?

- Trigonometric identities are statements that are true for all values of $x$ or $\theta$
- They are used to help simplify trigonometric equations before solving them
- Sometimes you may see identities written with the symbol ≡
  - This means 'identical to'

### What trigonometric identities do I need to know?

- The two trigonometric identities you must know are
  - $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
    - This is the identity for $\tan\theta$
  - $\sin^2\theta + \cos^2\theta = 1$
    - This is the Pythagorean identity
    - Note that the notation $\sin^2\theta$ is the same as $(\sin\theta)^2$
- Both identities can be found **in the formula booklet**
- Rearranging the second identity often makes it easier to work with
  - $\sin^2\theta = 1 - \cos^2\theta$
  - $\cos^2\theta = 1 - \sin^2\theta$

### Where do the trigonometric identities come from?

- You do not need to know the proof for these identities but it is a good idea to know where they come from
- From SOHCAHTOA we know that
  - $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{O}{H}$
  - $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{A}{H}$
  - $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{O}{A}$
- The identity for $\tan\theta$ can be seen by diving $\sin\theta$ by $\cos\theta$?
  - $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{O}{H}}{\frac{A}{H}} = \dfrac{O}{A} = \tan\theta$
- The Pythagorean identity can be seen by considering a right-triangle with a hypotenuse of 1
  - Then $(\text{opposite})^2 + (\text{adjacent})^2 = 1$
  - Therefore $\sin^2\theta + \cos^2\theta = 1$
- Considering the equation of the unit circle also shows the Pythagorean identity
  - The equation of the unit circle is $x^2 + y^2 = 1$
  - The coordinates on the unit circle are $(\cos\theta, \sin\theta)$
  - Therefore the equation of the unit circle could be written $\cos^2\theta + \sin^2\theta = 1$

www.thrivingscholars.com

- A third very useful identity is $\sin\theta = \cos(90° - \theta)$ or $\sin\theta = \cos(\frac{\pi}{2} - \theta)$
  - This is not included in the formula booklet but is useful to remember

## How are the trigonometric identities used?

- Most commonly trigonometric identities are used to change an equation into a form that allows it to be solved
- They can also be used to prove further identities such as the **double angle formulae**

### Exam Tip

- If you are asked to show that one thing is identical (≡) to another, look at what parts are missing – for example, if tan x has gone it must have been substituted

### Worked Example

Show that the equation $2\sin^2 x - \cos x = 0$ can be written in the form $a\cos^2 x + b\cos x + c = 0$, where $a$, $b$ and $c$ are integers to be found.

$2\sin^2 x - \cos x = 0$

Equation has both sinx and cosx so will need changing before it can be solved.

Use the identity $\sin^2 x = 1 - \cos^2 x$

Substitute: $2(1 - \cos^2 x) - \cos x = 0$

Expand: $2 - 2\cos^2 x - \cos x = 0$

Rearrange: $2\cos^2 x + \cos x - 2 = 0$

**a = 2, b = 1, c = -2**

www.thrivingscholars.com



# Compound Angle Formulae

## What are the compound angle formulae?

- There are six **compound angle formulae** (also known as **addition formulae**), two each for **sin**, **cos** and **tan**:
- For **sin** the +/- sign on the left-hand side **matches** the one on the right-hand side
  - sin(A+B)≡sinAcosB + cosAsinB
  - sin(A-B)≡sinAcosB - cosAsinB
- For **cos** the +/- sign on the left-hand side is **opposite to** the one on the right-hand side
  - cos(A+B)≡cosAcosB - sinAsinB
  - cos(A-B)≡cosAcosB + sinAsinB
- For **tan** the +/- sign on the left-hand side **matches** the one in the **numerator** on the right-hand side, and is **opposite to** the one in the **denominator**
  - $\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$
  - $\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$
- The compound angle formulae can all the found in the formula booklet, you do not need to remember them

## When are the compound angle formulae used?

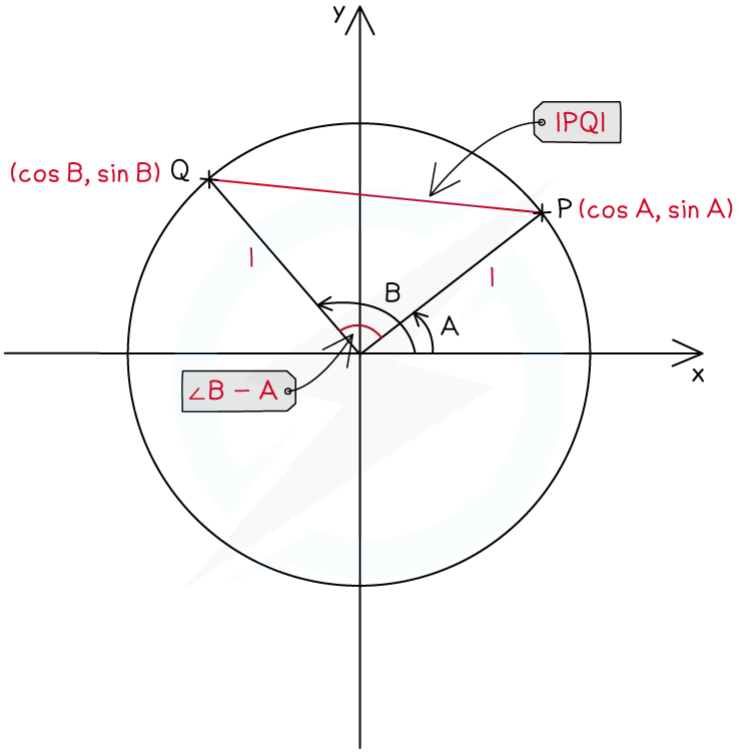
- The compound angle formulae are particularly useful when find the values of trigonometric ratios without the use of a calculator
  - For example to find the value of sin15° rewrite it as sin (45 – 30)° and then
    - apply the compound formula for sin(A – B)
    - use your knowledge of exact values to calculate the answer
- The compound angle formulae are also used…
  - … to derive further multiple angle trig identities such as the double angle formulae
  - … in trigonometric proof
  - … to simplify complicated trigonometric equations before solving

## How are the compound angle formulae for cosine proved?

- The proof for the compound angle identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$ can be seen by considering two coordinates on a unit circle, $P(\cos A, \sin A)$ and $Q(\cos B, \sin B)$
  - The angle between the positive $x$- axis and the point $P$ is $A$
  - The angle between the positive $x$- axis and the point $Q$ is $B$
  - The angle between $P$ and $Q$ is $B - A$
- Using the distance formula (Pythagoras) the distance $PQ$ can be given as
  - $|PQ|^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$
- Using the cosine rule the distance $PQ$ can be given as
  - $|PQ|^2 = 1^2 + 1^2 - 2(1)(1)\cos(B - A) = 2 - 2\cos(B - A)$

www.thrivingscholars.com

- Equating these two formulae, expanding and rearranging gives
  - $2 - 2\cos(B - A) = \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos A \cos B - 2\sin A \sin B$
  - $2 - 2\cos(B - A) = 2 - 2(\cos A \cos B + \sin A \sin B)$
- Therefore $\cos(B - A) = \cos A \cos B + \sin A \sin B$
- Changing $-A$ for $A$ in this identity and rearranging proves the identity for $\cos(A + B)$
  - $\cos(B - (-A)) = \cos(-A)\cos B + \sin(-A)\sin B = \cos A \cos B - \sin A \sin B$




Copyright © Save My Exams. All Rights Reserved

## How are the compound angle formulae for sine proved?

- The proof for the compound angle identity $\sin(A + B)$ can be seen by using the above proof for $\cos(B - A)$ and
  - Considering $\cos(\pi/2 - (A + B)) = \cos(\pi/2)\cos(A + B) + \sin(\pi/2)\sin(A + B)$
  - Therefore $\cos(\pi/2 - (A + B)) = \sin(A + B)$
  - Rewriting $\cos(\pi/2 - (A + B))$ as $\cos((\pi/2 - A) + B)$ gives
    - $\cos(\pi/2 - (A + B)) = \cos(\pi/2 - A)\cos B + \sin(\pi/2 - A)\sin B$
  - Using $\cos(\pi/2 - A) = \sin A$ and $\sin(\pi/2 - A) = \cos A$ and equating gives
    - $\sin(A + B) = \sin A \cos B + \cos A \cos B$
- Substituting $B$ for $-B$ proves the result for $\sin(A - B)$

## How are the compound angle formulae for tan proved?

- The proof for the compound angle identities $\tan(A \pm B)$ can be seen by
  - Rewriting $\tan(A \pm B)$ as $\dfrac{\sin(A \pm B)}{\cos(A \pm B)}$
  - Substituting the compound angle formulae in

www.thrivingscholars.com

- Dividing the numerator and denominator by $\cos A \cos B$

### Exam Tip

- All these formulae are in the **Topic 3: Geometry and Trigonometry** section of the formula booklet – make sure that you use them correctly paying particular attention to any negative/positive signs

www.thrivingscholars.com

## Worked Example

a)

Show that $\tan\left(x+\frac{\pi}{4}\right)-\tan\left(x-\frac{\pi}{4}\right)=\frac{2(\tan^2 x+1)}{1-\tan^2 x}$

Use the compound angle formula for tan:

$$\tan\left(x+\tfrac{\pi}{4}\right)=\frac{\tan x+\tan\frac{\pi}{4}}{1-\tan x\tan\frac{\pi}{4}}=\frac{\tan x+1}{1-\tan x}$$

$$\tan\left(x-\tfrac{\pi}{4}\right)=\frac{\tan x-\tan\frac{\pi}{4}}{1+\tan x\tan\frac{\pi}{4}}=\frac{\tan x-1}{1+\tan x}$$

Put together and simplify:

$$\frac{\tan x+1}{1-\tan x}-\frac{\tan x-1}{1+\tan x}=\frac{(\tan x+1)(1+\tan x)-(\tan x-1)(1-\tan x)}{(1-\tan x)(1+\tan x)}$$

$$=\frac{\tan^2 x+2\tan x+1-(-\tan^2 x+2\tan x-1)}{(1-\tan x)(1+\tan x)}$$

$$=\frac{2\tan^2 x+2}{1-\tan^2 x}$$

$$\boxed{\frac{2(\tan^2 x+1)}{1-\tan^2 x}}$$

b)

Hence, solve $\tan\left(x+\frac{\pi}{4}\right)-\tan\left(x-\frac{\pi}{4}\right)=-4$ for $0\le x\le\frac{\pi}{2}$

Use the answer found in (a) to write a new equation:

$$\frac{2(\tan^2 x+1)}{1-\tan^2 x}=-4$$

Rearrange and bring all terms in tanx to one side:

$$2(\tan^2 x+1)=-4(1-\tan^2 x)$$

$$2\tan^2 x+2=-4+4\tan^2 x$$

$$2\tan^2 x-6=0$$

$$\tan^2 x=3$$

$$\tan x=\pm\sqrt{3}$$

$$x=\frac{\pi}{3},\ -\frac{\pi}{3}\leftarrow\text{outside of given range}$$

$$\boxed{x=\frac{\pi}{3}}$$

www.thrivingscholars.com





# Double Angle Formulae

## What are the double angle formulae?

- The **double angle formulae** for **sine** and **cosine** are:
  - $\sin 2\theta = 2\sin\theta\cos\theta$
  - $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
  - $\tan 2\theta \equiv \frac{2\tan\theta}{1-\tan^2\theta}$
- These can be found in the formula booklet
  - The formulae for sin and cos can be found in the SL section
  - The formula for tan can be found in the HL section

## How are the double angle formulae derived?

- The double angle formulae can be derived from the compound angle formulae
- Simply replace *B* for *A* in each of the formulae and simplify
- For example
  - Sin 2*A* = sin (*A* + *A*) = sin*A*cos*A* + sin*A*cos*A* = 2sin*A*cos*A*

## How are the double angle formulae used?

- Double angle formulae will often be used with…
  - … trigonometry exact values
  - … graphs of trigonometric functions
  - … relationships between trigonometric ratios
- To help solve trigonometric equations which contain $\sin\theta\cos\theta$:
  - Substitute $\sin\theta\cos\theta$ for $\frac{1}{2}\sin 2\theta$
  - Solve for $2\theta$, finding all values in the range for $2\theta$
    - The range will need adapting for $2\theta$
  - Find the solutions for $\theta$
- To help solve trigonometric equations which contain $\sin 2\theta$ and $\sin\theta$ or $\cos\theta$
  - Substitute $\sin 2\theta$ for $2\sin\theta\cos\theta$
  - Isolate all terms in $\theta$
  - Factorise or use another identity to write the equation in a form which can be solved
- To help solve trigonometric equations which contain $\cos 2\theta$ and $\sin\theta$ or $\cos\theta$
  - Substitute $\cos 2\theta$ for either $2\cos^2\theta - 1$ or $1 - 2\sin^2\theta$
    - Choose the trigonometric ratio that is already in the equation
  - Isolate all terms in $\theta$
  - Solve
    - The equation will most likely be in the form of a quadratic
- To help solve trigonometric equations which contain tan 2θ
  - Substitute tan 2θ for the double angle identity
  - Rearrange, often this will lead to a quadratic equation in terms of tan θ
  - Solve
- Double angle formulae can be used in proving other trigonometric identities

www.thrivingscholars.com

## Exam Tip

- All these formulae are in the **Topic 3: Geometry and Trigonometry** section of the formula booklet
- If you are asked to show that one thing is identical (≡) to another, look at what parts are missing – for example, if $\sin\theta$ has disappeared you may want to choose the equivalent expression for $\cos 2\theta$ that does not include $\sin\theta$

## Worked Example

Without using a calculator, solve the equation $\sin 2\theta = \sin\theta$ for $0° \leq \theta \leq 360°$. Show all working clearly.

Double angle identity: $\sin 2\theta = 2\sin\theta\cos\theta$

$2\sin\theta\cos\theta = \sin\theta$

Bring both identities to one side:

$2\sin\theta\cos\theta - \sin\theta = 0$

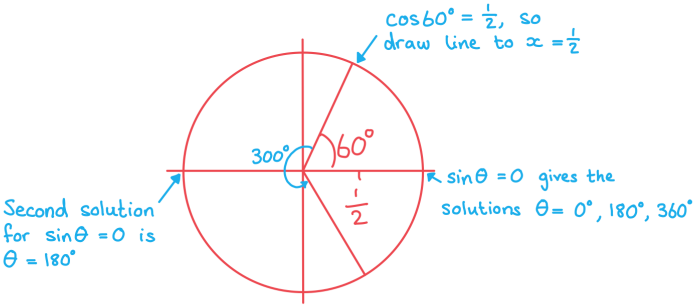
Factorise: $\sin\theta(2\cos\theta - 1) = 0$

Find solutions: $\sin\theta = 0$ $\quad 2\cos\theta - 1 = 0$

$\theta = 0$ $\quad \cos\theta = \frac{1}{2}$

$\theta = 60°$

Find secondary values within range:




$\theta = 0°, 60°, 180°, 300°, 360°$

www.thrivingscholars.com





# Relationship Between Trigonometric Ratios

## What relationships between trigonometric ratios should I know?

- If you know a value for one trig ratio you can often use this to work out the value for the others without needing to find $\theta$
- If you know that $\sin\theta = \frac{a}{b}$, where $a, b \in \mathbb{N}$, you can:
  - Sketch a right-triangle with *a* opposite $\theta$ and *b* on the hypotenuse
  - Use Pythagoras' theorem to find the value of the adjacent side
  - Use SOHCAHTOA to find the values of cos $\theta$ and tan $\theta$
- If you know a value for sin $\theta$ or cos $\theta$ you can use the Pythagorean relationship
  - $\sin^2\theta + \cos^2\theta = 1$
  - to find the value of the other
- If you know a value for sin $\theta$ or cos $\theta$ you can use the double angle formulae to find the value of sin $2\theta$ or cos $2\theta$
- If you know a value for tan $\theta$ you can use the double angle formulae to find the value of tan $2\theta$
- If you know two out of the three values for sin $\theta$, cos $\theta$ or tan $\theta$ you can use the identity in tan
  - $\tan\theta = \frac{\sin\theta}{\cos\theta}$
  - to find the value of the third ratio

## How do we determine whether a trigonometric ratio will be positive or negative?

- It is possible to determine whether a trigonometric ratio will be positive or negative by looking at the size of the angle and considering the **unit circle**
  - Angles in the range $0° < \theta° < 90°$ will be positive for all three ratios
  - Angles in the range $90° < \theta° < 180°$ will be positive for sin and negative for cos and tan
  - Angles in the range $180° < \theta° < 270°$ will be positive for tan and negative for sin and cos
  - Angles in the range $270° < \theta° < 360°$ will be positive for cos and negative for sin and tan
- The ratios for angles of 0°, 90°, 180°, 270° and 360° are either 0, 1, -1 or undefined
  - You should know these ratios or know how to derive them without a calculator

### Exam Tip

- Being able to sketch out the unit circle and remembering CAST can help you to find all solutions to a problem in an exam question

www.thrivingscholars.com

## Worked Example

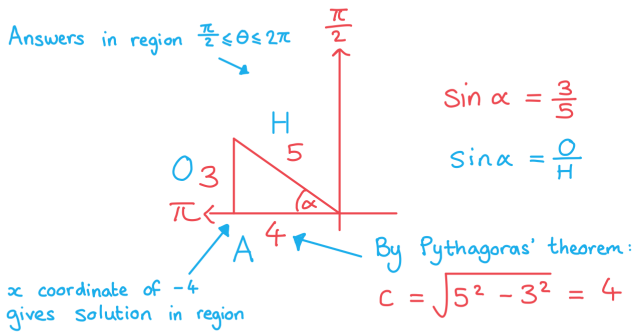
The value of $\sin\alpha = \frac{3}{5}$ for $\frac{\pi}{2} \leq \alpha \leq \pi$. Find:

i)

$\cos\alpha$

Method 1: Use right-triangle:

$\frac{\pi}{2} \leq \alpha \leq \pi$




$\sin\alpha = \frac{3}{5}$

$\sin\alpha = \frac{O}{H}$

By Pythagoras' theorem:

$c = \sqrt{5^2 - 3^2} = 4$

$\cos\alpha = \frac{A}{H} = -\frac{4}{5}$

$$\cos\alpha = -\frac{4}{5}$$

Method 2: Use Pythagorean identity:

$\cos^2\alpha = 1 - \sin^2\alpha = 1 - \left(\frac{3}{5}\right)^2$

$\cos\alpha = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$

↖ Check which solution is in range.

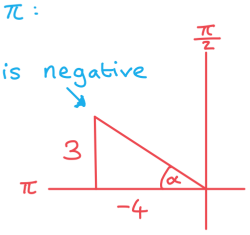
ii)

$\tan\alpha$

www.thrivingscholars.com

Use $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4}$

Check if $\tan\alpha = -\frac{3}{4}$ is in the correct range for $\frac{\pi}{2} \leq \alpha \leq \pi$:




$$\tan\alpha = -\frac{3}{4}$$

iii)

$\sin 2\alpha$

Double angle identity: $\sin 2\theta = 2\sin\theta\cos\theta$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)$$
$$= -\frac{24}{25}$$

$\sin\alpha = \frac{3}{5}$
$\cos\alpha = -\frac{4}{5}$

$$\sin 2\alpha = -\frac{24}{25}$$

iv)

$\cos 2\alpha$

Double angle identity: $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$$\cos 2\alpha = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\cos 2\alpha = \frac{7}{25}$$

v)

$\tan 2\alpha$

www.thrivingscholars.com



Using identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}$$

$$\boxed{\tan 2\alpha = -\frac{24}{7}}$$

www.thrivingscholars.com



# Trigonometric Equations: sinx = k

## How are trigonometric equations solved?

- Trigonometric equations can have an infinite number of solutions
  - For an equation in sin or cos you can add 360° or 2π to each solution to find more solutions
  - For an equation in tan you can add 180° or π to each solution
- When solving a trigonometric equation you will be given a range of values within which you should find all the values
- Solving the equation normally and using the inverse function on your calculator or your knowledge of **exact values** will give you the **primary value**
- The **secondary values** can be found with the help of:
  - The **unit circle**
  - The **graphs of trigonometric functions**

## How are trigonometric equations of the form sin x = k solved?

- It is a good idea to sketch the graph of the trigonometric function first
  - Use the given range of values as the domain for your graph
  - The intersections of the graph of the function and the line y = k will show you
    - The location of the solutions
    - The number of solutions
  - You will be able to use the symmetry properties of the graph to find all secondary values within the given range of values
- The method for finding secondary values are:
  - For the equation $\sin x = k$ the primary value is $x_1 = \sin^{-1} k$
    - A secondary value is $x_2 = 180° - \sin^{-1} k$
    - Then all values within the range can be found using $x_1 \pm 360n$ and $x_2 \pm 360n$ where $n \in \mathbb{N}$
  - For the equation $\cos x = k$ the primary value is $x_1 = \cos^{-1} k$
    - A secondary value is $x_2 = -\cos^{-1} k$
    - Then all values within the range can be found using $x_1 \pm 360n$ and $x_2 \pm 360n$ where $n \in \mathbb{N}$
  - For the equation $\tan x = k$ the primary value is $x = \tan^{-1} k$
    - All secondary values within the range can be found using $x \pm 180n$ where $n \in \mathbb{N}$

### Exam Tip

- If you are using your GDC it will only give you the principal value and you need to find all other solutions for the given interval
- Sketch out the CAST diagram and the trig graphs on your exam paper to refer back to as many times as you need to

www.thrivingscholars.com

## Worked Example

Solve the equation $2\cos x = -1$, finding all solutions in the range $-\pi \leq x \leq \pi$.

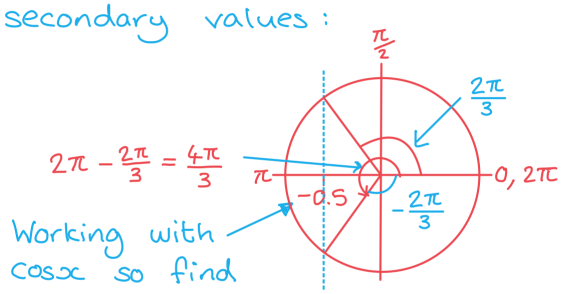
Isolate $\cos x$ : $\cos x = \frac{-1}{2}$

use GDC or knowledge of exact values

$x = \cos^{-1}\left(-\frac{1}{2}\right)$

$= \frac{2\pi}{3}$ ← Primary value

Find secondary values :




Working with $\cos x$ so find $x$-coordinate

$\frac{2\pi}{3} \pm 2\pi n$ and $\frac{4\pi}{3} \pm 2\pi n$

Find all answers in range $-\pi \leq x \leq 3\pi$

$$-\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$$

www.thrivingscholars.com

# Trigonometric Equations: sin(ax + b) = k

## How can I solve equations with transformations of trig functions?

- Trigonometric equations in the form $\sin(ax + b)$ can be solved in more than one way
- The easiest method is to consider the transformation of the angle as a substitution
  - For example let $u = ax + b$
- Transform the given interval for the solutions in the same way as the angle
  - For example if the given interval is $0° \leq x \leq 360°$ the new interval will be
  - $(a(0°) + b) \leq u \leq (a(360°) + b)$
- Solve the function to find the primary value for $u$
- Use either the unit circle or sketch the graph to find all the other solutions in the range for $u$
- Undo the substitution to convert all of the solutions back into the corresponding solutions for $x$
- Another method would be to sketch the transformation of the function
  - If you use this method then you will not need to use a substitution for the range of values

### Exam Tip

- If you transform the interval, remember to convert the found angles back to the final values at the end!
- If you are using your GDC it will only give you the principal value and you need to find all other solutions for the given interval
- Sketch out the CAST diagram and the trig graphs on your exam paper to refer back to as many times as you need to

www.thrivingscholars.com

## Worked Example

Solve the equation $2\cos(2x - 30°) = -1$, finding all solutions in the range $-360° \leq x \leq 360°$.

$2\cos(2x - 30°) = -1$ $\quad -360° \leq x \leq 360°$

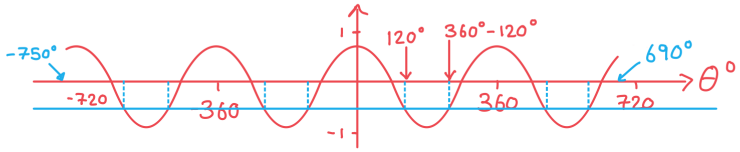
Start by changing the range: $-750° \leq 2x - 30 \leq 690°$

Substitute $\theta = 2x - 30$:

$2\cos\theta = -1$ $\quad -750° \leq \theta \leq 690°$

$\cos\theta = -\frac{1}{2}$

$\theta = \cos^{-1}(-\frac{1}{2}) = 120°$ ← Primary value




From the sketch you can see there are 8 solutions:

$\theta = 120° \pm 360°$ and $\theta = 240° \pm 360°$

$\theta = -600°, -480°, -240°, -120°, 120°, 240°, 480°, 600°$

Solve for $x$: $x = \frac{\theta° + 30}{2}$

$x = -285°, -225°, -105°, -45°$
$75°, 135°, 255°, 315°$

www.thrivingscholars.com



# Quadratic Trigonometric Equations

## How are quadratic trigonometric equations solved?

- A quadratic trigonometric equation is one that includes either $\sin^2\theta$, $\cos^2\theta$ or $\tan^2\theta$
- Often the **identity** $\sin^2\theta + \cos^2\theta = 1$ can be used to rearrange the equation into a form that is possible to solve
  - If the equation involves both sine and cosine then the **Pythagorean identity** should be used to write the equation in terms of just one of these functions
- Solve the **quadratic equation** using your GDC, the quadratic equation or factorisation
  - This can be made easier by changing the function to a single letter
    - Such as changing $2\cos^2\theta - 3\cos\theta - 1 = 0$ to $2c^2 - 3c - 1 = 0$
- A quadratic can give up to two solutions
  - You must consider both solutions to see whether a real value exists
  - Remember that solutions for $\sin\theta = k$ and $\cos\theta = k$ only exist for $-1 \le k \le 1$
  - Solutions for $\tan\theta = k$ exist for all values of k
- Find all solutions within the given interval
  - There will often be more than two solutions for one quadratic equation
  - The best way to check the number of solutions is to sketch the graph of the function

### Exam Tip

- Sketch the trig graphs on your exam paper to refer back to as many times as you need to!
- Be careful to make sure you have found **all** of the solutions in the given interval, being super-careful if you get a negative solution but have a positive interval

www.thrivingscholars.com

## Worked Example

Solve the equation $11\sin x - 7 = 5\cos^2 x$, finding all solutions in the range $0 \leq x \leq 2\pi$.

Use the identity $\cos^2 x = 1 - \sin^2 x$ to write equation in terms of $\sin x$:

$11\sin x - 7 = 5(1 - \sin^2 x)$ in formula booklet.

$= 5 - 5\sin^2 x$

Move all terms to one side:

$11\sin x - 7 - (5 - 5\sin^2 x) = 0$

Spot the hidden quadratic:

$11\sin x - 7 - 5 + 5\sin^2 x = 0$

$5\sin^2 x + 11\sin x - 12 = 0$

$\sin x = \frac{4}{5}$ or $\sin x = -3$

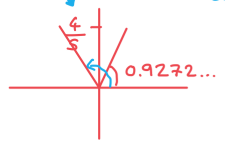
$\sin x$ so use y coordinate.

no solution so disregard.

$\sin x = \frac{4}{5}$

$x = 0.9272\ldots$

primary solution




$x = \pi - 0.9272\ldots$

$= 2.214\ldots$

secondary solution

$x = 0.927, 2.21$ (3 s.f.)

www.thrivingscholars.com

# 3.7 Inverse & Reciprocal Trig Functions

## 3.7.1 Reciprocal Trig Functions

### Reciprocal Trig Functions

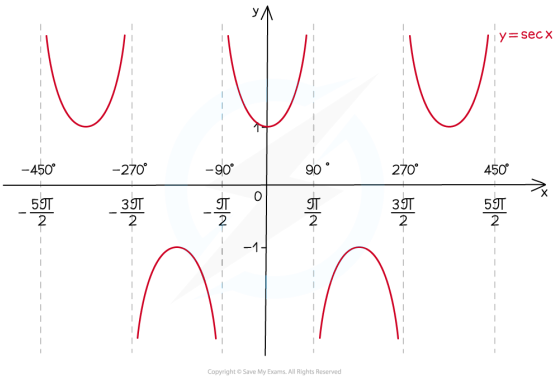
### What are the reciprocal trig functions?

- There are three reciprocal trig functions that each correspond to either sin, cos or tan
  - Secant (sec *x*)
    - $\sec x = \frac{1}{\cos x}$
  - Cosecant (cosec *x*)
    - $\operatorname{cosec} x = \frac{1}{\sin x}$
  - Cotangent (cot *x*)
    - $\cot x = \frac{1}{\tan x}$
  - The identities above for sec *x* and cosec *x* are given in the formula booklet
  - The identity for cot *x* is **not given**, you will need to remember it
  - A good way to remember which function is which is to look at the **third** letter in each of the reciprocal trig functions
    - co**t** *x* is 1 over **t**an *x* etc
- Each of the reciprocal trig functions are undefined for certain values of *x*
  - sec *x* is undefined for values of *x* for which cos *x* = 0
  - cosec *x* is undefined for values of *x* for which sin *x* = 0
  - cot *x* is undefined for values of *x* for which tan *x* = 0
    - When tan *x* is undefined, cot *x* = 0
- Rearranging the identity $\tan x = \frac{\sin x}{\cos x}$ gives
  - $\cot x = \frac{\cos x}{\sin x}$
    - This is not in the formula booklet but is easily derived
- Be careful not to confuse the reciprocal trig functions with the inverse trig functions
  - $\sin^{-1} x \neq \frac{1}{\sin x}$

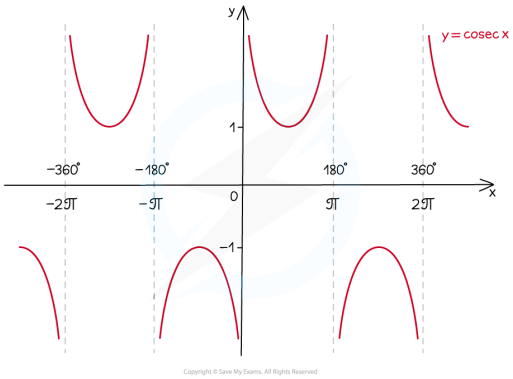
### What do the graphs of the reciprocal trig functions look like?

- The graph of ***y* = sec*x*** has the following properties:
  - *The* ***y*-axis** is a **line of symmetry**
  - It has a **period** of **360°** (**2π** radians)
  - There are vertical **asymptotes** wherever **cos** *x* **= 0**
    - **If drawing the graph without the help of a GDC it is a good idea to sketch cos *x* first and draw these in**
  - **The domain** is all *x* **except odd multiples of 90°** (90°, -90°, 270°, -270°, etc.)
    - in **radians** this is all *x* **except odd multiples of π/2** (π/2, - π/2, 3π/2, -3π/2, etc.)
  - **The range** is ***y* ≤ -1 or *y* ≥ 1**

www.thrivingscholars.com

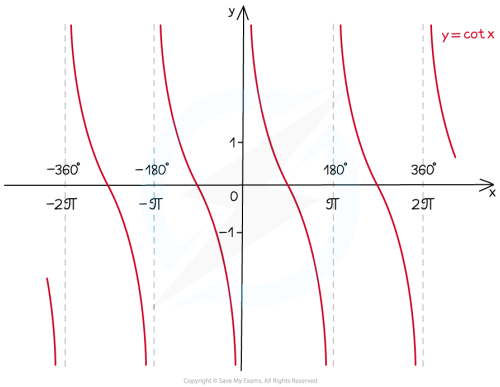



- The graph of ***y* = cosec *x*** has the following properties:
  - It has a **period** of **360°** (**2π** radians)
  - There are vertical **asymptotes** wherever **sin *x* = 0**
    - **If drawing the graph it is a good idea to sketch sin *x* first and draw these in**
  - The **domain** is all *x* **except multiples of 180°** (0°, 180°, -180°, 360°, -360°, etc.)
    - in radians this is all *x* **except multiples of π** (0, π, - π, 2π, -2π, etc.)
  - The range is $y \leq -1$ or $y \geq 1$




- The graph of ***y* = cot *x*** has the following properties
  - It has a **period** of **180°** or **π** radians
  - There are vertical **asymptotes** wherever **tan *x* = 0**
  - The **domain** is all *x* **except multiples of 180°** (0°, 180°, -180°, 360°, -360°, etc.)
    - In radians this is all *x* **except multiples of π** (0, π, - π, 2π, -2π, etc.)
  - The **range** is $y \in \mathbb{R}$ (i.e. cot can take *any* real number value)

www.thrivingscholars.com




## Exam Tip

- To solve equations with the reciprocal trig functions, convert them into the regular trig functions and solve in the usual way
- Don't forget that both **tan** and **cot** can be written in terms of **sin** and **cos**
- You will sometimes see **csc** instead of **cosec** for cosecant

www.thrivingscholars.com

## Worked Example

Without the use of a calculator, find the values of

a)

$\sec \frac{\pi}{6}$

$\sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)}$

the third letter is c so sec is related to cos

$\cos\left(\frac{\pi}{6}\right)$ is an exact value you should know.

$= \frac{1}{\frac{\sqrt{3}}{2}}$

$\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$

b)

$\cot 45°$

$\cot 45° = \frac{1}{\tan 45°}$

the third letter is t so cot is related to tan

$\tan 45°$ is an exact value you should know.

$= \frac{1}{1}$

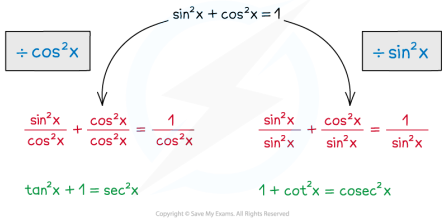
$\cot 45° = 1$

www.thrivingscholars.com

# Pythagorean Identities

## What are the Pythagorean Identities?

- Aside from the Pythagorean identity $\sin^2 x + \cos^2 x = 1$ there are two further Pythagorean identities you will need to learn
  - $1 + \tan^2 \theta = \sec^2 \theta$
  - $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$
  - Both can be found in the formula booklet
- Both of these identities can be derived from $\sin^2 x + \cos^2 x = 1$
  - To derive the identity for $\mathbf{sec^2 x}$ divide $\sin^2 x + \cos^2 x = 1$ by $\mathbf{\cos^2 x}$
  - To derive the identity for $\mathbf{cosec^2 x}$ divide $\sin^2 x + \cos^2 x = 1$ by $\mathbf{\sin^2 x}$




### Exam Tip

All the Pythagorean identities can be found in the **Topic 3: Geometry and Trigonometry** section of the formula booklet

www.thrivingscholars.com

## Worked Example

Solve the equation $9\sec^2\theta - 11 = 3\tan\theta$ in the interval $0 \leq \theta \leq 2\pi$.

$9\sec^2\theta - 11 = 3\tan\theta$ , $0 \leq \theta \leq 2\pi$

consider how this could be changed to use $\tan^2 + 1 = \sec^2$

Range is given in terms of $\pi$ so work in radians

$(9\sec^2\theta - 9) - 2 = 3\tan\theta$

$9(\sec^2\theta - 1) - 2 = 3\tan\theta$

$9\tan^2\theta - 3\tan\theta - 2 = 0$

$(3\tan\theta - 2)(3\tan\theta + 1) = 0$

$\tan\theta = \frac{2}{3} \Rightarrow \theta = 0.5880\ldots$

or $\theta = \pi + 0.5880\ldots = 3.729\ldots$

or $\tan\theta = -\frac{1}{3} \Rightarrow \theta = -0.3217\ldots$

or $\theta = \pi + (-0.3217\ldots) = 2.819\ldots$

and $\theta = 2\pi + (-0.3217\ldots) = 5.961\ldots$

$\theta = 0.588, 2.82, 3.73, 5.96$ (3 s.f.)

www.thrivingscholars.com



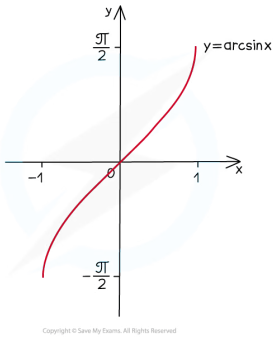
# Inverse Trig Functions

## What are the inverse trig functions?

- The functions **arcsin**, **arccos** and **arctan** are the **inverse functions** of **sin**, **cos** and **tan** respectively when their domains are restricted
  - $\sin(\arcsin x) = x$ for $-1 \leq x \leq 1$
  - $\cos(\arccos x) = x$ for $-1 \leq x \leq 1$
  - $\tan(\arctan x) = x$ for all $x$
- You will have seen and used the inverse trig **operations** many times already
  - Arcsin is the operation $\mathbf{\sin^{-1}}$
  - Arccos is to the operation $\mathbf{\cos^{-1}}$
  - Arctan is the operation $\mathbf{\tan^{-1}}$
- The domains of **sin**, **cos**, and **tan** must first be restricted to make them **one-to-one functions**
  - A function can only have an inverse if it is a one-to-one function
- The domain of $\mathbf{\sin x}$ is restricted to $\mathbf{-\pi/2 \leq x \leq \pi/2}$ $(\mathbf{-90° \leq x \leq 90°})$
- The domain of $\mathbf{\cos x}$ is restricted to $\mathbf{0 \leq x \leq \pi}$ $(\mathbf{0° \leq x \leq 180°})$
- The domain of $\mathbf{\tan x}$ is restricted to $\mathbf{-\pi/2 < x < \pi/2}$ $(\mathbf{-90° < x < 90°})$
- Be aware that $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$ are **not** the same as the reciprocal trig functions
  - They are used to solve trig equations such as $\sin x = 0.5$ for all values of $x$
  - $\arcsin x$ is the same as $\sin^{-1} x$ but not the same as $(\sin x)^{-1}$

## What do the graphs of the inverse trig functions look like?

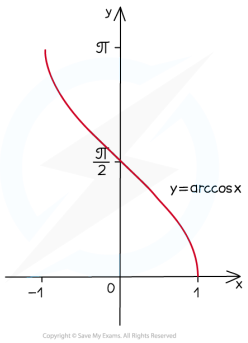
- The graphs of **arcsin**, **arccos** and **arctan** are the **reflections** of the graphs of **sin**, **cos** and **tan** (after their domains have been restricted) in the line $y = x$
  - The **domains** of $\arcsin x$ and $\arccos x$ are both $-1 \leq x \leq 1$
  - The **range** of $\arcsin x$ is $-\pi/2 \leq y \leq \pi/2$



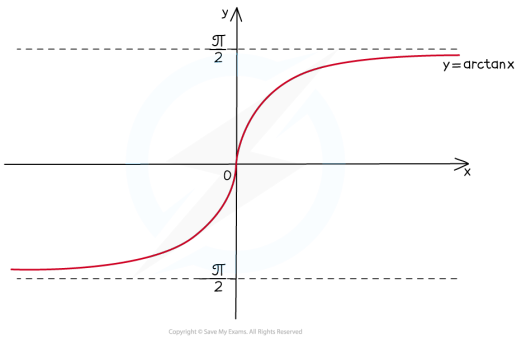

Copyright © Save My Exams. All Rights Reserved

  - The **range** of $\arccos x$ is $0 \leq y \leq \pi$

www.thrivingscholars.com




- The **domain** of arctan $x$ is $x \in \mathbb{R}$
- The **range** of arctan $x$ is $-\pi/2 < y < \pi/2$
  - Note that there are horizontal asymptotes at $\pi/2$ and $-\pi/2$




## How are the inverse trig functions used?

- The functions **arcsin**, **arccos** and **arctan** are used to evaluate trigonometric equations such as $\sin x = 0.5$
  - If $\sin x = 0.5$ then $\arcsin 0.5 = x$ for values of $x$ between $-\pi/2 \leq x \leq \pi/2$
    - You can then use symmetries of the trig function to find solutions over other intervals
- The inverse trig functions are also used to help evaluate algebraic expressions
  - From $\sin(\arcsin x) = x$ we can also say that $\sin^n(\arcsin x) = x^n$ for $-1 \leq x \leq 1$
  - If using an inverse trig function to evaluate an algebraic expression then remember to consider the domain and range of the function
    - $\arcsin(\sin x) = x$ only for $-\pi/2 \leq x \leq \pi/2$
    - $\arccos(\cos x) = x$ only for $0 \leq x \leq \pi$
    - $\arctan(\tan x) = x$ only for $-\pi/2 < x < \pi/2$
  - The symmetries of the trig functions can be used when values lie outside of the domain or range
    - Using $\sin(x) = \sin(\pi - x)$ you get $\arcsin(\sin(2\pi/3)) = \arcsin(\sin(\pi/3)) = \pi/3$

### Exam Tip

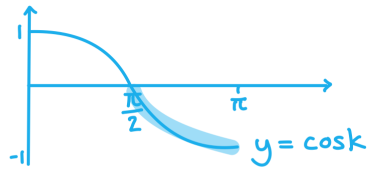
- Make sure you know the shapes of the graphs for **sin**, **cos** and **tan** so that you can easily reflect them in the line $y = x$ and hence sketch the graphs of **arcsin**, **arccos** and **arctan**

www.thrivingscholars.com

## Worked Example

Given that $x$ satisfies the equation $\arccos x = k$ where $\frac{\pi}{2} < k < \pi$, state the range of possible values of $x$.

If $\arccos x = k$, then $x = \cos k$ $(\cos(\arccos x) = x)$




For $\frac{\pi}{2} < k < \pi$, $-1 < \cos k < 0$

$-1 < x < 0$

www.thrivingscholars.com



## 3.8 Further Trigonometry

### 3.8.1 Trigonometric Proof

# Trigonometric Proof

## How do I prove new trigonometric identities?

- You can use trigonometric identities you already know to prove new identities
- Make sure you know how to find all of the trig identities in the formula booklet
  - The identity for tan, simple Pythagorean identity and the double angle identities for in and cos are in the SL section
    - $\tan\theta = \frac{\sin\theta}{\cos\theta}$
    - $\cos^2\theta + \sin^2\theta = 1$
    - $\sin 2\theta = 2\sin\theta\cos\theta$
    - $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
  - The reciprocal trigonometric identities for sec and cosec, further Pythagorean identities, compound angle identities and the double angle formula for tan
    - $\sec\theta = \frac{1}{\cos\theta}$
    - $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$
    - $1 + \tan^2\theta = \sec^2\theta$
    - $1 + \cot^2\theta = \operatorname{cosec}^2\theta$
    - $\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$
    - $\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B$
    - $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
    - $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$
  - The identity for cot is **not in the formula booklet**, you will need to remember it
    - $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$
- To prove an identity start on one side and proceed step by step until you get to the other side
  - It is more common to start on the left hand side but you can start a proof from either end
  - Occasionally it is easier to show that one side subtracted from the other is zero
  - You should not work on both sides simultaneously

## What should I look out for when proving new trigonometric identities?

- Look for anything that could be a part of one of the above identities on either side
  - For example if you see $\sin 2\theta$ you can replace it with $2\sin\theta\cos\theta$
  - If you see $2\sin\theta\cos\theta$ you can replace it with $\sin 2\theta$

www.thrivingscholars.com

- Look for ways of reducing the number of different trigonometric functions there are within the identity
  - For example if the identity contains $\tan\theta$, $\cot\theta$ and $\operatorname{cosec}\theta$ you could try
    - using the identities $\tan\theta = 1/\cot\theta$ and $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ to write it all in terms of $\cot\theta$
    - or rewriting it all in terms of $\sin\theta$ and $\cos\theta$ and simplifying
- Often you may need to trial a few different methods before finding the correct one
- Clever substitution into the **compound angle formulae** can be a useful tool for proving identities
  - For example rewriting $\cos\frac{\theta}{2}$ as $\cos\left(\theta - \frac{\theta}{2}\right)$ doesn't change the ratio but could make an identity easier to prove
- You will most likely need to be able to work with fractions and fractions-within-fractions
- Always keep an eye on the 'target' expression – this can help suggest what identities to use

### Exam Tip

- Don't forget that you can start a proof from either end – sometimes it might be easier to start from the left-hand side and sometimes it may be easier to start from the right-hand side
- Make sure you use the formula booklet as all of the relevant trigonometric identities are given to you
- Look out for special angles (0°, 90°, etc) as you may be able to quickly simplify or cancel parts of an expression (e.g. $\cos 90° = 0$)

www.thrivingscholars.com

## Worked Example

Prove that $8\cos^4\theta - 8\cos^2\theta + 1 = \cos 4\theta$.

It is easiest to start on the right-hand side and apply the double angle formula for $\cos 2\theta$.

$$8\cos^4\theta - 8\cos^2\theta + 1 = \cos 4\theta$$

The form of the left-hand side suggests that the identity $\cos 2A = 2\cos^2 A - 1$ would be more useful than the other options.

$$\begin{aligned}
\cos 4\theta &= 2\cos^2 2\theta - 1 \\
&= 2(2\cos^2\theta - 1)^2 - 1 \\
&= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1 \\
&= 8\cos^4\theta - 8\cos^2\theta + 2 - 1
\end{aligned}$$

$$\therefore \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

www.thrivingscholars.com



# Strategy for Trigonometric Equations

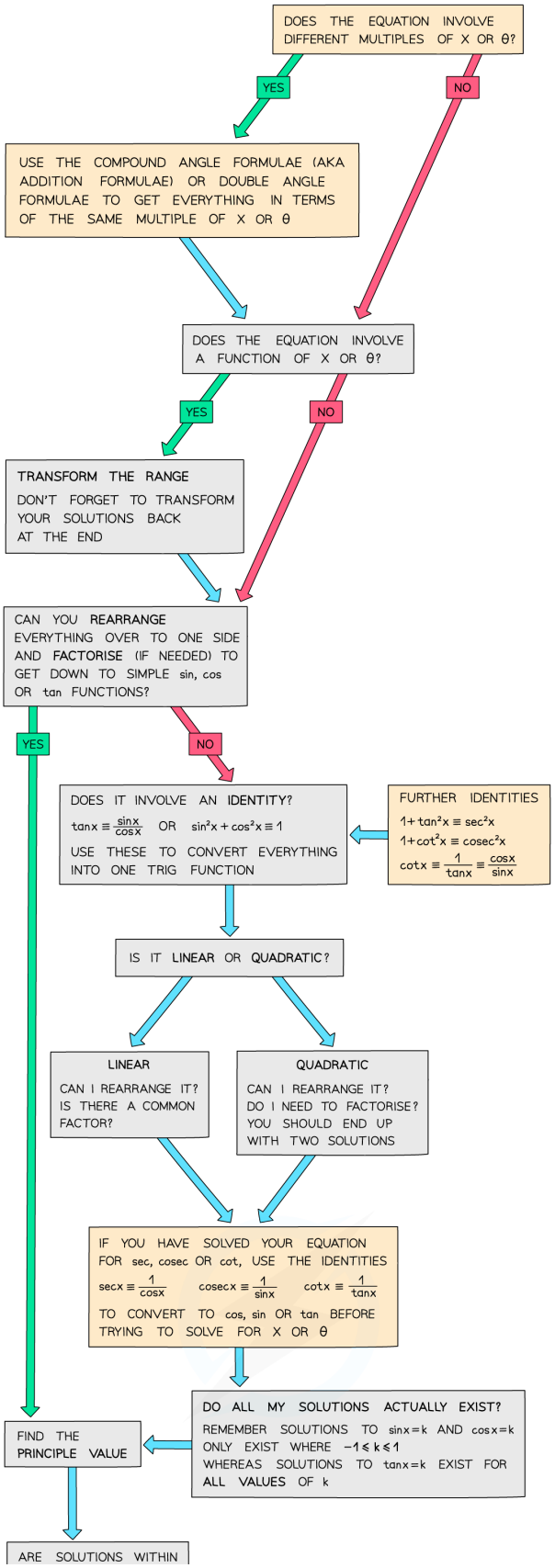
## How do I approach solving trig equations?

- You can solve trig equations in a variety of different ways
  - **Sketching a graph**
    - If you have your GDC it is always worth sketching the graph and using this to analyse its features
  - Using **trigonometric** identities**, Pythagorean** identities**,** the **compound** or **double angle** identities
    - Almost all of these are in the formula booklet, make sure you have it open at the right page
  - Using the **unit circle**
  - Factorising **quadratic** trig equations
    - Look out for quadratics such as $5\tan^2 x - 3\tan x - 4 = 0$
- The final rearranged equation you solve will involve **sin**, **cos** or **tan**
  - Don't try to solve an equation with **cosec**, **sec**, or **cot** directly

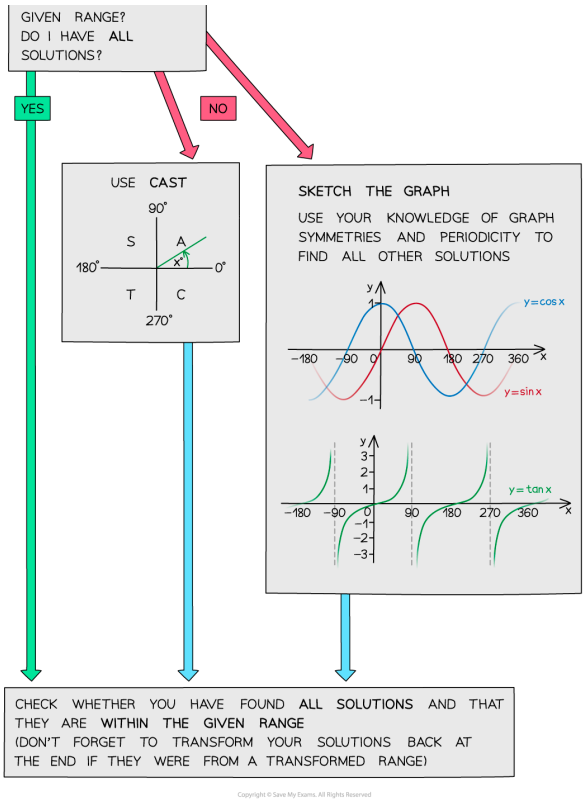
## What should I look for when solving trig equations?

- Check the value of $x$ or $\theta$
  - If it is just $x$ or $\theta$ you can begin solving
  - If there are **different multiples** of $x$ or $\theta$ you will need to use the **double angle formulae** to get everything in terms of the same multiple of $x$ or $\theta$
  - If it is a **function** of $x$ or $\theta$, e.g. $2x - 15$, you will need to **transform the range** first
    - You must remember to transform your solutions back again at the end
- Does it involve more than one trigonometric function?
  - If it does, try to **rearrange** everything to bring it to one side, you may need to **factorise**
  - If not, can you use an identity to reduce the number of different trigonometric functions?
    - You should be able to use identities to reduce everything to just one simple trig function (either sin, cos or tan)
- Is it **linear** or **quadratic**?
  - If it is linear you should be able to rearrange and solve it
  - If it is quadratic you may need to factorise first
    - You will most likely get two solutions, consider whether they both **exist**
    - Remember solutions to $\sin x = k$ and $\cos x = k$ only exist for $-1 \leq k \leq 1$ whereas solutions to $\tan x = k$ exist for all values of $k$
- Are my solutions within the given range and do I need to find more solutions?
  - Be extra careful if your solutions are negative but the given range is positive only
  - Use a sketch of the graph or the unit circle to find the other solutions within the range
  - If you have a function of $x$ or $\theta$ make sure you are finding the solutions within the **transformed range**
    - Don't forget to transform the solutions back so that they are in the required range at the end

www.thrivingscholars.com



www.thrivingscholars.com




Copyright © Save My Exams. All Rights Reserved

## Exam Tip

- Try to use identities and formulas to reduce the equation into its simplest terms.
- Don't forget to check the function range and ensure you have included all possible solutions.
- If the question involves a function of x or θ ensure you transform the range first (and ensure you transform your solutions back again at the end!).

www.thrivingscholars.com

## Worked Example

Find the solutions of the equation $(1+\cot^2 2\theta)(5\cos^2\theta - 1) = \cot^2 2\theta$ in the interval $0 \leq \theta \leq 2\pi$.

Move equivalent trig functions to the same sides:

$5\cos^2\theta - 1 = \dfrac{\cot^2 2\theta}{1+\cot^2 2\theta}$ divide both sides by $1+\cot^2 2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$\therefore \cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$

$5\left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right) - 1 = \dfrac{\cot^2 2\theta}{\operatorname{cosec}^2 2\theta}$

$\dfrac{5}{2}\cos 2\theta + \dfrac{3}{2} = \dfrac{\frac{\cos^2 2\theta}{\sin^2 2\theta}}{\frac{1}{\sin^2 2\theta}}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

$\frac{1}{2}(5\cos 2\theta + 3) = \cos^2 2\theta$ Rearrange to form a quadratic in $\cos 2\theta$

$2\cos^2 2\theta - 5\cos 2\theta - 3 = 0$

$(2\cos 2\theta + 1)(\cos 2\theta - 3) = 0$

$\cos 2\theta = -\frac{1}{2}$ or $\cos 2\theta = 3$ ← no solutions

We are solving the equation for $2\theta$ so we must transform the range first: $0 \leq 2\theta \leq 4\pi$

$2\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ (primary value)

so $2\theta = \frac{2\pi}{3}$, $2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$, $\frac{2\pi}{3} + 2\pi = \frac{8\pi}{3}$, $\frac{4\pi}{3} + 2\pi = \frac{10\pi}{3}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

www.thrivingscholars.com

# 3.9 Vector Properties

## 3.9.1 Introduction to Vectors

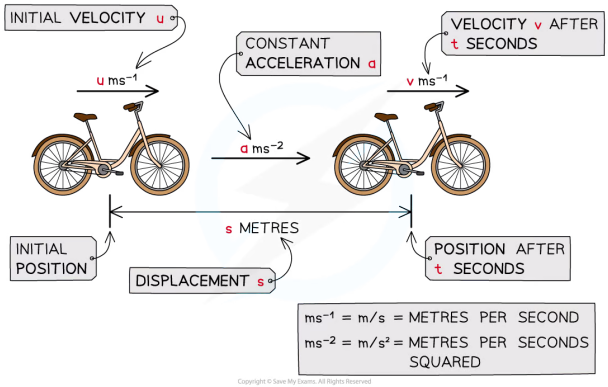
### Scalars & Vectors

### What are scalars?

- **Scalars** are quantities without **direction**
  - They have only a **size (magnitude)**
  - For example: **speed, distance, time, mass**
- **Most scalar quantities** can never be **negative**
  - You cannot have a negative speed or distance

### What are vectors?

- **Vectors** are quantities which also have a **direction**, this is what makes them more than just a scalar
  - For example: two objects with **velocities** of 7 m/s and -7 m/s are travelling at the **same speed** but in **opposite directions**
- A **vector quantity** is described by both its **magnitude** and **direction**
- A vector has **components** in the direction of the *x*- , *y*-, and *z*- axes
  - Vector quantities can have **positive** or **negative** components
- Some examples of vector quantities you may come across are **displacement**, **velocity**, **acceleration**, **force/weight**, **momentum**
  - **Displacement** is the position of an object from a starting point
  - **Velocity** is a speed in a given direction (displacement over time)
  - **Acceleration** is the change in velocity over time
- Vectors may be given in either 2– or 3– dimensions




### Exam Tip

- Make sure you fully understand the definitions of all the words in this section so that you can be clear about what your exam question is asking of you

www.thrivingscholars.com

## Worked Example

State whether each of the following is a scalar or a vector quantity.

a)

A speed boat travels at 3 m/s on a bearing of 052°

Speed with a given direction → velocity

**Vector**

b)

A garden is 1.7 m wide

Length with no direction

**Scalar**

c)

A car accelerates forwards at 5.4 $ms^{-2}$

Acceleration has direction

**Vector**

d)

A film lasts 2 hours 17 minutes

Time has no direction

**Scalar**

e)

An athlete runs at an average speed of 10.44 $ms^{-1}$

Speed with no direction is a scalar

**Scalar**

f)

A ball rolls forwards 60 cm before stopping

www.thrivingscholars.com



Displacement has direction

**Vector**

www.thrivingscholars.com

# Vector Notation

## How are vectors represented?

- **Vectors** are usually represented using an arrow in the direction of movement
  - The length of the arrow represents its magnitude
- They are written as lowercase letters either in **bold** or underlined
  - For example a vector from the point O to A will be written **a** or $\underline{a}$
    - The vector from the point A to O will be written **–a** or $-\underline{a}$
- If the start and end point of the vector is known, it is written using these points as capital letters with an arrow showing the direction of movement
  - For example: $\overrightarrow{AB}$ or $\overrightarrow{BA}$
- Two vectors are equal only if their corresponding components are equal
- Numerically, vectors are either represented using **column vectors** or **base vectors**
  - Unless otherwise indicated, you may carry out all working and write your answers in either of these two types of vector notation

## What are column vectors?

- **Column vectors** are where one number is written above the other enclosed in brackets
- **In 2-dimensions the top number represents movement in the horizontal direction (right/left) and the bottom number represents movement in the vertical direction (up/down)**
- **A positive value represents movement in the positive direction (right/up) and a negative value represents movement in the negative direction (left/down)**
  - For example: The column vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ represents **3 units** in the **positive horizontal** ($x$) direction (i.e., **right**) and **2 units** in the **negative vertical** ($y$) direction (i.e., **down**)
- **In 3-dimensions the top number represents the movement in the *x* direction (length), the middle number represents movement in the *y* direction (width) and the bottom number represents the movement in the *z* direction (depth)**
  - For example: The column vector $\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$ represents **3 units** in the **positive *x* direction**, **4 units** in the **negative *y* direction** and **2 units** in the **positive *z* direction**

## What are base vectors?

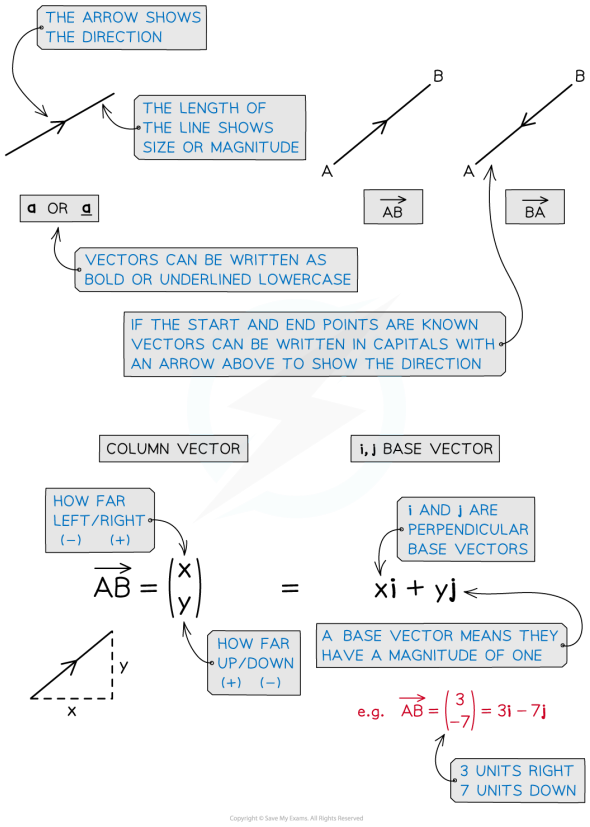
- **Base vectors use i**, **j** and **k** notation **where i**, **j** and **k** are **unit vectors** in the positive $x$, $y$, and $z$ directions respectively
  - **This is sometimes also called unit vector notation**
  - **A unit vector has a magnitude of 1**
- **In 2-dimensions i represents movement in the horizontal direction (right/left) and j represents the movement in the vertical direction (up/down)**
  - **For example:** The vector $(-4\mathbf{i} + 3\mathbf{j})$ would mean **4 units** in the **negative horizontal** ($x$) direction (i.e., **left**) and **3 units** in the **positive vertical** ($y$) direction (i.e., **up**)
- **In 3-dimensions i represents movement in the *x* direction (length), j represents movement in the *y* direction (width) and k represents the movement in the *z* direction**

www.thrivingscholars.com

**(depth)**

- **For example:** The vector $(-4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ would mean **4 units** in the **negative *x* direction**, **3 units** in the **positive *y* direction** and **1 unit** in the **negative *z* direction**

- As they are vectors, **i, j and k** are displayed in **bold** in textbooks and online but in handwriting they would be underlined ($\underline{i}$, $\underline{j}$ and $\underline{k}$)




## Exam Tip

- Practice working with all types of vector notation so that you are prepared for whatever comes up in the exam
  - Your working and answer in the exam can be in any form unless told otherwise
  - It is generally best to write your final answer in the same form as given in the question, however you will not lose marks for not doing this unless it is specified in the question
- Vectors appear in **bold** (non-italic) font in textbooks and on exam papers, etc (i.e. $\boldsymbol{F}$, $\boldsymbol{\alpha}$) but in handwriting should be underlined (i.e. $\underline{F}$, $\underline{\alpha}$)

www.thrivingscholars.com

## Worked Example

a)

Write the vector $\begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$ using **base vector** notation.

$\begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix} = -4\underline{i} + 0\underline{j} + 5\underline{k}$

$0\underline{j}$ is not needed when giving answer in base vector form.

$5\underline{k} - 4\underline{i}$

b)

Write the vector $\mathbf{k} - 2\mathbf{j}$ using **column vector** notation.

$\underline{k} - 2\underline{j} = 0\underline{i} - 2\underline{j} + 1\underline{k}$

Be careful with negative components and missing terms when working with base vectors

$\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ ← The zero term is needed when using column vector notation

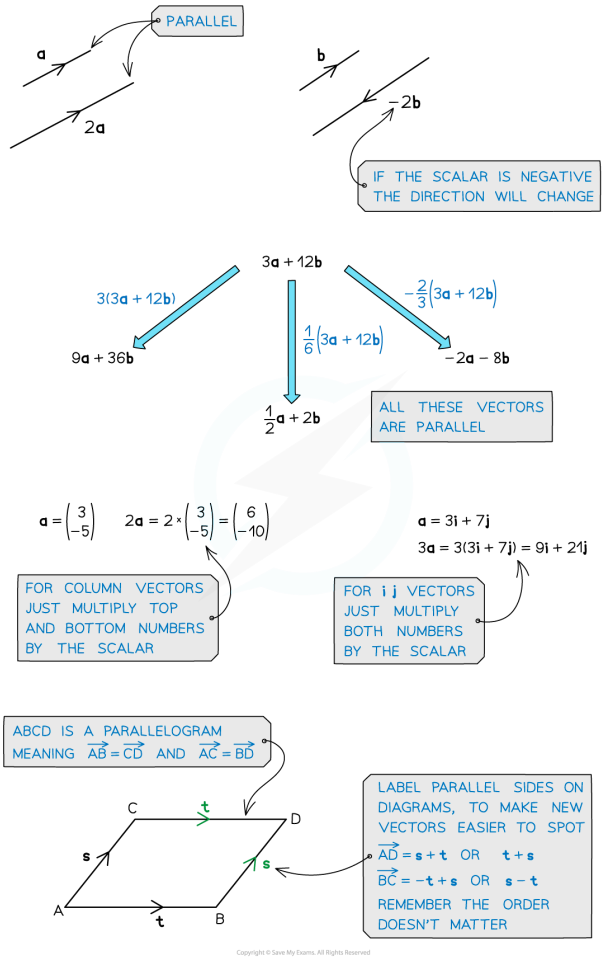
www.thrivingscholars.com

# Parallel Vectors

## How do you know if two vectors are parallel?

- Two vectors are parallel if one is a **scalar multiple** of the other
  - This means that all components of the vector have been multiplied by a **common constant (scalar)**
- Multiplying every component in a vector by a **scalar** will change the **magnitude** of the vector but not the **direction**
  - For example: the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{b} = 2\mathbf{a} = 2\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$ will have the **same direction** but the vector **b** will have twice the magnitude of **a**
    - They are **parallel**
- If a vector can be factorised by a **scalar** then it is parallel to any **scalar multiple** of the factorised vector
  - For example: The vector $9\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ can be factorised by the scalar 3 to $3(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ so the vector $9\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ is parallel to any **scalar multiple** of $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
- If a vector is multiplied by a **negative scalar** its direction will be **reversed**
  - It will still be **parallel** to the original vector
- Two vectors are **parallel** if they have the same or reverse **direction** and **equal** if they have the same **size and direction**

www.thrivingscholars.com




Copyright © Save My Exams. All Rights Reserved

## Exam Tip

- It is easiest to spot that two vectors are parallel when they are in column vector notation
  - in your exam by writing vectors in column vector form and looking for a scalar multiple you will be able to quickly determine whether they are parallel or not

www.thrivingscholars.com

## Worked Example

Show that the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$ and $\mathbf{b} = 6\mathbf{k} - 3\mathbf{i}$ are parallel and find the scalar multiple that maps **a** onto **b**.

Convert both vectors into the same form and then look for a value of k such that $\underline{a} = k\,\underline{b}$, where k is a scalar.

$\underline{a} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

$\underline{b} = 6\underline{k} - 3\underline{i} = -3\underline{i} + 0\underline{j} + 6\underline{k}$

$= \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

$= -\frac{3}{2}\underline{a}$

$\underline{b} = -\frac{3}{2}\underline{a}$, $k = -\frac{3}{2}$

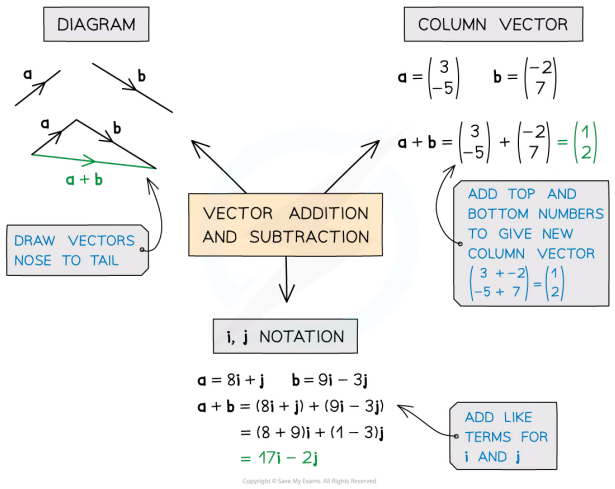
www.thrivingscholars.com



# Adding & Subtracting Vectors

## How are vectors added and subtracted numerically?

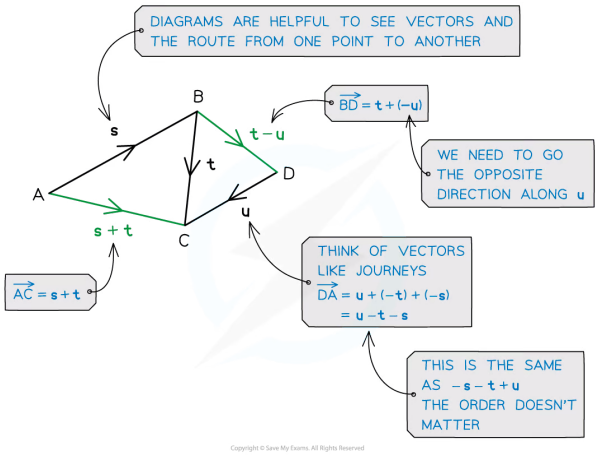
- To **add** or **subtract** vectors numerically simply add or subtract each of the corresponding components
- In **column vector** notation just add the top, middle and bottom parts together
  - For example: $\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -8 \end{pmatrix}$
- In **base vector** notation add each of the **i**, **j**, and **k** components together separately
  - For example: $(2\mathbf{i} + \mathbf{j} - 5\mathbf{k}) - (\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) = (\mathbf{i} - 3\mathbf{j} - 8\mathbf{k})$




## How are vectors added and subtracted geometrically?

- Vectors can be **added** geometrically by joining the end of one vector to the start of the next one
- The **resultant** vector will be the shortest route from the start of the first vector to the end of the second
  - A **resultant** vector is a vector that results from **adding** or **subtracting** two or more vectors
- If the two vectors have the same **starting position**, the second vector can be **translated** to the end of the first vector to find the resultant vector
  - This results in a **parallelogram** with the resultant vector as the diagonal
- To **subtract** vectors, consider this as **adding on the negative vector**
  - For example: $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$
  - The end of the **resultant vector a – b** will not be anywhere near the end of the vector **b**
    - Instead, it will be at the point where the end of the vector **–b** would be

www.thrivingscholars.com




Copyright © Save My Exams. All Rights Reserved

## Exam Tip

- Working in column vectors tends to be easiest when adding and subtracting
  - in your exam, it can help to convert any vectors into column vectors before carrying out calculations with them
- If there is no diagram, drawing one can be helpful to help you visualise the problem

## Worked Example

Find the resultant of the vectors $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$.

$\underline{a} = 5\underline{i} - 2\underline{j} + 0\underline{k} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

Writing as a column vector makes adding and subtracting easier.

$\underline{a} + \underline{b} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

Resultant vector $= 2\underline{i} - \underline{j} + 2\underline{k}$

www.thrivingscholars.com

# Position Vectors

## What is a position vector?

- A position vector describes the **position** of a point in relation to the **origin**
  - It describes the **direction** and the **distance** from the point O: 0**i** + 0**j** + 0**k** or $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
  - It is different to a **displacement vector** which describes the direction and distance between any two points
- The position vector of point A is written with the notation **a** = $\overrightarrow{OA}$
  - The origin is always denoted O
- The individual components of a position vector are the coordinates of its end point
  - For example the point with coordinates (3, -2, -1) has position vector 3**i** – 2**j** – **k**

### Worked Example

Determine the position vector of the point with coordinates (4, -1, 8).

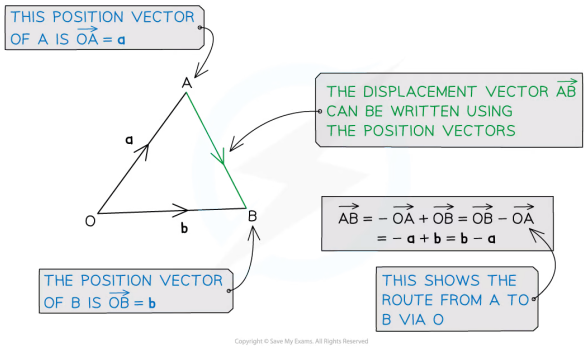
$4\underline{i} - \underline{j} + 8\underline{k}$

www.thrivingscholars.com

# Displacement Vectors

## What is a displacement vector?

- A **displacement vector** describes the shortest route between any two points
  - It describes the **direction** and the **distance** between any two points
  - It is different to a **position vector** which describes the direction and distance from the point O: 0**i** + 0**j** or $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- The displacement vector of point B from the point A is written with the notation $\overrightarrow{AB}$
- A displacement vector between two points can be written in terms of the displacement vectors of a third point
  - $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$
- A displacement vector can be written in terms of its position vectors
  - For example the displacement vector $\overrightarrow{AB}$ can be written in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$
  - $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$
  - For position vector **a** = $\overrightarrow{OA}$ and **b** = $\overrightarrow{OB}$ the displacement vector $\overrightarrow{AB}$ can be written **b** – **a**




### Exam Tip

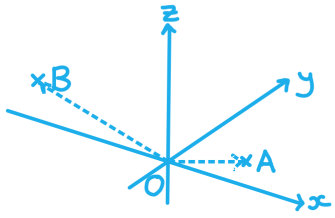
- In an exam, sketching a quick diagram can help to make working out a displacement vector easier

www.thrivingscholars.com

## Worked Example

The point A has coordinates (3, 0, -1) and the point B has coordinates (-2, -5, 7).
Find the displacement vector $\overrightarrow{AB}$.

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \qquad \overrightarrow{OB} = \begin{pmatrix} -2 \\ -5 \\ 7 \end{pmatrix}$$




$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$= -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} -2 \\ -5 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} -5 \\ -5 \\ 8 \end{pmatrix}$$

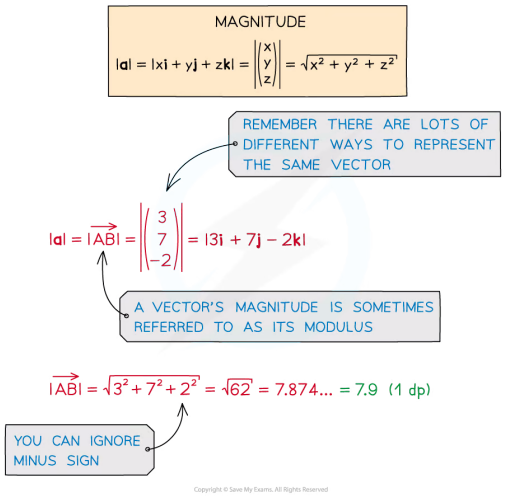
www.thrivingscholars.com



# Magnitude of a Vector

## How do you find the magnitude of a vector?

- The **magnitude** of a vector tells us its **size** or **length**
  - For a **displacement** vector it tells us the **distance** between the two points
  - For a **position** vector it tells us the **distance** of the point from the **origin**
- The magnitude of the vector $\overrightarrow{AB}$ is denoted $|\overrightarrow{AB}|$
  - The magnitude of the vector **a** is denoted $|\mathbf{a}|$
- The magnitude of a vector can be found using **Pythagoras' Theroem**
- The magnitude of a vector $\boldsymbol{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is found using
  - $|\boldsymbol{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
  - where $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
  - This is **given in the formula booklet**




## How do I find the distance between two points?

- **Vectors** can be used to find the distance (or displacement) between two points
  - It is the **magnitude** of the vector between them
- Given the **position vectors** of two points:
  - Find the displacement vector between them
  - Find the magnitude of the displacement vector between them

### Exam Tip

- Finding the magnitude of a vector is the same as finding the distance between two coordinates, it is a useful formula to commit to memory in order to save time in the exam, however it is in your formula booklet if you need it

www.thrivingscholars.com

## Worked Example

Find the magnitude of the vector AB = 4**i** – **j** + 2**k**.

| Magnitude of a vector | $\lvert \mathbf{v} \rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$, where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ |
|---|---|

$$|\overrightarrow{AB}| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$$

$$|\overrightarrow{AB}| = \sqrt{21}$$

## Unit Vectors

**What is a unit vector?**

- A **unit vector** has a **magnitude** of 1
- It can be found by dividing a vector by its **magnitude**
  - This will result in a vector with a size of 1 unit in the direction of the original vector
- A unit vector in the direction of **a** is denoted $\frac{\mathbf{a}}{|\mathbf{a}|}$
  - For example a unit vector in the direction 3**i** – 4**j** is $\frac{(3\mathbf{i} - 4\mathbf{j})}{\sqrt{3^2 + 4^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

## Exam Tip

- Finding the unit vector will not be a question on its own but will be a useful skill for further vectors problems so it is important to be confident with it

## Worked Example

Find the unit vector in the direction 2**i** – 2**j** + **k**.

Let $\underline{a} = 2\underline{i} - 2\underline{j} + \underline{k}$

Find the magnitude of $\underline{a}$

| Magnitude of a vector | $\lvert \mathbf{v} \rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$, where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ |
|---|---|

$$|\underline{a}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

Divide $\underline{a}$ by its magnitude:

$$\text{Unit vector} = \frac{\underline{a}}{|\underline{a}|} = \frac{2\underline{i} - 2\underline{j} + \underline{k}}{3}$$

$$\frac{2}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{1}{3}\underline{k}$$

www.thrivingscholars.com



# The Scalar ('Dot') Product

## What is the scalar product?

- The scalar product (also known as the dot product) is one form in which two vectors can be combined together
- The scalar product between two vectors **a** and **b** is denoted $\mathbf{a} \cdot \mathbf{b}$
- The result of taking the scalar product of two vectors is a **real number**
  - i.e. a scalar
- The scalar product of two vectors gives information about the angle between the two vectors
  - If the scalar product is **positive** then the angle between the two vectors is **acute** (less than 90°)
  - If the scalar product is **negative** then the angle between the two vectors is **obtuse** (between 90° and 180°)
  - If the scalar product is **zero** then the angle between the two vectors is **90°** (the two vectors are **perpendicular**)

## How is the scalar product calculated?

- There are **two methods** for calculating the scalar product
- The most common method used to find the scalar product between the two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is to find the **sum of the product of each component** in the two vectors
  - $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$
  - Where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
  - This is **given in the formula booklet**
- The scalar product is also equal to the **product of the magnitudes** of the two vectors and the **cosine of the angle between them**
  - $\boldsymbol{v} \cdot \boldsymbol{w} = |v||w| \cos\theta$
  - Where $\theta$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
    - The two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are joined at the start and pointing away from each other
- The scalar product can be used in the second formula to find the angle between the two vectors

## What properties of the scalar product do I need to know?

- The order of the vectors doesn't change the result of the scalar product (it is **commutative**)
  - $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$
- The **distributive law** over addition can be used to 'expand brackets'
  - $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$
- The scalar product is **associative** with respect to multiplication by a scalar
  - $(k\boldsymbol{v}) \cdot (\boldsymbol{w}) = k(\boldsymbol{v} \cdot \boldsymbol{w})$
- The scalar product between a vector and itself is equal to the **square of its magnitude**
  - $\boldsymbol{v} \cdot \boldsymbol{v} = |\boldsymbol{v}|^2$

www.thrivingscholars.com

- If two vectors, $\boldsymbol{v}$ and $\boldsymbol{w}$, are **parallel** then the magnitude of the scalar product is equal to the **product** of the magnitudes of the vectors
  - $|\boldsymbol{v} \cdot \boldsymbol{w}| = |\boldsymbol{w}||\boldsymbol{v}|$
  - This is because cos 0° = 1 and cos 180° = -1
- If two vectors are **perpendicular** the scalar product is **zero**
  - This is because cos 90° = 0

### Exam Tip

- Whilst the formulae for the scalar product are given in the formula booklet, the properties of the scalar product are not, however they are important and it is likely that you will need to recall them in your exam so be sure to commit them to memory

www.thrivingscholars.com

## Worked Example

Calculate the scalar product between the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ and $\boldsymbol{w} = 3\mathbf{j} - 2\mathbf{k} - \mathbf{i}$ using:

i)
the formula $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$,

$\underline{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} = 2\underline{i} + 0\underline{j} - 5\underline{k}$

$\underline{w} = 3\underline{j} - 2\underline{k} - \underline{i} = -1\underline{i} + 3\underline{j} - 2\underline{k}$

Be aware of the order of the terms.

| Scalar product | $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ |
|---|---|

$\underline{v} \cdot \underline{w} = (2 \times -1) + (0 \times 3) + (-5 \times -2) = -2 + 10$

$\underline{v} \cdot \underline{w} = 8$

ii)
the formula $\boldsymbol{v} \cdot \boldsymbol{w} = |v||w|\cos\theta$, given that the angle between the two vectors is 66.6°.

$\underline{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} = 2\underline{i} + 0\underline{j} - 5\underline{k}$ $\quad \underline{w} = -1\underline{i} + 3\underline{j} - 2\underline{k}$

| Scalar product | $\boldsymbol{v} \cdot \boldsymbol{w} = \lvert\boldsymbol{v}\rvert \lvert\boldsymbol{w}\rvert \cos\theta$ |
|---|---|

Find the magnitude of both vectors:

$|\underline{v}| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$ $\quad |\underline{w}| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$

$\underline{v} \cdot \underline{w} = \sqrt{29} \times \sqrt{14} \cos 66.6°$

$\underline{v} \cdot \underline{w} = 8$

www.thrivingscholars.com

# Angle Between Two Vectors

## How do I find the angle between two vectors?

- If two vectors with different directions are placed at the same starting position, they will form an angle between them
- The two formulae for the scalar product can be used together to find this angle
  - $\cos\theta = \dfrac{v_1w_1 + v_2w_2 + v_3w_3}{|\boldsymbol{v}||\boldsymbol{w}|}$
  - This is given in the formula booklet
- To find the angle between two vectors:
  - Calculate the scalar product between them
  - Calculate the magnitude of each vector
  - Use the formula to find $\cos\theta$
  - Use inverse trig to find $\theta$

### Exam Tip

- The formula for this is given in the formula booklet so you do not need to remember it but make sure that you can find it quickly and easily in your exam

www.thrivingscholars.com

## Worked Example

Calculate the angle formed by the two vectors $\boldsymbol{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\boldsymbol{w} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

$\underline{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ , $\underline{w} = 3\underline{i} + 4\underline{j} - \underline{k} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

Start by finding the scalar product:

$\underline{v} \cdot \underline{w} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

$= (-1 \times 3) + (3 \times 4) + (2 \times -1) = 7$

Find the magnitude of both vectors:

$|\underline{v}| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{14}$

$|\underline{w}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26}$

| Angle between two vectors | $\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert}$ |
|---|---|

$\cos\theta = \frac{7}{\sqrt{14} \times \sqrt{26}} = 0.3668...$

$\theta = \cos^{-1}(0.3668...)$

$\theta = 68.5^\circ$ (3sf)

www.thrivingscholars.com

# Perpendicular Vectors

## How do I know if two vectors are perpendicular?

- If the **scalar product** of two (non-zero) vectors is **zero** then they are **perpendicular**
  - If $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ then $\boldsymbol{v}$ and $\boldsymbol{w}$ must be perpendicular to each other
- Two vectors are **perpendicular** if their **scalar product** is **zero**
  - The value of $\cos\theta = 0$ therefore $|\boldsymbol{v}||\boldsymbol{w}|\cos\theta = 0$

### Worked Example

Find the value of $t$ such that the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ t \\ 5 \end{pmatrix}$ and $\boldsymbol{w} = (t-1)\mathbf{i} - \mathbf{j} + \mathbf{k}$ are perpendicular to each other.

The two vectors $\underline{v}$ and $\underline{w}$ are perpendicular if $\underline{v} \cdot \underline{w} = 0$.

$\underline{v} = \begin{pmatrix} 2 \\ t \\ 5 \end{pmatrix}$, $\underline{w} = \begin{pmatrix} t-1 \\ -1 \\ 1 \end{pmatrix}$

$\underline{v} \cdot \underline{w} = 2(t-1) + t(-1) + 5(1)$

$= 2t - 2 - t + 5$

Therefore $\underline{v}$ and $\underline{w}$ are perpendicular if

$t + 3 = 0$

$t = -3$

www.thrivingscholars.com





# Geometric Proof with Vectors

## How can vectors be used to prove geometrical properties?

- If two vectors can be shown to be **parallel** then this can be used to prove parallel lines
  - If two vectors are **scalar multiples** of each other then they are **parallel**
  - To prove that two vectors are parallel simply show that one is a scalar multiple of the other
- If two vectors can be shown to be **perpendicular** then this can be used to prove perpendicular lines
  - If the **scalar product** is zero then the two vectors are **perpendicular**
- If two vectors can be shown to have equal **magnitude** then this can be used to prove two lines are the **same length**
- To prove a 2D shape is a **parallelogram** vectors can be used to
  - Show that there are two pairs of **parallel sides**
  - Show that the **opposite sides** are of **equal length**
    - The vectors opposite each other with be **equal**
  - If the angle between two of the vectors is shown to be 90° then the parallelogram is a **rectangle**
- To prove a 2D shape is a **rhombus** vectors can be used to
  - Show that there are two pairs of **parallel sides**
    - The vectors opposite each other with be **equal**
  - Show that **all four sides** are of **equal length**
  - If the angle between two of the vectors is shown to be 90° then the rhombus is a **square**

## How are vectors used to follow paths through a diagram?

- In a geometric diagram the vector $\overrightarrow{AB}$ forms a path from the point A to the point B
  - This is specific to the path AB
  - If the vector $\overrightarrow{AB}$ is labelled **a** then any other vector with the same **magnitude** and **direction** as **a** could also be labelled **a**
- The vector $\overrightarrow{BA}$ would be labelled **–a**
  - It is **parallel** to **a** but pointing in the **opposite direction**
- If the point M is exactly halfway between A and B it is called the midpoint of A and the vector $\overrightarrow{AM}$ could be labelled $\frac{1}{2}\mathbf{a}$
- If there is a point X on the line AB such that $\overrightarrow{AX} = 2\overrightarrow{XB}$ then X is two-thirds of the way along the line $\overrightarrow{AB}$
  - Other ratios can be found in similar ways
  - A diagram often helps to visualise this
- If a point X divides a line segment AB into the ratio p : q then
  - $\overrightarrow{AX} = \frac{p}{p+q}\overrightarrow{AB}$
  - $\overrightarrow{XB} = \frac{q}{p+q}\overrightarrow{AB}$

www.thrivingscholars.com

## How can vectors be used to find the midpoint of two vectors?

- If the point A has position vector **a** and the point B has position vector **b** then the **position vector** of the **midpoint** of $\overrightarrow{AB}$ is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$
  - The **displacement vector** $\overrightarrow{AB} = \mathbf{b}-\mathbf{a}$
  - Let **M** be the midpoint of $\overrightarrow{AB}$ then $\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB}) = \frac{1}{2}(\mathbf{b}-\mathbf{a})$
  - The **position vector** $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \mathbf{a} + \frac{1}{2}(\mathbf{b}-\mathbf{a}) = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$

## How can vectors be used to prove that three points are collinear?

- Three points are collinear if they all **lie on the same line**
  - The vectors between the three points will be **scalar multiples** of each other
- The points A, B and C are collinear if $\overrightarrow{AB} = k\overrightarrow{BC}$
- If the points A, B and M are collinear and $\overrightarrow{AM} = \overrightarrow{MB}$ then M is the **midpoint** of $\overrightarrow{AB}$

### Exam Tip

- Think of vectors like a journey from one place to another
  - You may have to take a detour e.g. A to B might be A to O then O to B
- Diagrams can help, if there isn't one, draw one
  - If a diagram has been given begin by labelling all known quantities and vectors

www.thrivingscholars.com

## Worked Example

Use vectors to prove that the points *A*, *B*, *C* and *D* with position vectors $\mathbf{a} = (3\mathbf{i} - 5\mathbf{j} - 4\mathbf{k})$, $\mathbf{b} = (8\mathbf{i} - 7\mathbf{j} - 5\mathbf{k})$, $\mathbf{c} = (3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$ and $\mathbf{d} = (5\mathbf{k} - 2\mathbf{i})$ are the vertices of a parallelogram.

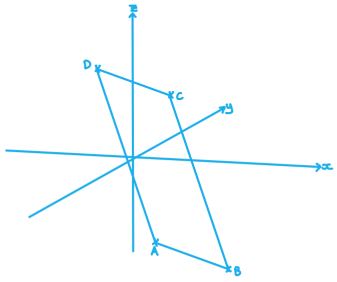
Find the displacement vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$$

$$\overrightarrow{BC} = \underline{c} - \underline{b} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 9 \end{pmatrix}$$

$$\overrightarrow{CD} = \underline{d} - \underline{c} = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$$

$$\overrightarrow{DA} = \underline{a} - \underline{d} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ -9 \end{pmatrix}$$




$\overrightarrow{AB} = -\overrightarrow{CD}$ and $\overrightarrow{BC} = -\overrightarrow{DA}$ $\therefore$ ABCD must be a parallelogram

www.thrivingscholars.com

# 3.10 Vector Equations of Lines

## 3.10.1 Vector Equations of Lines

### Equation of a Line in Vector Form

### How do I find the vector equation of a line?

- The formula for finding the **vector equation** of a line is
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
    - Where $\boldsymbol{r}$ is the **position vector** of any point on the line
    - $\boldsymbol{a}$ is the **position vector** of a known point on the line
    - $\boldsymbol{b}$ is a **direction** (displacement) **vector**
    - $\lambda$ is a scalar
  - This is **given in the formula booklet**
  - This equation can be used for vectors in both 2– and 3– dimensions
- This formula is similar to a regular equation of a straight line in the form $y = mx + c$ but with a vector to show both a point on the line and the direction (or gradient) of the line
  - In 2D the gradient can be found from the direction vector
  - In 3D a numerical value for the direction cannot be found, it is given as a vector
- As $\boldsymbol{a}$ could be the position vector of **any** point on the line and $\boldsymbol{b}$ could be **any scalar multiple** of the direction vector there are infinite vector equations for a single line
- Given any two points on a line with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ the **displacement** vector can be written as $\boldsymbol{b} - \boldsymbol{a}$
  - So the formula $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$ can be used to find the vector equation of the line
  - This is **not given in the formula booklet**

### How do I determine whether a point lies on a line?

- Given the equation of a line $\boldsymbol{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ the point $\mathbf{c}$ with position vector $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ is on the line if there exists a value of $\lambda$ such that
  - $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$
  - This means that there exists a single value of $\lambda$ that satisfies the three equations:
    - $c_1 = a_1 + \lambda b_1$
    - $c_2 = a_2 + \lambda b_2$
    - $c_3 = a_3 + \lambda b_3$
- A GDC can be used to solve this system of linear equations for
  - The point only lies on the line if a single value of $\lambda$ exists for all three equations
- Solve one of the equations first to find a value of $\lambda$ that satisfies the first equation and then check that this value also satisfies the other two equations
- If the value of $\lambda$ does not satisfy all three equations, then the point $\mathbf{c}$ does not lie on the line

www.thrivingscholars.com

## Exam Tip

- Remember that the vector equation of a line can take many different forms
  - This means that the answer you derive might look different from the answer in a mark scheme
- You can choose whether to write your vector equations of lines using unit vectors or as column vectors
  - Use the form that you prefer, however column vectors is generally easier to work with

## Worked Example

a)

Find a vector equation of a straight line through the points with position vectors **a** = 4**i** – 5**k** and **b** = 3**i** – 3**k**

Use the position vectors to find the displacement vector between them.

$\vec{OA} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$, $\vec{OB} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \Rightarrow \vec{AB} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

| Vector equation of a line | $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ |
|---|---|

position vector of point a; direction vector

$\boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ or $\boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

position vector of point b; direction vector

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

b)

Determine whether the point C with coordinate (2, 0, -1) lies on this line.

Let $C = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, then check to see if there exists a value of $\lambda$ such that

$$\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

From the 'i' component: $4 - \lambda = 2$ ①

From the 'j' component: $0 + 0\lambda = 0$ ② (✓) Works for all $\lambda$

From the 'k' component: $-5 + 2\lambda = -1$ ③

① $\Rightarrow \lambda = 2$ sub into ③ $\Rightarrow -5 + (2 \times 2) = -5 + 4 = -1$ ✓

Point C lies on the line

www.thrivingscholars.com

# Equation of a Line in Parametric Form

## How do I find the vector equation of a line in parametric form?

- By considering the three separate components of a vector in the $x$, $y$ and $z$ directions it is possible to write the **vector equation** of a line as **three separate equations**
  - Letting $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ becomes
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
    - Where $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ is a position vector and $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ is a direction vector
  - This vector equation can then be split into its three separate component forms:
    - $x = x_0 + \lambda l$
    - $y = y_0 + \lambda m$
    - $z = z_0 + \lambda n$
  - These are **given in the formula booklet**

### Worked Example

Write the parametric form of the equation of the line which passes through the point (-2, 1, 0) with direction vector $\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$.

| | |
|---|---|
| Parametric form of the equation of a line | $x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$ |

Use $r = a + \lambda b$ to write the equation in vector form first:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$$

position vector of a point

direction vector

Separate the components into their 3 separate equations.

$$x = -2 + 3\lambda$$
$$y = 1 + \lambda$$
$$z = -4\lambda$$

www.thrivingscholars.com

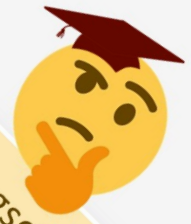
# Equation of a Line in Cartesian Form

- The **Cartesian** equation of a line can be found from the **vector equation of a line** by
  - Finding the vector equation of the line in parametric form
  - Eliminating $\lambda$ from the parametric equations
    - $\lambda$ can be eliminated by **making it the subject** of each of the parametric equations
    - For example: $x = x_0 + \lambda l$ gives $\lambda = \dfrac{x - x_0}{l}$
- In **2D** the **cartesian equation of a line** is a regular equation of a straight line simply given in the form
  - $y = mx + c$
  - $ax + by + d = 0$
  - $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ by rearranging $y - y_1 = m(x - x_1)$
- In **3D** the **cartesian equation of a line** also includes $z$ and is given in the form
  - $\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}$
  - where $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
  - This is **given in the formula booklet**
- If one of your variables **does not depend on** $\lambda$ then this part can be written as a separate equation
  - For example: $m = 0 \Rightarrow y = y_0$ gives $\dfrac{x - x_0}{l} = \dfrac{z - z_0}{n}$ , $y = y_0$

## How do I find the vector equation of a line given the Cartesian form?

- If you are given the **Cartesian** equation of a line in the form
  - $\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}$
- A vector equation of the line can be found by
  - STEP 1: Set each part of the equation equal to $\lambda$ individually
  - STEP 2: Rearrange each of these three equations (or two if working in 2D) to make $x$, $y$, and $z$ the subjects
    - This will give you the three **parametric equations**
    - $x = x_0 + \lambda l$
    - $y = y_0 + \lambda m$
    - $z = z_0 + \lambda n$
  - STEP 3: Write this in the vector form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

www.thrivingscholars.com



- STEP 4: Set $r$ to equal $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

- If one part of the cartesian equation is given separately and is not in terms of $\lambda$ then the corresponding component in the direction vector is equal to zero

## Worked Example

A line has the vector equation $r = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$. Find the Cartesian equation of the line.

| | |
|---|---|
| Cartesian equations of a line | $\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$ |

Begin by writing the equation of the line in parametric form:

$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \Rightarrow$

$x = 1 + 4\lambda$ ①

$y = -2\lambda$ ②

$z = 2 + \lambda$ ③

Rearrange each equation to make $\lambda$ the subject:

① $\lambda = \frac{x-1}{4}$

② $\lambda = \frac{y}{-2}$

③ $\lambda = z - 2$

Set each expression for $\lambda$ equal to each other:

$$\frac{x-1}{4} = \frac{y}{-2} = z - 2$$

www.thrivingscholars.com





# Kinematics using Vectors

## How are vectors related to kinematics?

- Vectors are often used in questions in the context of forces, acceleration or velocity
- If an object is moving in **one dimension** then its velocity, displacement and time are related using the formula $\boldsymbol{s} = \boldsymbol{v}t$
  - where $s$ is **displacement**, $v$ is **velocity** and $t$ is the **time taken**
- If an object is moving in **more than one dimension** then **vectors** are needed to represent its **velocity** and **displacement**
  - Whilst **time** is a **scalar quantity**, **displacement** and **velocity** are both **vector quantities**
- For an object moving at a **constant speed** in a **straight line** its velocity, displacement and time can be related using the vector equation of a line
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
  - Letting
    - $\boldsymbol{r}$ be the position of the object at the time, $t$
    - $\boldsymbol{a}$ be the position vector, $\boldsymbol{r_0}$ at the start ($t = 0$)
    - $\lambda$ represent the time, $t$
    - $\boldsymbol{b}$ be the **velocity** vector, $\boldsymbol{v}$
  - Then the position of the object at the time, $t$ can be given by
    - $\boldsymbol{r} = \boldsymbol{r_0} + t\,\mathbf{v}$
  - The speed of the object will be the magnitude of the velocity $|\boldsymbol{v}|$



### Exam Tip

- Kinematics questions can have a lot of information in, read them carefully and pick out the parts that are essential to the question
- Look out for where variables used are the same and/or different within vector equations, you will need to use different techniques to find these

www.thrivingscholars.com

## Worked Example

A car, moving at constant speed, takes 2 minutes to drive in a straight line from point A (-4, 3) to point B (6, -5).

At time $t$, in minutes, the position vector ($\boldsymbol{p}$) of the car relative to the origin can be given in the form $\boldsymbol{p} = \boldsymbol{a} + t\boldsymbol{b}$.

Find the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Vector $\underline{a}$ represents the initial position and vector $\underline{b}$ represents the direction vector per minute.

Position vector $\overrightarrow{OA} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

At $t = 0$ minutes, $\underline{p} = \underline{a}$ so $\underline{a} = \overrightarrow{OA} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

Position vector $\overrightarrow{OB} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$

At $t = 2$ minutes, the car is at the point B and so $\overrightarrow{OB} = \underline{a} + 2\underline{b}$

$\begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + 2\underline{b}$

Direction vector $2\underline{b} = \begin{pmatrix} 6 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \end{pmatrix}$

$\underline{a} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$

www.thrivingscholars.com



# Coincident, Parallel, Intersecting & Skew Lines

## How do I tell if two lines are parallel?

- Two lines are parallel if, and only if, their **direction vectors** are **parallel**
  - This means the direction vectors will be **scalar multiples** of each other
  - For example, the lines whose equations are $\mathbf{r}=\begin{pmatrix}2\\1\\-7\end{pmatrix}+\lambda_1\begin{pmatrix}2\\0\\-8\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}1\\-1\\5\end{pmatrix}+\lambda_2\begin{pmatrix}-1\\0\\4\end{pmatrix}$ are parallel
    - This is because $\begin{pmatrix}2\\0\\-8\end{pmatrix}=-2\begin{pmatrix}-1\\0\\4\end{pmatrix}$

## How do I tell if two lines are coincident?

- **Coincident lines** are two lines that lie directly on top of each other
  - They are indistinguishable from each other
- Two parallel lines will either **never intersect** or they are **coincident (identical)**
  - Sometimes the vector equations of the lines may look different
    - for example, the lines represented by the equations $\mathbf{r}=\begin{pmatrix}1\\-8\end{pmatrix}+s\begin{pmatrix}-4\\8\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}-3\\0\end{pmatrix}+t\begin{pmatrix}1\\-2\end{pmatrix}$ are **coincident**,
  - To check whether two lines are **coincident**:
    - First check that they are **parallel**
      - They are because $\begin{pmatrix}-4\\8\end{pmatrix}=-4\begin{pmatrix}1\\-2\end{pmatrix}$ and so their direction vectors are parallel
    - Next, determine whether **any point** on one of the lines also lies on the other
      - $\begin{pmatrix}1\\-8\end{pmatrix}$ is the position vector of a point on the first line and $\begin{pmatrix}1\\-8\end{pmatrix}=\begin{pmatrix}-3\\0\end{pmatrix}+4\begin{pmatrix}1\\-2\end{pmatrix}$ so it also lies on the second line
    - If two parallel lines share **any point**, then they share **all points** and are **coincident**

## What are skew lines?

- Lines that are **not parallel** and which **do not intersect** are called **skew lines**
  - This is only possible in **3-dimensions**

## How do I determine whether lines in 3 dimensions are parallel, skew, or intersecting?

- First, look to see if the direction vectors are parallel:
  - if the **direction vectors are parallel**, then the **lines are parallel**

www.thrivingscholars.com



- if the **direction vectors are not parallel**, the **lines are not parallel**
- If the lines are **parallel**, check to see if the lines are **coincident**:
  - If they **share any point**, then they are **coincident**
  - If **any point** on one line is **not on the other line**, then the lines are **not coincident**
- If the lines are **not parallel**, check whether they **intersect**:
  - STEP 1: Set the vector equations of the two lines equal to each other with **different variables**
    - e.g. $\lambda$ and $\mu$, for the parameters
  - STEP 2: Write the three separate equations for the **i**, **j**, and **k** components in terms of $\lambda$ and $\mu$
  - STEP 3: **Solve** two of the equations to find a value for $\lambda$ and $\mu$
  - STEP 4: **Check** whether the values of $\lambda$ and $\mu$ you have found satisfy the third equation
    - If **all three** equations are satisfied, then the lines **intersect**
    - If **not all three** equations are satisfied, then the lines are **skew**

## How do I find the point of intersection of two lines?

- If a pair of lines are **not parallel** and **do intersect**, a unique point of intersection can be found
  - If the two lines intersect, there will be a single point that will lie on both lines
- Follow the steps above to find the values of $\lambda$ and $\mu$ that satisfy **all three equations**
  - STEP 5: Substitute either the value of $\lambda$ or the value of $\mu$ into one of the vector equations to find the position vector of the point where the lines intersect
  - It is always a good idea to **check** in the other equations as well, you should get the same point for each line

### Exam Tip

- Make sure that you use different letters, e.g. $\lambda$ and $\mu$, to represent the parameters in vector equations of different lines
  - Check that the variable you are using has not already been used in the question

www.thrivingscholars.com

## Worked Example

Determine whether the following pair of lines are parallel, intersect, or are skew.

$\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} + s(5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and $\boldsymbol{r} = -5\mathbf{i} + 4\mathbf{j} + \mathbf{k} + t(2\mathbf{i} - \mathbf{j})$.

STEP 1: Check to see if the lines are parallel:

$$\mathbf{r}_1 = \begin{pmatrix}4\\3\\0\end{pmatrix} + \lambda\begin{pmatrix}5\\2\\3\end{pmatrix} \qquad \mathbf{r} = \begin{pmatrix}-5\\4\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\0\end{pmatrix}$$

direction vectors

The lines are not parallel because there is no value of k such that $\begin{pmatrix}5\\2\\3\end{pmatrix} = k\begin{pmatrix}2\\-1\\0\end{pmatrix}$

STEP 2: Check to see if the lines intersect:

$4 + 5\lambda = -5 + 2\mu$ ①

$3 + 2\lambda = 4 - \mu$ ②

$3\lambda = 1$ ③

Set up three equations for each of the $\underline{i}$, $\underline{j}$ and $\underline{k}$ components.

Equation ③: $\lambda = \frac{1}{3}$ Sub into ②: $3 + 2\left(\frac{1}{3}\right) = 4 - \mu$

$\frac{11}{3} = 4 - \mu$

$\mu = \frac{1}{3}$

Sub into ①: $4 + 5\left(\frac{1}{3}\right) = -5 + 2\left(\frac{1}{3}\right)$

$\frac{17}{3} \neq -\frac{13}{3}$ contradiction

There is no point of intersection.

**The lines are skew**

www.thrivingscholars.com

# Angle Between Two Lines

## How do we find the angle between two lines?

- The angle between two lines is equal to the angle between their **direction vectors**
  - It can be found using the **scalar product** of their direction vectors
- Given two lines in the form $\boldsymbol{r} = \boldsymbol{a}_1 + \lambda \boldsymbol{b}_1$ and $\boldsymbol{r} = \boldsymbol{a}_2 + \lambda \boldsymbol{b}_2$ use the formula
  - $\theta = \cos^{-1}\left(\dfrac{\boldsymbol{b}_1 \bullet \boldsymbol{b}_2}{\left|\boldsymbol{b}_1\right|\left|\boldsymbol{b}_2\right|}\right)$
- If you are given the equations of the lines in a different form or two points on a line you will need to find their direction vectors first
- To find the angle ABC the vectors BA and BC would be used, both starting from the point B
- The intersection of two lines will always create **two angles**, an acute one and an obtuse one
  - These two angles will add to 180°
  - You may need to subtract your answer from 180° to find the angle you are looking for
  - A **positive scalar product** will result in the **acute angle** and a **negative scalar product** will result in the **obtuse angle**
    - Using the absolute value of the scalar product will always result in the acute angle

### Exam Tip

- In your exam read the question carefully to see if you need to find the acute or obtuse angle
  - When revising, get into the practice of double checking at the end of a question whether your angle is acute or obtuse and whether this fits the question

www.thrivingscholars.com

## Worked Example

Find the acute angle, in radians between the two lines defined by the equations:

$$l_1:\ \boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \text{ and } l_2:\ \boldsymbol{b} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}$$

STEP 1: Find the scalar product of the direction vectors:

$\begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix} = (1 \times -3) + (-4 \times 2) + (-3 \times 5) = -3 + (-8) + (-15) = -26$

negative, so the angle will be the obtuse angle.

STEP 2: Find the magnitudes of the direction vectors:

$\sqrt{(1)^2 + (-4)^2 + (-3)^2} = \sqrt{26}$ $\quad$ $\sqrt{(-3)^2 + (2)^2 + (5)^2} = \sqrt{38}$

STEP 3: Find the angle: $\cos\theta = \dfrac{|-26|}{\sqrt{26}\sqrt{38}}$

Using the absolute value will result in the acute angle

$$\theta = \cos^{-1}\left(\frac{26}{\sqrt{26}\sqrt{38}}\right)$$

$\theta = 0.597$ radians (3sf)

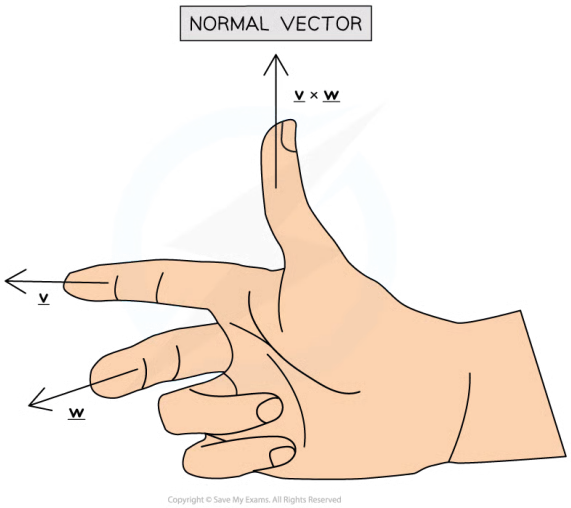
www.thrivingscholars.com



# The Vector ('Cross') Product

## What is the vector (cross) product?

- The **vector product** (also known as the **cross** product) is a form in which two vectors can be combined together
- The vector product between two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is denoted $\boldsymbol{v} \times \boldsymbol{w}$
- The result of taking the vector product of two vectors is a **vector**
- The **vector product** is a vector **in a plane** that is **perpendicular** to the two vectors from which it was calculated
  - This could be in either direction, depending on the angle between the two vectors
  - The **right-hand** rule helps you see which direction the vector product goes in
    - By pointing your index finger and your middle finger in the direction of the two vectors your thumb will automatically go in the direction of the vector product




## How do I find the vector (cross) product?

- There are **two methods** for calculating the vector product
- The **vector product** of the two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ can be written in **component form** as follows:
  - $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} \boldsymbol{v}_2\boldsymbol{w}_3 - \boldsymbol{v}_3\boldsymbol{w}_2 \\ \boldsymbol{v}_3\boldsymbol{w}_1 - \boldsymbol{v}_1\boldsymbol{w}_3 \\ \boldsymbol{v}_1\boldsymbol{w}_2 - \boldsymbol{v}_2\boldsymbol{w}_1 \end{pmatrix}$
  - Where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
  - This is **given in the formula booklet**
- The vector product can also be found in terms of its **magnitude** and **direction**
- The **magnitude of the vector product** is equal to the **product of the magnitudes** of the two vectors and the **sine of the angle between them**

www.thrivingscholars.com



- $|\boldsymbol{v} \times \boldsymbol{w}| = |v||w|\sin\theta$
- Where $\theta$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
  - The two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are joined at the start and pointing away from each other
- This is **given in the formula booklet**

- The **direction of the vector product** is **perpendicular** to both $\boldsymbol{v}$ and $\boldsymbol{w}$

## What properties of the vector product do I need to know?

- The order of the vectors **is important** and **changes the result** of the vector product
  - $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{w} \times \boldsymbol{v}$
  - However
  - $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$
- The **distributive law** can be used to 'expand brackets'
  - $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$
  - Where $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are all vectors
- Multiplying a **scalar** by a vector gives the result:
  - $(k\boldsymbol{v}) \times \boldsymbol{w} = \boldsymbol{v} \times (k\boldsymbol{w}) = k(\boldsymbol{v} \times \boldsymbol{w})$
- The vector product between a vector and itself is equal to **zero**
  - $\boldsymbol{v} \times \boldsymbol{v} = 0$
- If two vectors are **parallel** then the vector product is **zero**
  - This is because sin 0° = sin 180° = 0
- If $\boldsymbol{v} \times \boldsymbol{w} = 0$ then $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel if they are non-zero
- If two vectors, $\boldsymbol{v}$ and $\boldsymbol{w}$, are **perpendicular** then the magnitude of the vector product is equal to the **product** of the magnitudes of the vectors
  - $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{w}||\boldsymbol{v}|$
  - This is because sin 90° = 1

### Exam Tip

- The formulae for the vector product are given in the formula booklet, make sure you use them as this is an easy formula to get wrong
- The properties of the vector product are not given in the formula booklet, however they are important and it is likely that you will need to recall them in your exam so be sure to commit them to memory

www.thrivingscholars.com

## Worked Example

Calculate the magnitude of the vector product between the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ and $\boldsymbol{w} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ using

i)

the formula $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$,

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ $\quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$

Use the formula to find the cross-product:

$$\underline{v} \times \underline{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} = \begin{pmatrix} (0)(-1) - (-5)(-2) \\ (-5)(3) - (2)(-1) \\ (2)(-2) - (0)(3) \end{pmatrix} = \begin{pmatrix} -10 \\ -13 \\ -4 \end{pmatrix}$$

Find the magnitude of $\underline{v} \times \underline{w}$:

$$|\underline{v} \times \underline{w}| = \sqrt{(-10)^2 + (-13)^2 + (-4)^2} = \sqrt{285}$$

$$|\underline{v} \times \underline{w}| = 16.9 \text{ (3sf)}$$

ii)

the formula , given that the angle between them is 1 radian.

Find the magnitude of $\underline{v}$ and $\underline{w}$:

$$|\underline{v}| = \sqrt{2^2 + 0^2 + (-5)^2} = \sqrt{29}$$

$$|\underline{w}| = \sqrt{3^2 + (-2)^2 + (-1)^2} = \sqrt{14}$$

$$|\underline{v} \times \underline{w}| = |\underline{v}||\underline{w}| \sin\theta$$

$$= \sqrt{29} \times \sqrt{14} \sin(1^c)$$

$$|\underline{v} \times \underline{w}| = 17.0 \text{ (3sf)}$$

www.thrivingscholars.com

# Areas using Vector Product

## How do I use the vector product to find the area of a parallelogram?

- The **area of the parallelogram** with two adjacent sides formed by the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is equal to **the magnitude of the vector product** of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$
  - $A = |\boldsymbol{v} \times \boldsymbol{w}|$ where $\boldsymbol{v}$ and $\boldsymbol{w}$ form two **adjacent sides** of the parallelogram
    - This is **given in the formula booklet**

## How do I use the vector product to find the area of a triangle?

- The **area of the triangle** with two sides formed by the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is equal to **half of the magnitude of the vector product** of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$
  - $A = \frac{1}{2}|\boldsymbol{v} \times \boldsymbol{w}|$ where $\boldsymbol{v}$ and $\boldsymbol{w}$ form two **sides** of the triangle
    - This is **not** given in the formula booklet

### Exam Tip

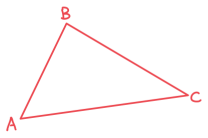
- The formula for the area of the parallelogram is given in the formula booklet but the formula for the area of a triangle is not
  - Remember that the area of a triangle is half the area of a parallelogram

www.thrivingscholars.com

## Worked Example

Find the area of the triangle enclosed by the coordinates (1, 0, 5), (3, -1, 2) and (2, 0, -1).

Let A be (1, 0, 5), B be (3, -1, 2) and C be (2, 0, -1)




You can use any two direction vectors moving away from any vertex.

Find the two direction vectors $\vec{AB}$ and $\vec{AC}$

$$\vec{AB} = \begin{pmatrix}3\\-1\\2\end{pmatrix} - \begin{pmatrix}1\\0\\5\end{pmatrix} = \begin{pmatrix}2\\-1\\-3\end{pmatrix} \qquad \vec{AC} = \begin{pmatrix}2\\0\\-1\end{pmatrix} - \begin{pmatrix}1\\0\\5\end{pmatrix} = \begin{pmatrix}1\\0\\-6\end{pmatrix}$$

Find the cross product of the two direction vectors:

$$\vec{AB} \times \vec{AC} = \begin{pmatrix}2\\-1\\-3\end{pmatrix} \times \begin{pmatrix}1\\0\\-6\end{pmatrix} = \begin{pmatrix}(-1)(-6) - (-3)(0)\\(-3)(1) - (2)(-6)\\(2)(0) - (-1)(1)\end{pmatrix} = \begin{pmatrix}6\\9\\1\end{pmatrix}$$

Find the magnitude of the cross product

$$|\vec{AB} \times \vec{AC}| = \sqrt{6^2 + 9^2 + 1^2} = \sqrt{118}$$

Area of the triangle is half the magnitude

$$\text{Area} = \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{118}$$

**Area = 5.43 $u^2$ (3sf)**

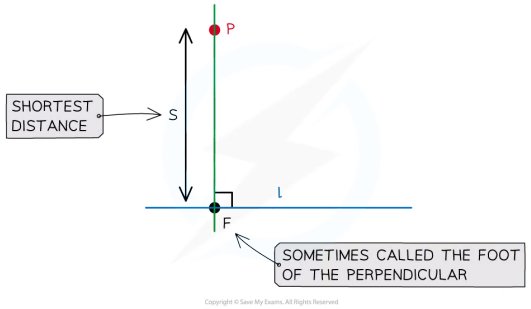
www.thrivingscholars.com



# Shortest Distance Between a Point and a Line

## How do I find the shortest distance from a point to a line?

- The shortest distance from any point to a line will always be the **perpendicular** distance
  - Given a line *l* with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a point *P* not on *l*
  - The **scalar product** of the direction vector, **b**, and the vector in the direction of the **shortest distance** will be zero
- The shortest distance can be found using the following steps:
  - STEP 1: Let the vector equation of the line be ***r*** and the point not on the line be *P*, then the point on the line closest to *P* will be the point *F*
    - The point *F* is sometimes called the foot of the perpendicular
  - STEP 2: Sketch a diagram showing the line *l* and the points *P* and *F*
    - The vector $\overrightarrow{FP}$ will be **perpendicular** to the line *l*
  - STEP 3: Use the equation of the line to find the position vector of the point *F* in terms of $\lambda$
  - STEP 4: Use this to find the displacement vector $\overrightarrow{FP}$ in terms of $\lambda$
  - STEP 5: The scalar product of the direction vector of the line *l* and the displacement vector $\overrightarrow{FP}$ will be zero
    - Form an equation $\overrightarrow{FP} \cdot \mathbf{b} = 0$ and solve to find $\lambda$
  - STEP 6: Substitute $\lambda$ into $\overrightarrow{FP}$ and find the magnitude $|\overrightarrow{FP}|$
    - The shortest distance from the point to the line will be the magnitude of $\overrightarrow{FP}$
- Note that the shortest distance between the point and the line is sometimes referred to as the **length of the perpendicular**




## How do we use the vector product to find the shortest distance from a point to a line?

- The vector product can be used to find the shortest distance from any point to a line on a 2–dimensional plane
- Given a point, P, and a line $\mathbf{r} = a + \lambda b$
  - The shortest distance from P to the line will be $\dfrac{|\overrightarrow{AP} \times b|}{|b|}$
  - Where *A* is a point on the line
  - This is **not** given in the formula booklet

www.thrivingscholars.com

## Exam Tip

- Column vectors can be easier and clearer to work with when dealing with scalar products.

## Worked Example

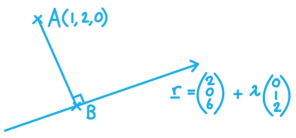
Point $A$ has coordinates (1, 2, 0) and the line $l$ has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

Point $B$ lies on the $l$ such that $[AB]$ is perpendicular to $l$.

Find the shortest distance from $A$ to the line $l$.

B is on $l$ so can be written in terms of $\lambda$:

$$\vec{OB} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ \lambda \\ 6+2\lambda \end{pmatrix}$$




Find $\vec{AB}$ using $\vec{AB} = \vec{OB} - \vec{OA}$

$$\vec{AB} = \begin{pmatrix} 2 \\ \lambda \\ 6+2\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda - 2 \\ 6+2\lambda \end{pmatrix}$$

$\vec{AB}$ is perpendicular to $l$: $\vec{AB} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$

$$\begin{pmatrix} 1 \\ \lambda - 2 \\ 6+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$$

$$\lambda - 2 + 2(6+2\lambda) = 0$$

$$5\lambda + 10 = 0$$

$$\lambda = -2$$

Substitute back into $\vec{AB}$ and find the magnitude:

$$\vec{AB} = \begin{pmatrix} 1 \\ -2-2 \\ 6+2(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$$

$$|\vec{AB}| = \sqrt{1^2 + (-4)^2 + 2^2} = \sqrt{21}$$

**Shortest distance $= \sqrt{21}$ units**

www.thrivingscholars.com

# Shortest Distance Between Two Lines

## How do we find the shortest distance between two parallel lines?

- Two **parallel** lines will never intersect
- The shortest distance between two **parallel lines** will be the **perpendicular distance** between them
- Given a line $l_1$ with equation $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and a line $l_2$ with equation $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$ then the shortest distance between them can be found using the following steps:
  - STEP 1: Find the vector between $\mathbf{a}_1$ and a general coordinate from $l_2$ in terms of $\mu$
  - STEP 2: Set the scalar product of the vector found in STEP 1 and the direction vector $\mathbf{d}_1$ equal to zero
    - Remember the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are scalar multiples of each other and so either can be used here
  - STEP 3: Form and solve an equation to find the value of $\mu$
  - STEP 4: Substitute the value of $\mu$ back into the equation for $l_2$ to find the coordinate on $l_2$ closest to $l_1$
  - STEP 5: Find the distance between $\mathbf{a}_1$ and the coordinate found in STEP 4
- Alternatively, the formula $\frac{|\overrightarrow{AB} \times \mathbf{d}|}{|\mathbf{d}|}$ can be used
  - Where $\overrightarrow{AB}$ is the vector connecting the two given coordinates $\mathbf{a}_1$ and $\mathbf{a}_2$
  - $\mathbf{d}$ is the simplified vector in the direction of $\mathbf{d}_1$ and $\mathbf{d}_2$
  - This is **not** given in the formula booklet

## How do we find the shortest distance from a given point on a line to another line?

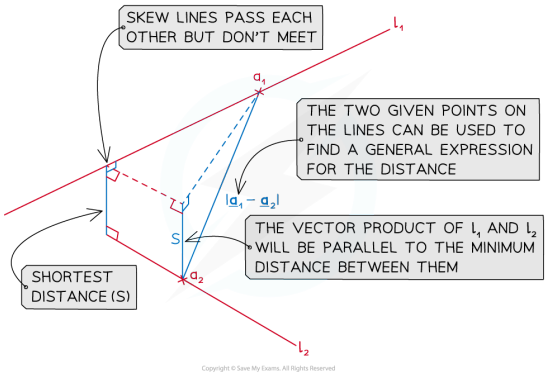
- The shortest distance from any point on a line to another line will be the **perpendicular** distance from the point to the line
- If the angle between the two lines is known or can be found then right-angled trigonometry can be used to find the perpendicular distance
  - The formula $\frac{|\overrightarrow{AB} \times \mathbf{d}|}{|\mathbf{d}|}$ given above is derived using this method and can be used
- Alternatively, the equation of the line can be used to find a general coordinate and the steps above can be followed to find the shortest distance

## How do we find the shortest distance between two skew lines?

- Two **skew** lines are not parallel but will never intersect
- The shortest distance between two **skew lines** will be perpendicular to **both** of the lines
  - This will be at the point where the two lines pass each other with the perpendicular distance where the point of intersection would be
  - The **vector product** of the two direction vectors can be used to find a vector in the direction of the shortest distance
  - The shortest distance will be a vector **parallel** to the vector product
- To find the shortest distance between two skew lines with equations $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$,

www.thrivingscholars.com

- STEP 1: Find the vector product of the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$
  - $\mathbf{d} = \mathbf{d}_1 \times \mathbf{d}_2$
- STEP 2: Find the vector in the direction of the line between the two general points on $l_1$ and $l_2$ in terms of $\lambda$ and $\mu$
  - $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
- STEP 3: Set the two vectors parallel to each other
  - $\mathbf{d} = k\overrightarrow{AB}$
- STEP 4: Set up and solve a system of linear equations in the three unknowns, $k$, $\lambda$ and $\mu$




## Exam Tip

- Exam questions will often ask for the shortest, or minimum, distance within vector questions
- If you're unsure start by sketching a quick diagram
- Sometimes calculus can be used, however usually vector methods are required

www.thrivingscholars.com

## Worked Example

Consider the skew lines $l_1$ and $l_2$ as defined by:

$$l_1 : \boldsymbol{r} = \begin{pmatrix} 6 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

$$l_2 : \boldsymbol{r} = \begin{pmatrix} -5 \\ 4 \\ -8 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Find the minimum distance between the two lines.

Find the vector product of the direction vectors.

$$\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} (-3)(1) - (4)(2) \\ (4)(-1) - (2)(1) \\ (2)(2) - (-3)(-1) \end{pmatrix} = \begin{pmatrix} -11 \\ -6 \\ 1 \end{pmatrix}$$

Find the vector in the direction of the line between the general coordinates.

$$\overrightarrow{AB} = \underbrace{\begin{pmatrix} -5 - \mu \\ 4 + 2\mu \\ -8 + \mu \end{pmatrix}}_{\text{A point on } l_2} - \underbrace{\begin{pmatrix} 6 + 2\lambda \\ -4 - 3\lambda \\ 3 + 4\lambda \end{pmatrix}}_{\text{A point on } l_1} = \begin{pmatrix} -11 - \mu - 2\lambda \\ 8 + 2\mu + 3\lambda \\ -11 + \mu - 4\lambda \end{pmatrix}$$

$$\begin{pmatrix} -11 - \mu - 2\lambda \\ 8 + 2\mu + 3\lambda \\ -11 + \mu - 4\lambda \end{pmatrix} = k \begin{pmatrix} -11 \\ -6 \\ 1 \end{pmatrix}$$

$\overrightarrow{AB}$ is parallel to $\begin{pmatrix} -11 \\ -6 \\ 1 \end{pmatrix}$

so $\overrightarrow{AB} = k \begin{pmatrix} -11 \\ -6 \\ 1 \end{pmatrix}$

Set up and solve a system of equations.

$$\left.\begin{aligned} 11k - 2\lambda - \mu &= 11 \\ 6k + 3\lambda + 2\mu &= -8 \\ \mu - 4\lambda - k &= 11 \end{aligned}\right\}$$

Solve using GDC:

$k = \frac{31}{79} \quad \lambda = -\frac{238}{79} \quad \mu = -\frac{52}{79}$

Substitute back into the expression for $\overrightarrow{AB}$ and find the magnitude:

$$|\overrightarrow{AB}| = \left| \begin{pmatrix} -11 - \left(-\frac{52}{79}\right) - 2\left(-\frac{238}{79}\right) \\ 8 + 2\left(-\frac{52}{79}\right) + 3\left(-\frac{238}{79}\right) \\ -11 + \left(-\frac{52}{79}\right) - 4\left(-\frac{238}{79}\right) \end{pmatrix} \right| = \left| \begin{pmatrix} -\frac{341}{79} \\ -\frac{186}{79} \\ \frac{31}{79} \end{pmatrix} \right| = \sqrt{\left(-\frac{341}{79}\right)^2 + \left(-\frac{186}{79}\right)^2 + \left(\frac{31}{79}\right)^2}$$

**Shortest distance = 4.93 units (3 s.f.)**

www.thrivingscholars.com

# 3.11 Vector Planes

## 3.11.1 Vector Equations of Planes

### Equation of a Plane in Vector Form

### How do I find the vector equation of a plane?

- A plane is a flat surface which is two-dimensional
  - Imagine a flat piece of paper that continues on forever in both directions
- A plane in often denoted using the capital Greek letter $\Pi$
- The vector form of the equation of a plane can be found using **two direction vectors** on the plane
  - The direction vectors must be
    - **parallel** to the plane
    - **not parallel** to each other
    - therefore they will **intersect** at some point on the plane
- The formula for finding the **vector equation** of a plane is
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
    - Where $\boldsymbol{r}$ is the **position vector** of any point on the plane
    - $\boldsymbol{a}$ is the **position vector** of a known point on the plane
    - $\boldsymbol{b}$ and $\boldsymbol{c}$ are two **non-parallel direction** (displacement) **vectors** parallel to the plane
    - $\lambda$ and $\mu$ are scalars
  - The formula is **given in the formula booklet** but you must make sure you know what each part means
- As $\boldsymbol{a}$ could be the position vector of **any** point on the plane and $\boldsymbol{b}$ and $\boldsymbol{c}$ could be **any non-parallel** direction vectors on the plane there are infinite vector equations for a single plane

### How do I determine whether a point lies on a plane?

- Given the equation of a plane $\boldsymbol{r} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix} + \mu \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix}$ then the point $\boldsymbol{r}$ with position vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is on the plane if there exists a value of $\lambda$ and $\mu$ such that
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix} + \mu \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix}$
  - This means that there exists a single value of $\lambda$ and $\mu$ that satisfy the three **parametric** equations:
    - $x = a_1 + \lambda b_1 + \mu c_1$
    - $y = a_2 + \lambda b_2 + \mu c_2$
    - $z = a_3 + \lambda b_3 + \mu c_3$

www.thrivingscholars.com

- Solve two of the equations first to find the values of $\lambda$ and $\mu$ that satisfy the first two equation and then check that this value also satisfies the third equation
- If the values of $\lambda$ and $\mu$ do not satisfy all three equations, then the point $\boldsymbol{r}$ does not lie on the plane

### Exam Tip

- The formula for the vector equation of a plane is given in the formula booklet, make sure you know what each part means
- Be careful to use different letters, e.g. $\lambda$ and $\mu$ as the scalar multiples of the two direction vectors

www.thrivingscholars.com

## Worked Example

The points A, B and C have position vectors $\boldsymbol{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\boldsymbol{b} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, and $\boldsymbol{c} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ respectively, relative to the origin O.

(a) Find the vector equation of the plane.

Start by finding the direction vectors $\vec{AB}$ and $\vec{AC}$

$$\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix}$$

$$\vec{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

All three points lie on the plane, so choose the position vector of one point, e.g. $\vec{OA}$, to use as 'a' in the vector equation of a plane formula.

Check that $\vec{AB}$ and $\vec{AC}$ are not parallel.

$$r = \underline{a} + \lambda\vec{AB} + \mu\vec{AC}$$

$$\underline{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

(This is one of many correct answers)

(b) Determine whether the point D with coordinates (-2, -3, 5) lies on the plane.

www.thrivingscholars.com

Let D have position vector $\underline{d} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$, then the point D lies on the plane if there exists a value of $\lambda$ and $\mu$ for which: $\begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$

Find the parametric equations:

$-2 = 3 - 2\lambda + \mu \Rightarrow \mu - 2\lambda = -5$ ①

$-3 = 2 - 4\lambda - 3\mu \Rightarrow 3\mu + 4\lambda = 5$ ②

} solve two equations for $\lambda$ and $\mu$.

$5 = -1 + 5\lambda + 4\mu \Rightarrow 4\mu + 5\lambda = 6$ ③

Find the value of $\lambda$ and $\mu$ from two equations:

$2\text{①}: \quad 2\mu - 4\lambda = -10$

$+\text{②}: \quad 3\mu + 4\lambda = 5$

$5\mu = -5$

$\mu = -1$ sub into ①: $(-1) - 2\lambda = -5$

$\lambda = 2$

Check to see if $\lambda$ and $\mu$ satisfy the third equation:

$4(-1) + 5(2) = -4 + 10 = 6$ ✓

**The point D lies on the plane.**

www.thrivingscholars.com

# Equation of a Plane in Cartesian Form

## How do I find the vector equation of a plane in cartesian form?

- The **cartesian** equation of a plane is given in the form
  - $ax + by + cz = d$
  - This is **given in the formula booklet**
- A **normal vector** to the plane can be used along with a **known point on the plane** to find the cartesian equation of the plane
  - The normal vector will be a vector that is **perpendicular** to the plane
- The **scalar product** of the normal vector and any **direction vector** on the plane will the **zero**
  - The two vectors will be perpendicular to each other
  - The **direction vector** from a fixed-point *A* to any point on the plane, *R* can be written as $\boldsymbol{r} - \boldsymbol{a}$
  - Then $\boldsymbol{n} \bullet (\boldsymbol{r} - \boldsymbol{a}) = 0$ and it follows that $(\boldsymbol{n} \bullet \boldsymbol{r}) - (\boldsymbol{n} \bullet \boldsymbol{a}) = 0$
- This gives the **equation of a plane using the normal vector:**
  - $\boldsymbol{n} \bullet \boldsymbol{r} = \boldsymbol{a} \bullet \boldsymbol{n}$
    - Where $\boldsymbol{r}$ is the **position vector** of any point on the plane
    - $\boldsymbol{a}$ is the **position vector** of a known point on the plane
    - $\boldsymbol{n}$ is a vector that is **normal** to the plane
  - This is **given in the formula booklet**
- If the vector $\boldsymbol{r}$ is given in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\boldsymbol{a}$ and $\boldsymbol{n}$ are both known vectors given in the form $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ then the Cartesian equation of the plane can be found using:
  - $\boldsymbol{n} \cdot \boldsymbol{r} = n_1 x + n_2 y + n_3 z$
  - $\boldsymbol{a} \cdot \boldsymbol{n} = a_1 n_1 + a_2 n_2 + a_3 n_3$
  - Therefore $n_1 x + n_2 y + n_3 z = a_1 n_1 + a_2 n_2 + a_3 n_3$
  - This simplifies to the form $ax + by + cz = d$

## How do I find the equation of a plane in Cartesian form given the vector form?

- The **Cartesian** equation of a plane can be found if you know
  - the **normal vector** and
  - a **point** on the plane
- The **vector equation of a plane** can be used to find the **normal vector** by finding the **vector product** of the two direction vectors
  - A vector product is always perpendicular to the two vectors from which it was calculated
- The vector $\boldsymbol{a}$ given in the vector equation of a plane is a **known point** on the plane
  - Once you have found the normal vector then the point $\boldsymbol{a}$ can be used in the formula $\boldsymbol{n} \bullet \boldsymbol{r} = \boldsymbol{a} \bullet \boldsymbol{n}$ to find the equation in Cartesian form

www.thrivingscholars.com

- To find $ax + by + cz = d$ given $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$:
  - Let $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \boldsymbol{b} \times \boldsymbol{c}$ then $d = \boldsymbol{n} \cdot \boldsymbol{a}$

## Exam Tip

- In an exam, using whichever form of the equation of the plane to write down a normal vector to the plane is always a good starting point

## Worked Example

A plane $\Pi$ contains the point $A(2, 6, -3)$ and has a normal vector $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$.

a)
Find the equation of the plane in its Cartesian form.

The components of the normal vector are the x-, y- and z- coefficients of the Cartesian form:

$3x - y + 4z = d$

The point $(2, 6, -3)$ is on the plane so

$d = 3(2) - (6) + 4(-3) = 6 - 6 - 12 = -12$

Therefore $3x - y + 4z = -12$

b)
Determine whether point B with coordinates $(-1, 0, -2)$ lies on the same plane.

Test by putting the coordinates into the equation:

$3(-1) - (0) + 4(-2) = -3 - 8 = -11 \neq -12$

The point with coordinates (-1, 0, 2) does not lie on the plane

www.thrivingscholars.com



# Intersection of Line & Plane

## How do I tell if a line is parallel to a plane?

- A line is parallel to a plane if its **direction vector** is **perpendicular** to the plane's **normal vector**
- If you know the Cartesian equation of the plane in the form $ax + by + cz = d$ then the values of *a*, *b*, and c are the individual components of a normal vector to the plane
- The **scalar product** can be used to check in the direction vector and the normal vector are perpendicular
  - If two vectors are perpendicular their scalar product will be zero

## How do I tell if the line lies inside the plane?

- If the line is parallel to the plane then it will either **never intersect** or it will lie inside the plane
  - Check to see if they have a common point
- If a line is parallel to a plane and they share **any point**, then the line lies inside the plane

## How do I find the point of intersection of a line and a plane?

- If a line is **not parallel** to a plane it will **intersect** it at a single point
- If both the **vector equation of the line** and the **Cartesian equation of the plane** is known then this can be found by:
- STEP 1: Set the position vector of the point you are looking for to have the individual components *x*, *y*, and *z* and substitute into the vector equation of the line
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
- STEP 2: Find the parametric equations in terms of *x*, *y*, and *z*
  - $x = x_0 + \lambda l$
  - $y = y_0 + \lambda m$
  - $z = z_0 + \lambda n$
- STEP 3: Substitute these parametric equations into the Cartesian equation of the plane and solve to find λ
  - $a(x_0 + \lambda l) + b(y_0 + \lambda m) + c(z_0 + \lambda n) = d$
- STEP 4: Substitute this value of λ back into the vector equation of the line and use it to find the position vector of the point of intersection
- STEP 5: Check this value in the Cartesian equation of the plane to make sure you have the correct answer

www.thrivingscholars.com

## Worked Example

Find the point of intersection of the line $r = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ with the plane $3x - 4y + z = 8$.

Find the parametric form of the equation of the line:

Let $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ then $x = 1 + 2\lambda$, $y = -3 - \lambda$, $z = 2 - \lambda$

Substitute into the equation of the plane:

$3(1 + 2\lambda) - 4(-3 - \lambda) + (2 - \lambda) = 8$

Solve to find $\lambda$:

$3 + 6\lambda + 12 + 4\lambda + 2 - \lambda = 8$

$\lambda = -1$

Substitute $\lambda = -1$ into the vector equation of the line:

$r = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 - 2 \\ -3 + 1 \\ 2 + 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$

**(-1, -2, 3)**

www.thrivingscholars.com

# Intersection of Planes

## How do we find the line of intersection of two planes?

- Two planes will either be **parallel** or they will intersect along a **line**
  - Consider the point where a wall meets a floor or a ceiling
  - You will need to find the **equation of the line** of intersection
- If you have the Cartesian forms of the two planes then the equation of the line of intersection can be found by solving the two equations simultaneously
  - As the solution is a vector equation of a line rather than a unique point you will see below how the equation of the line can be found by part solving the equations
  - For example:
    - $2x - y + 3z = 7$ (1)
    - $x - 3y + 4z = 11$ (2)
- STEP 1: Choose one variable and substitute this variable for $\lambda$ in both equations
  - For example, letting $x = \lambda$ gives:
    - $2\lambda - y + 3z = 7$ (1)
    - $\lambda - 3y + 4z = 11$ (2)
- STEP 2: Rearrange the two equations to bring $\lambda$ to one side
  - Equations (1) and (2) become
    - $y - 3z = 2\lambda - 7$ (1)
    - $3y - 4z = \lambda - 11$ (2)
- STEP 3: Solve the equations simultaneously to find the two variables in terms of $\lambda$
  - 3(1) – (2) Gives
    - $z = 2 - \lambda$
  - Substituting this into (1) gives
    - $y = -1 - \lambda$
- STEP 4: Write the three parametric equations for $x$, $y$, and $z$ in terms of $\lambda$ and convert into the vector equation of a line in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
  - The parametric equations
    - $x = \lambda$
    - $y = -1 - \lambda$
    - $z = 2 - \lambda$
  - Become
    - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$
- If you have fractions in your direction vector you can change its magnitude by multiplying each one by their common denominator
  - The magnitude of the direction vector can be changed without changing the equation of a line
- An alternative method is to find two points on both planes by setting either $x$, $y$, or $z$ to zero and solving the system of equations using your GDC or row reduction
  - Repeat this twice to get two points on both planes

www.thrivingscholars.com



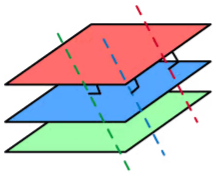
- These two points can then be used to find the vector equation of the line between them
- This will be the line of intersection of the planes
- This method relies on the line of intersection having points where the chosen variables are equal to zero

## How do we find the relationship between three planes?

- Three planes could either be **parallel**, intersect at one **point**, or intersect along a **line**
- If the three planes have a **unique point of intersection** this point can be found by using your GDC (or row reduction) to solve the three equations in their Cartesian form
  - Make sure you know how to use your GDC to solve a **system of linear equations**
  - Enter all three equations in for the three variables $x$, $y$, and $z$
  - Your GDC will give you the unique solution which will be the coordinates of the point of intersection
- If the three planes do not intersect at a unique point you will not be able to use your GDC to solve the equations
  - If there are no solutions to the system of Cartesian equations then there is no unique point of intersection
- If the three planes are all **parallel** their **normal vectors** will be parallel to each other
  - Show that the normal vectors all have equivalent **direction vectors**
  - These direction vectors may be **scalar multiples** of each other
- If the three planes have **no point of intersection** and are **not all parallel** they may have a relationship such as:
  - Each plane intersects two other planes such that they form a **prism** (none are parallel)
  - Two planes are parallel with the third plane intersecting each of them
  - Check the normal vectors to see if any two of the planes are parallel to decide which relationship they have
- If the three planes intersect along a line there will not be a unique solution to the three equations but there will be a **vector equation of a line** that will satisfy the three equations
- The system of equations will need to be solved by **elimination** or **row reduction**
  - Choose one variable to substitute for $\lambda$
  - Solve two of the equations simultaneously to find the other two variables in terms of $\lambda$
  - Write $x$, $y$, and $z$ in terms of $\lambda$ in the parametric form of the equation of the line and convert into the vector form of the equation of a line

www.thrivingscholars.com

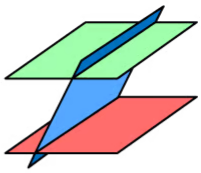
3 PARALLEL PLANES



3 NORMALS ARE PARALLEL

NO POINT OF INTERSECTION

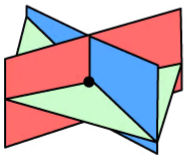
2 PARALLEL PLANES



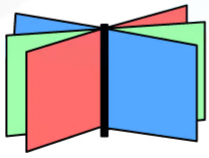
2 NORMALS ARE PARALLEL

2 LINES OF INTERSECTION

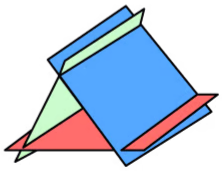
NO PARALLEL PLANES



A UNIQUE POINT OF INTERSECTION



ONE LINE OF INTERSECTION



EACH PLANE INTERSECTS WITH TWO OTHERS

Copyright © Save My Exams. All Rights Reserved

## Exam Tip

- In an exam you may need to decide the relationship between three planes by using row reduction to determine the number of solutions
  - Make sure you are confident using row reduction to solve systems of linear equations
  - Make sure you remember the different forms three planes can take

www.thrivingscholars.com

## Worked Example

Two planes $\Pi_1$ and $\Pi_2$ are defined by the equations:

$\Pi_1: 3x+4y+2z=7$

$\Pi_2: x-2y+3z=5$

Find the vector equation of the line of intersection of the two planes.

**STEP 1:** Let $z=\lambda$, then $3x+4y+2\lambda=7$ ①

$x-2y+3\lambda=5$ ②

You can substitute any variable here, look at the equations to see which is easiest.

**STEP 2:** ①: $3x+4y=7-2\lambda$

②: $x-2y=5-3\lambda$

Write the two equations as simultaneous equations for the two remaining constants.

**STEP 3:** Find $x$ and $z$ in terms of $\lambda$:

① − 2②: $(3x+4y=7-2\lambda)$

$+(2x-4y=10-6\lambda)$

$5x = 17-8\lambda$

$x=\frac{17}{5}-\frac{8\lambda}{5}$

sub into ②: $\frac{17}{5}-\frac{8\lambda}{5}-2y+3\lambda=5$

$y=\frac{7\lambda}{10}-\frac{8}{10}$

**STEP 4:** $x=\frac{17}{5}-\frac{8\lambda}{5}$, $y=\frac{7\lambda}{10}-\frac{4}{5}$, $z=\lambda$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{17}{5}\\-\frac{4}{5}\\0\end{pmatrix}+\lambda\begin{pmatrix}-\frac{8}{5}\\\frac{7}{10}\\1\end{pmatrix}$$

The components of the direction vector can be multiplied by a scalar without changing the direction.

$$\mathbf{r}=\begin{pmatrix}17/5\\-4/5\\0\end{pmatrix}+\lambda\begin{pmatrix}16\\7\\10\end{pmatrix}$$

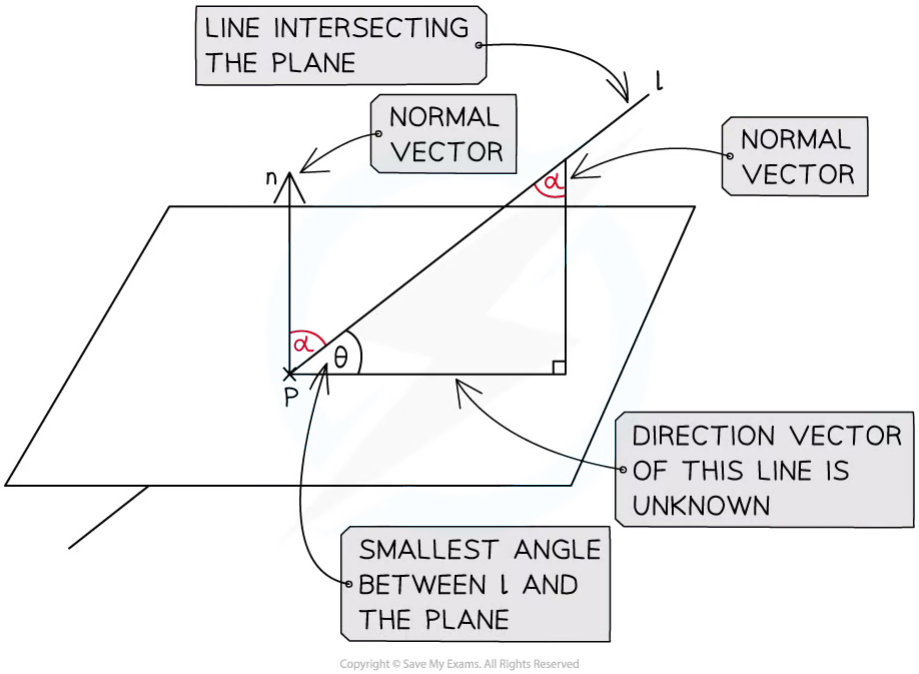
www.thrivingscholars.com



# Angle Between Line & Plane

## How do I find the angle between a line and a plane?

- When you find the angle between a line and a plane you will be finding the angle between the line itself and the line on the plane that creates the smallest angle with it
  - This means the line on the plane directly under the line as it joins the plane
- It is easiest to think of these two lines making a right-triangle with the normal vector to the plane
  - The line joining the plane will be the **hypotenuse**
  - The line on the plane will be **adjacent** to the angle
  - The normal will the **opposite** the angle
- As you do not know the angle of the line on the plane you can instead find the angle between the **normal** and the **hypotenuse**
  - This is the angle **opposite** the angle you want to find
  - This angle **can be found** because you will know the direction vector of the line joining the plane and the normal vector to the plane
  - This angle is also equal to the angle made by the line at the point it joins the plane and the normal vector at this point
- The smallest angle between the line and the plane will be 90° minus the angle between the normal vector and the line
  - In radians this will be $\frac{\pi}{2}$ minus the angle between the normal vector and the line




Copyright © Save My Exams. All Rights Reserved

www.thrivingscholars.com

## Exam Tip

- Remember that if the scalar product is negative your answer will result in an obtuse angle
  - Taking the absolute value of the scalar product will ensure that you get the acute angle as your answer

## Worked Example

Find the angle in radians between the line $L$ with vector equation $\boldsymbol{r}=(2-\lambda)\boldsymbol{i}+(\lambda+1)\boldsymbol{j}+(1-2\lambda)\boldsymbol{k}$ and the plane $\Pi$ with Cartesian equation $x-3y+2z=5$.

Rewrite line equation in standard vector form:

$$\mathbf{r}=\begin{pmatrix}2-\lambda\\1+\lambda\\1-2\lambda\end{pmatrix}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\-2\end{pmatrix}$$

$\begin{pmatrix}-1\\1\\-2\end{pmatrix}$ ← direction vector of the line

Find the normal vector of the plane:

$x-3y+2z=5 \Rightarrow$ normal vector $=\begin{pmatrix}1\\-3\\2\end{pmatrix}$

components of the normal vector

Find the angle between the direction vector and the normal vector, $\alpha$:

| Angle between two vectors | $\cos\theta=\frac{v_1w_1+v_2w_2+v_3w_3}{\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert}$ |
|---|---|

$$\cos\alpha=\frac{\left|\begin{pmatrix}-1\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\2\end{pmatrix}\right|}{\sqrt{(-1)^2+(1)^2+(-2)^2}\times\sqrt{1^2+(-3)^2+2^2}}=\frac{|(-1)(1)+(1)(-3)+(-2)(2)|}{\sqrt{6}\sqrt{14}}$$

$$\theta=\frac{\pi}{2}-\cos^{-1}\alpha$$

$$\theta=\frac{\pi}{2}-\cos^{-1}\left(\frac{|-8|}{\sqrt{6}\sqrt{14}}\right)$$

Using the absolute value ensures we find the acute angle.

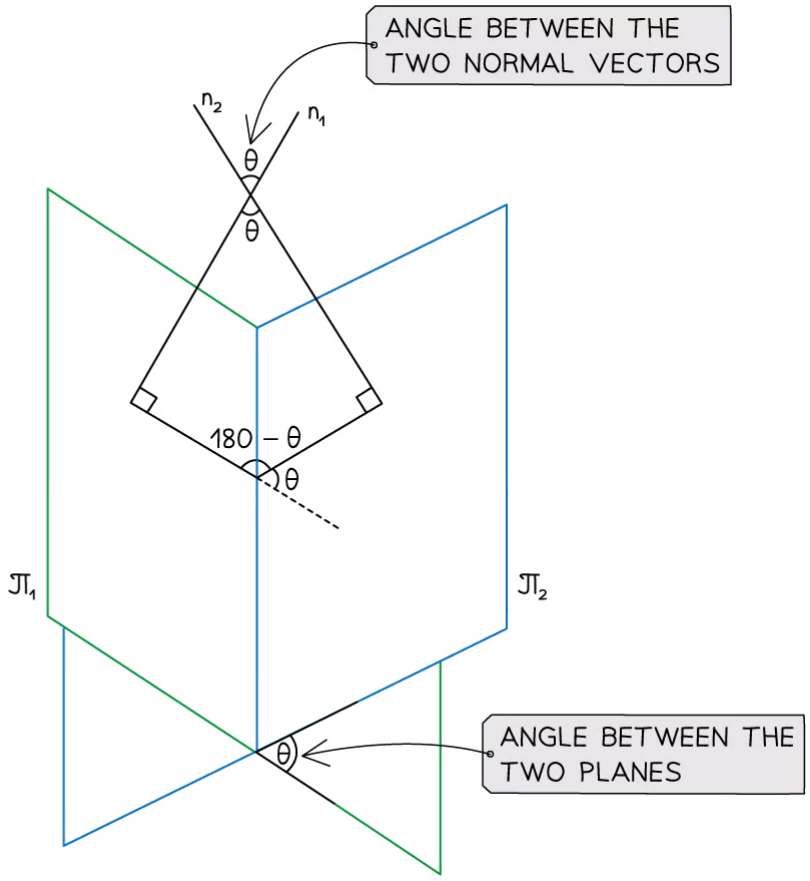
$\theta = 1.06$ radians (3s.f.)

www.thrivingscholars.com

# Angle Between Two Planes

## How do we find the angle between two planes?

- The angle between two planes is equal to the angle between their **normal vectors**
  - It can be found using the **scalar product** of their normal vectors
- If two planes $\Pi_1$ and $\Pi_2$ with normal vectors $n_1$ and $n_2$ meet at an angle then the two planes and the two normal vectors will form a quadrilateral
  - The angles between the planes and the normal will both be 90°
  - The angle between the two planes and the angle opposite it (between the two normal vectors) will add up to 180°




### Exam Tip

- In your exam read the question carefully to see if you need to find the acute or obtuse angle
  - When revising, get into the practice of double checking at the end of a question whether your angle is acute or obtuse and whether this fits the question

www.thrivingscholars.com

## Worked Example

Find the acute angle between the two planes which can be defined by equations $\Pi_1$: $2x - y + 3z = 7$ and $\Pi_2$: $x + 2y - z = 20$.

Find the normal vectors of each of the planes:

$\pi_1$: $2x - y + 3z = 7$ $\Rightarrow$ normal vector, $n_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$\pi_2$: $x + 2y - z = 20$ $\Rightarrow$ normal vector, $n_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Find the angle between the two normal vectors:

| | |
|---|---|
| Angle between two vectors | $\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert}$ |

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{|(2)(1) + (-1)(2) + (3)(-1)|}{\sqrt{2^2 + (-1)^2 + 3^2} \times \sqrt{1^2 + 2^2 + (-1)^2}} = \frac{|-3|}{\sqrt{14} \times \sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right)$$

Using the absolute value ensures we find the acute angle.

$\theta = 1.24$ radians (3 s.f.)

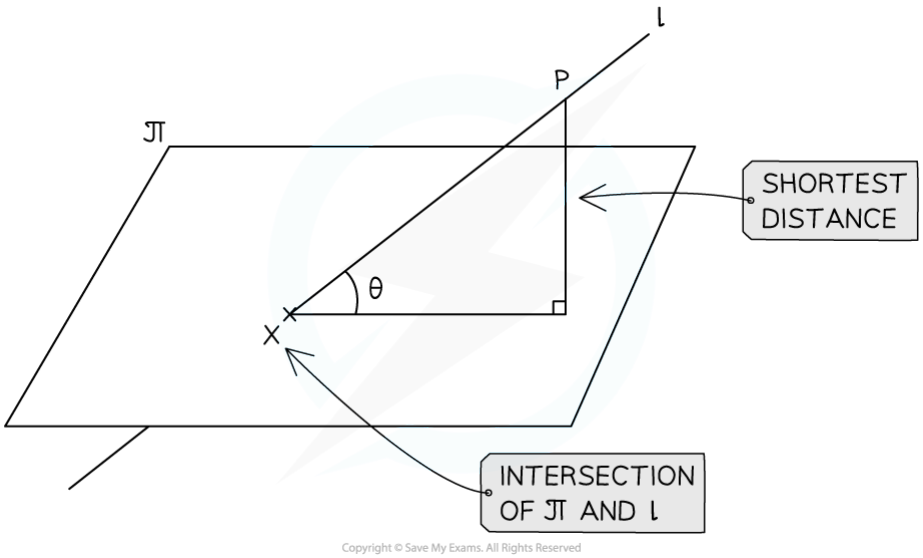
www.thrivingscholars.com



# Shortest Distance Between a Line and a Plane

## How do I find the shortest distance between a given point on a line and a plane?

- The shortest distance from any point on a line to a plane will always be the **perpendicular** distance from the point to the plane
- Given a point, $P$, on the line $l$ with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a plane $\Pi$ with equation $\mathbf{r} \cdot \mathbf{n} = d$
  - STEP 1: Find the vector equation of the line perpendicular to the plane that goes through the point, $P$, on $l$
    - This will have the position vector of the point, $P$, and the direction vector $\mathbf{n}$
  - STEP 2: Find the coordinates of the point of intersection of this new line with $\Pi$ by substituting the equation of the line into the equation of the plane
  - STEP 3: Find the distance between the given point on the line and the point of intersection
    - This will be the shortest distance from the plane to the point
- A question may provide the acute angle between the line and the plane
  - Use right-angled trigonometry to find the perpendicular distance between the point on the line and the plane
    - Drawing a clear diagram will help



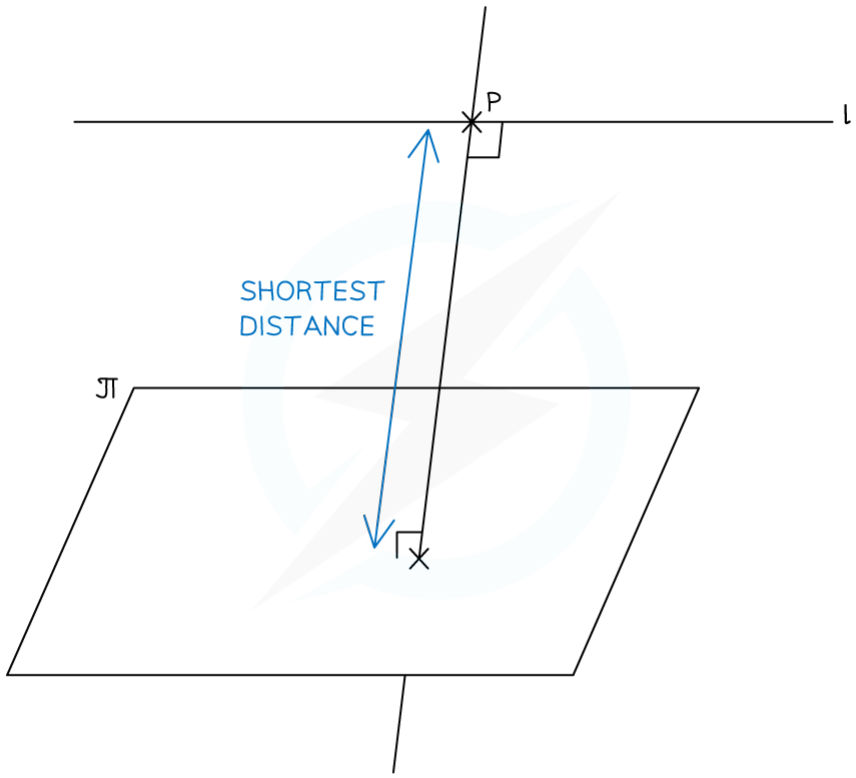

Copyright © Save My Exams. All Rights Reserved

## How do I find the shortest distance between a plane and a line parallel to the plane?

- The shortest distance between a line and a plane that are parallel to each other will be the **perpendicular** distance from the line to the plane
- Given a line $l_1$ with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a plane $\Pi$ parallel to $l_1$ with equation $\mathbf{r} \cdot \mathbf{n} = d$
  - Where $\mathbf{n}$ is the **normal vector** to the plane
  - STEP 1: Find the equation of the line $l_2$ perpendicular to $l_1$ and $\Pi$ going through the point $\mathbf{a}$ in the form $\mathbf{r} = \mathbf{a} + \mu\mathbf{n}$

www.thrivingscholars.com

- STEP 2: Find the point of intersection of the line $l_2$ and $\Pi$
- STEP 3: Find the distance between the point of intersection and the point,




Copyright © Save My Exams. All Rights Reserved

## Exam Tip

- Vector planes questions can be tricky to visualise, read the question carefully and sketch a very simple diagram to help you get started

www.thrivingscholars.com

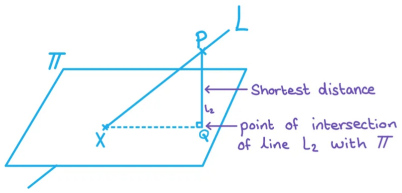
## Worked Example

The plane $\Pi$ has equation $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 6$.

The line $L$ has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$.

The point $P\,(-2,\ 11,\ -15)$ lies on the line $L$.

Find the shortest distance between the point P and the plane $\Pi$.




STEP 1: Use the given point, P and the known normal to the plane, $\underline{n}$ to write an equation for the line perpendicular to $\Pi$, $L_2$.

$$\underline{r} = \begin{pmatrix} -2 \\ 11 \\ -15 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

(P) (n)

STEP 2: Find the point of intersection, Q, of the new line, $L_2$, with $\Pi$.

$$\left( \begin{pmatrix} -2 \\ 11 \\ -15 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 6$$

$$2(-2+2\lambda) - (11-\lambda) + (\lambda - 15) = 6$$

$$-4 + 4\lambda - 11 + \lambda + \lambda - 15 = 6$$

$$6\lambda - 30 = 6$$

$$\lambda = 6 \Rightarrow \overrightarrow{OQ} = \begin{pmatrix} -2 \\ 11 \\ -15 \end{pmatrix} + 6\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ -9 \end{pmatrix}$$

STEP 3: Find the distance between P and Q.

$$|\overrightarrow{PQ}| = \sqrt{(10--2)^2 + (5-11)^2 + (-9--15)^2} = 6\sqrt{6} \text{ units}$$

**Shortest distance = $6\sqrt{6}$ units**

www.thrivingscholars.com

# Shortest Distance Between Two Planes

## How do I find the shortest distance between two parallel planes?

- Two **parallel** planes will never intersect
- The shortest distance between two **parallel planes** will be the **perpendicular distance** between them
- Given a plane $\Pi_1$ with equation $\mathbf{r}\cdot\mathbf{n} = d$ and a plane $\Pi_2$ with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ then the shortest distance between them can be found
  - STEP 1: The equation of the line perpendicular to both planes and through the point **a** can be written in the form $\mathbf{r} = \mathbf{a} + s\mathbf{n}$
  - STEP 2: Substitute the equation of the line into $\mathbf{r}\cdot\mathbf{n} = d$ to find the coordinates of the point where the line meets $\Pi_1$
  - STEP 3: Find the distance between the two points of intersection of the line with the two planes

## How do I find the shortest distance from a given point on a plane to another plane?

- The shortest distance from any point, *P* on a plane, $\Pi_1$, to another plane, $\Pi_2$ will be the **perpendicular** distance from the point to $\Pi_2$
  - STEP 1: Use the given coordinates of the point *P* on $\Pi_1$ and the normal to the plane $\Pi_2$ to find the vector equation of the line through *P* that is perpendicular to $\Pi_1$
  - STEP 2: Find the point of intersection of this line with the plane $\Pi_2$
  - STEP 3: Find the distance between the two points of intersection

### Exam Tip

- There are a lot of steps when answering these questions so set your methods out clearly in the exam

www.thrivingscholars.com

## Worked Example

Consider the parallel planes defined by the equations:

$$\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = 44,$$

$$\Pi_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find the shortest distance between the two planes $\Pi_1$ and $\Pi_2$.

Find the equation of the line perpendicular to the planes through the point $(0,0,3)$

$$L: r = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + s \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$$

Normal vector of $\pi_1$

position vector of $\pi_2$

Substitute the equation of L into the equation of $\pi_1$:

$$\begin{pmatrix} 3s \\ -5s \\ 3+2s \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = 44$$

$$3(3s) - 5(-5s) + 2(3+2s) = 44$$

$$38s + 6 = 44$$

$$s = 1$$

Substitute $s = 1$ back into the equation of $L$:

$$r = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 5 \end{pmatrix}$$

Find the distance between $(0,0,3)$ and $(3,-5,5)$

$$d = \sqrt{3^2 + (-5)^2 + (5-3)^2}$$

$$= \sqrt{38}$$

**Shortest distance $= \sqrt{38}$ units**

www.thrivingscholars.com

www.thrivingscholars.com

www.thrivingscholars.com

www.thrivingscholars.com

www.thrivingscholars.com

www.thrivingscholars.com

www.thrivingscholars.com

# IB Maths DP

## 4. Statistics & Probability

CONTENTS


4.1 Statistics Toolkit
- 4.1.1 Sampling & Data Collection
- 4.1.2 Statistical Measures
- 4.1.3 Frequency Tables
- 4.1.4 Linear Transformations of Data
- 4.1.5 Outliers
- 4.1.6 Univariate Data
- 4.1.7 Interpreting Data

4.2 Correlation & Regression
- 4.2.1 Bivariate Data
- 4.2.2 Correlation & Regression

4.3 Probability
- 4.3.1 Probability & Types of Events
- 4.3.2 Conditional Probability
- 4.3.3 Bayes' Theorem
- 4.3.4 Sample Space Diagrams

4.4 Probability Distributions
- 4.4.1 Discrete Probability Distributions
- 4.4.2 Mean & Variance

4.5 Binomial Distribution
- 4.5.1 The Binomial Distribution
- 4.5.2 Calculating Binomial Probabilities

4.6 Normal Distribution
- 4.6.1 The Normal Distribution
- 4.6.2 Calculations with Normal Distribution
- 4.6.3 Standardisation of Normal Variables

4.7 Further Probability Distributions
- 4.7.1 Probability Density Function

www.thrivingscholars.com



# 4.1 Statistics Toolkit

## 4.1.1 Sampling & Data Collection

### Types of Data

### What are the different types of data?

- **Qualitative** data is data that is usually given in words not numbers to **describe** something
  - For example: the colour of a teacher's car
- **Quantitative** data is data that is given using numbers which **counts or measures** something
  - For example: the number of pets that a student has
- **Discrete** data is quantitative data that needs to be **counted**
  - Discrete data can only take **specific values** from a set of (usually finite) values
  - For example: the number of times a coin is flipped until a 'tails' is obtained
- **Continuous** data is quantitative data that needs to be **measured**
  - Continuous data can take **any value** within a range of infinite values
  - For example: the height of a student
- **Age** can be **discrete or continuous** depending on the context or how it is defined
  - If you mean how many years old a person is then this is discrete
  - If you mean how long a person has been alive then this is continuous

### What is the difference between a population and a sample?

- The **population** refers to the **whole set** of things which you are interested in
  - For example: if a vet wanted to know how long a typical French bulldog slept for in a day then the population would be all the French bulldogs in the world
- A **sample** refers to a **subset of the population** which is used to collect data from
  - For example: the vet might take a sample of French bulldogs from different cities and record how long they sleep in a day
- A **sampling frame** is a **list** of all members of the **population**
  - For example: a list of employees' names within a company
- Using a **sample instead of a population**:
  - Is quicker and cheaper
  - Leads to less data needing to be analysed
  - Might not fully represent the population
  - Might introduce bias

www.thrivingscholars.com

# Sampling Techniques

## What is a random sample and a biased sample?

- A **random sample** is where every member of the population has an equal chance of being included in the sample
- A **biased sample** is where the sample is **not random**

## What sampling techniques do I need to know?

### Simple random sampling

- **Simple random sampling** is where every group of members from the population has an **equal probability** of being selected for the sample
- To carry this out you would...
  - uniquely number every member of a population
  - randomly select *n* different numbers using a random number generator or a form of lottery (where numbers are selected randomly)
- **Effectiveness**:
  - Useful when you have a small population or want a small sample (such as children in a class)
  - It can be time-consuming if the sample or population is large
  - This can not be used if it is not possible to number or list all the members of the population (such as fish in a lake)

### Systematic sampling

- **Systematic sampling** is where a sample is formed by choosing members of a population at regular intervals using a list
- To carry this out you would...
  - calculate the size of the interval $k = \frac{\text{size of population } (N)}{\text{size of sample } (n)}$
  - choose a random starting point between 1 and $k$
  - select every $k$th member after the first one
- **Effectiveness**:
  - Useful when there is a natural order (such as a list of names or a conveyor belt of items)
  - Quick and easy to use
  - This can not be used if it is not possible to number or list all the members of the population (such as penguins in Antarctica)

### Stratified sampling

- **Stratified sampling** is where the population is divided into disjoint groups (called strata) and then a random sample is taken from each group (stratum)
- The proportion of a stratum that is sampled is equal to the proportion of the population that belong to that stratum
- To carry this out you would...
  - Calculate the number of members sampled from each stratum

www.thrivingscholars.com



- $\frac{\text{size of sample } (n)}{\text{size of population } (N)} \times \text{number of members in the stratum}$
  - Take a random sample from each stratum
- **Effectiveness**:
  - Useful when there are very different groups of members within a population
  - The sample will be representative of the population structure
  - The members selected from each stratum are chosen randomly
  - This can not be used if the population can not be split into groups or if the groups overlap

### Quota sampling

- **Quota sampling** is where the population is split into groups (like stratified sampling) and members of the population are selected until each quota is filled
- To carry this out you would...
  - Calculate how many people you need from each group
  - Select members from each group until that quota is filled
    - The members do not have to be selected randomly
- **Effectiveness**:
  - Useful when collecting data by asking people who walk past you in a public place or when a sampling frame is not available
  - This can introduce bias as some members of the population might choose not to be included in the sample

### Convenience sampling

- **Convenience sampling** is where a sample is formed using available members of the population who fit the criteria
- To carry this out you would...
  - Select members that are easiest to reach
- **Effectiveness**:
  - Useful when a list of the population is not possible
  - This is unlikely to be representative of the population structure
  - This is likely to produce biased results

## What are the main criticisms of sampling techniques?

- Most sampling techniques can be improved by taking a larger sample
- Sampling can introduce bias - so you want to minimise the bias within a sample
  - To minimise bias the sample should be as close to random as possible
- A sample only gives information about those members
  - Different samples may lead to different conclusions about the population

www.thrivingscholars.com

## Worked Example

Mike is a biologist studying mice in an open enclosure. He has access to approximately 540 field mice and 260 harvest mice. Mike wants to sample 10 mice and he wants the proportions of the two types of mice in his sample to reflect their respective proportions of the population.

a)
Calculate the number of field mice and harvest mice that Mike should include in his sample.

Total number of mice

$540 + 260 = 800$

Field mice $\frac{540}{800} \times 10 = 6.75$

Fraction of field mice

Sample size

Fraction of harvest mice

Harvest mice $\frac{260}{800} \times 10 = 3.25$

**Include 7 field mice and 3 harvest mice**

b)
Given that Mike does not have a list of all mice in the enclosure, state the name of this sampling method.

No list of population so can not be a random sample

**Quota sampling**

c)
Suggest one way in which Mike could improve his sampling method.

**Mark could improve his sampling method by increasing his sample size**

www.thrivingscholars.com

# Reliability of Data

## How can I decide if data is reliable?

- Data from a sample is reliable if similar results would be obtained from a different sample from the same population
- The sample should be **representative** of the population
- The sample should be **big enough**
    - Sampling a small proportion of a population is unlikely to be reliable

## What can cause data to be unreliable?

- If the sample is **biased**
    - It is **not random**
- If **errors** are made when collecting data
    - Numbers could be recorded incorrectly, duplicated or missed out
- If the person collecting the data **favours some members** over others
    - They might seek out members who will lead to a desired outcome
    - They might exclude members if they would cause the sample to oppose the desired outcome
- If a significant proportion of **data is missing**
    - Some data may be unavailable
    - Some members might decide not to be part of the sample
        - This will mean the results are not necessarily representative of the population

www.thrivingscholars.com





# Mean, Mode, Median

## What are the mean, mode and median?

- Mean, median and mode are **measures of central tendency**
  - They describe where the centre of the data is
- They are all types of **averages**
- In statistics it is important to be specific about which average you are referring to
- The **units** for the mean, mode and median are the **same** as the units for the data

## How are the mean, mode, and median calculated for ungrouped data?

- The **mode** is the value that occurs **most often** in a data set
  - It is possible for there to be **more than one mode**
  - It is possible for there to be **no mode**
    - In this case **do not** say the mode is zero
- The **median** is the **middle** value when the data is in **order of size**
  - If there are two values in the middle then the median is the **midpoint** of the two values
- The **mean** is the **sum** of all the values **divided by the number of values**

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

  - Where $\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n$ is the sum of the $n$ pieces of data
  - The mean can be represented by the symbol $\mu$
- Your **GDC** can calculate these statistical measures if you input the data using the statistics mode

www.thrivingscholars.com

## Worked Example

Find the mode, median and mode for the data set given below.

43 29 70 51 64 43

Mode is the most common

**Mode = 43**

Median is the middle when in order

29 43 43 51 64 70

$\frac{43+51}{2} = 47$

**Median = 47**

Mean = $\frac{\Sigma x}{n}$

$\Sigma x = 300$ and $n = 6$ $\frac{300}{6} = 50$

**Mean = 50**

www.thrivingscholars.com

# Quartiles & Range

## What are quartiles?

- **Quartiles** are **measures of location**
- Quartiles divide a population or data set into **four equal sections**
  - The **lower quartile, $Q_1$** splits the lowest 25% from the highest 75%
  - The **median, $Q_2$** splits the lowest 50% from the highest 50%
  - The **upper quartile, $Q_3$** splits the lowest 75% from the highest 25%
- There are different methods for finding quartiles
  - Values obtained by hand and using technology may differ
- You will be expected to use your GDC to calculate the quartiles

## What are the range and interquartile range?

- The **range** and **interquartile range** are both **measures of dispersion**
  - They describe how spread out the data is
- The **range** is the largest value of the data minus the smallest value of the data
- The **interquartile range** is the range of the central 50% of data
  - It is the upper quartile minus the lower quartile

$$\text{IQR} = Q_3 - Q_1$$

    - This is given in the **formula booklet**
- The **units** for the range and interquartile range are the **same** as the units for the data

www.thrivingscholars.com

## Worked Example

Find the range and interquartile range for the data set given below.

43 29 70 51 64 43

Range = Maximum − Minimum

70 − 29

**Range = 41**

Find upper and lower quartiles using GDC

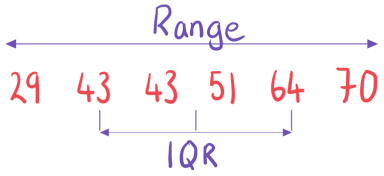
$Q_1 = 43$ and $Q_3 = 64$

IQR = $Q_3 - Q_1$

64 − 43

**IQR = 21**

By hand



www.thrivingscholars.com



# Standard Deviation & Variance

## What are the standard deviation and variance?

- The **standard deviation** and **variance** are both **measures of dispersion**
  - They describe how spread out the data is in relation to the mean
- The **variance** is the **mean** of the **squares** of the **differences** between **the values and the mean**
  - Variance is denoted $\sigma^2$
- The **standard deviation** is the **square-root** of the **variance**
  - Standard deviation is denoted $\sigma$
- The **units** for the standard deviation are the **same** as the units for the data
- The **units** for the variance are the **square** of the units for the data

## How are the standard deviation and variance calculated for ungrouped data?

- In the exam you will be expected to use the statistics function on your **GDC** to calculate the standard deviation and the variance
- Calculating the standard deviation and the variance by hand may deepen your understanding
- The formula for **variance** is $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$
  - This can be rewritten as

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

- The formula for **standard deviation** is $\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$
  - This can be rewritten as

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2}$$

- You **do not** need to learn these formulae as you will use your GDC to calculate these

www.thrivingscholars.com

## Worked Example

Find the variance and standard deviation for the data set given below.

43 29 70 51 64 43

Find variance and standard deviation using GDC

$\sigma_x^2 = 189.333...$ and $\sigma_x = 13.759...$

Variance = 189 (3sf)
Standard deviation = 13.8 (3sf)

By hand

$\sigma^2 = \frac{\Sigma x^2}{n} - \bar{x}^2$

$\Sigma x^2 = 16136 \quad \bar{x} = 50 \quad n = 6$

$\sigma^2 = \frac{16136}{6} - 50^2 = 189.333...$

$\sigma = \sqrt{189.333...} = 13.759...$

www.thrivingscholars.com





## Ungrouped Data

### How are frequency tables used for ungrouped data?

- Frequency tables can be used for ungrouped data when you have lots of the same values within a data set
  - They can be used to collect and present data easily
- If the value 4 has a frequency of 3 this means that there are three 4's in the data set

### How are measures of central tendency calculated from frequency tables with ungrouped data?

- The **mode** is the value that has the **highest frequency**
- The **median** is the **middle** value
  - Use cumulative frequencies (running totals) to find the median
- The **mean** can be calculated by
  - Multiplying each value $x_i$ by its frequency $f_i$
  - Summing to get $\Sigma f_i x_i$
  - Dividing by the total frequency $n = \Sigma f_i$
  - This is given in the formula booklet

$$\overline{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}$$

  - Your **GDC** can calculate these statistical measures if you input the values and their frequencies using the statistics mode

### How are measures of dispersion calculated from frequency tables with ungrouped data?

- The **range** is the largest value of the data minus the smallest value of the data
- The **interquartile range** is calculated by

$$\text{IQR} = Q_3 - Q_1$$

  - The **quartiles** can be found by using your GDC and inputting the values and their frequencies
- The **standard deviation** and **variance** can be calculated by hand using the formulae
  - **Variance**

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{k} f_i x_i^2 - \overline{x}^2$$

  - **Standard deviation**

www.thrivingscholars.com

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{k} f_i x_i^2 - \bar{x}^2}$$

- You **do not need to learn** these formulae as you will be expected to use your GDC to find the standard deviation and variance
  - You may want to see these formulae to deepen your understanding

### Exam Tip

- Always check whether your answers make sense when using your GDC
  - The value for a measure of central tendency should be within the range of data

www.thrivingscholars.com

## Worked Example

The frequency table below gives information number of pets owned by 30 students in a class.

| Number of pets | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Frequency | 11 | 5 | 8 | 6 |

Find

a)
the mode.

Mode = value with highest frequency

Mode = 0

b)
the median.

Median = middle value

n = 30 so median is midpoint of 15th and 16th

| Number of pets | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Cumulative frequency | 11 | 16 | 24 | 30 |

Median = 1

c)
the mean.

Formula Booklet

| | |
|---|---|
| Mean, $\bar{x}$, of a set of data | $\bar{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}$ $\quad n = \sum_{i=1}^{k} f_i$ |

$$\bar{x} = \frac{\Sigma fx}{n} = \frac{11\times0 + 5\times1 + 8\times2 + 6\times3}{11+5+8+6} \quad \frac{39}{30}$$

Mean = 1.3

d)
the standard deviation.

www.thrivingscholars.com



Use GDC $\sigma_x = 1.159...$

Standard deviation = 1.16 (3sf)

www.thrivingscholars.com



# Grouped Data

## How are frequency tables used for grouped data?

- Frequency tables can be used for grouped data when you have lots of the same values within the same interval
  - Class intervals will be written using inequalities and without gaps
    - $10 \leq x < 20$ and $20 \leq x < 30$
  - If the class interval $10 \leq x < 20$ has a frequency of 3 this means there are three values in that interval
    - You do not know the **exact data values** when you are given grouped data

## How are measures of central tendency calculated from frequency tables with grouped data?

- The **modal class** is the class that has the **highest frequency**
  - This is for equal class intervals only
- The **median** is the **middle** value
  - The exact value can not be calculated but it can be estimated by using a **cumulative frequency graph**
- The **exact mean** can not be calculated as you do not have the raw data
- The **mean** can be **estimated** by
  - Identifying the mid-interval value (midpoint) $x_i$ for each class
  - Multiplying each value by the class frequency $f_i$
  - Summing to get $\Sigma f_i x_i$
  - Dividing by the total frequency $n = \Sigma f_i$
  - This is given in the formula booklet

$$\overline{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}$$

  - Your **GDC** can estimate the mean if you input the mid-interval values and the class frequencies using the statistics mode

## How are measures of dispersion calculated from frequency tables with grouped data?

- The exact **range** can not be calculated as the largest and smallest values are unknown
- The **interquartile range** can be estimated by

$$\text{IQR} = Q_3 - Q_1$$

  - **Estimates** of the **quartiles** can be found by using a **cumulative frequency graph**
- The **standard deviation** and **variance** can be estimated using the mid-interval values $x_i$ in the formulae
  - **Variance**

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{k} f_i x_i^2 - \overline{x}^2$$

www.thrivingscholars.com



- **Standard deviation**

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{k} f_i x_i^2 - \bar{x}^2}$$

- You **do not need to learn** these formulae as you will be expected to use your GDC to estimate the standard deviation and variance using the mid-interval values
  - You may want to see these formulae to deepen your understanding



### Exam Tip

- As you can only estimate statistical measures from a grouped frequency table it is good practice to indicate that the values are not exact
  - You can do this by rounding values rather than leaving as surds and fractions
  - $\bar{x} = 0.333$ (3sf) rather than $\bar{x} = \frac{1}{3}$

www.thrivingscholars.com

## Worked Example

The table below shows the heights in cm of a group of 25 students.

| Height, $h$ | Frequency |
|---|---|
| $150 \le h < 155$ | 3 |
| $155 \le h < 160$ | 5 |
| $160 \le h < 165$ | 9 |
| $165 \le h < 170$ | 7 |
| $170 \le h < 175$ | 1 |

a)

Write down the modal class.

Modal class = class with highest frequency

**Modal class = $160 \le h < 165$**

b)

Write down the mid-interval value of the modal class.

Mid-interval value = $\frac{\text{Upper boundary + lower boundary}}{2}$

$\frac{160 + 165}{2}$

**Mid-interval value = 162.5 cm**

c)

Calculate an estimate for the mean height.

Use mid-interval values to estimate the mean

Formula Booklet

| | |
|---|---|
| Mean, $\bar{x}$, of a set of data | $\bar{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}$ $\quad n = \sum_{i=1}^{k} f_i$ |

$\bar{x} = \frac{3 \times 152.5 + 5 \times 157.5 + 9 \times 162.5 + 7 \times 167.5 + 1 \times 172.5}{3 + 5 + 9 + 7 + 1} = \frac{4052.5}{25}$

**Estimated mean = 162.1 cm**

www.thrivingscholars.com



# Linear Transformations of Data

## Why are linear transformations of data used?

- Sometimes data might be very large or very small
- You can apply a **linear transformation** to the data to make the values more manageable
  - You may have heard this referred to as:
    - Effects of constant changes
    - Linear coding
- Linear transformations of data can **affect the statistical measures**

## How is the mean affected by a linear transformation of data?

- Let $\overline{x}$ be the **mean** of some data
- If you **multiply each value** by a constant ***k*** then you will need to **multiply the mean by *k***
  - Mean is $k\overline{x}$
- If you **add or subtract** a constant ***a*** from all the **values** then you will need to **add or subtract** the constant ***a* to the mean**
  - Mean is $\overline{x} \pm a$

## How is the variance and standard deviation affected by a linear transformation of data?

- Let $\sigma^2$ be the **variance** of some data
  - $\sigma$ is the **standard deviation**
- If you **multiply** each value by a constant ***k*** then you will need to **multiply** the **variance by *k*²**
  - Variance is $k^2\sigma^2$
  - You will need to **multiply** the **standard deviation** by the **absolute value** of *k*
    - Standard deviation is $|k|\sigma$
  - If you **add or subtract** a constant ***a*** from all the **values** then the **variance** and the **standard deviation stay the same**
    - Variance is $\sigma^2$
    - Standard deviation is $\sigma$

### Exam Tip

- If you forget these results in an exam then you can look in the HL section of the formula booklet to see them written in a more algebraic way
  - Linear transformation of a single variable

$$\mathrm{E}(aX+b) = a\mathrm{E}(X) + b$$
$$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$$

  - where E(...) means the mean and Var(...) means the variance

www.thrivingscholars.com

## Worked Example

A teacher marks his students' tests. The raw mean score is 31 marks and the standard deviation is 5 marks. The teacher standardises the score by doubling the raw score and then adding 10.

a)
Calculate the mean standardised score.

If data is multiplied by k then mean is multiplied by k

If k is added to data then k is added to the mean

$31 \times 2 + 10$

**Mean of standardised scores = 72**

b)
Calculate the standard deviation of the standardised scores.

If data is multiplied by k then standard deviation is multiplied by $|k|$

If k is added to data then standard deviation is unchanged

$5 \times 2$

**Standard deviation of standardised scores = 10**

www.thrivingscholars.com





# Outliers

## What are outliers?

- Outliers are extreme data values that do not fit with the rest of the data
  - They are either a lot bigger or a lot smaller than the rest of the data
- Outliers are defined as values that are **more than 1.5 × IQR from the nearest quartile**
  - $x$ is an outlier if $x < Q_1 - 1.5 \times IQR$ or $x > Q_3 + 1.5 \times IQR$
- Outliers can have a big effect on some statistical measures

## Should I remove outliers?

- The decision to remove outliers will **depend on the context**
- Outliers **should be removed** if they are found to be **errors**
  - The data may have been recorded incorrectly
  - For example: The number 17 may have been recorded as 71 by mistake
- Outliers **should not be removed** if they are a **valid part of the sample**
  - The data may need to be checked to verify that it is not an error
  - For example: The annual salaries of employees of a business might appear to have an outlier but this could be the director's salary

www.thrivingscholars.com

## Worked Example

The ages, in years, of a number of children attending a birthday party are given below.

2, 7, 5, 4, 8, 4, 6, 5, 5, 29, 2, 5, 13

a)

Identify any outliers within the data set.

$x$ is an outlier if $x < Q_1 - 1.5 \times IQR$ or $x > Q_3 + 1.5 \times IQR$

Using GDC

$Q_1 = 4$ and $Q_3 = 7.5$ $\therefore IQR = 3.5$

$Q_1 - 1.5 \times IQR = 4 - 1.5 \times 3.5 = -1.25$

$Q_3 + 1.5 \times IQR = 7.5 + 1.5 \times 3.5 = 12.75$

Outliers are 13 and 29

b)

Suggest which value(s) should be removed. Justify your answer.

13 should not be removed as it is a valid age of a child.

29 should be removed as this is an age of an adult.

www.thrivingscholars.com

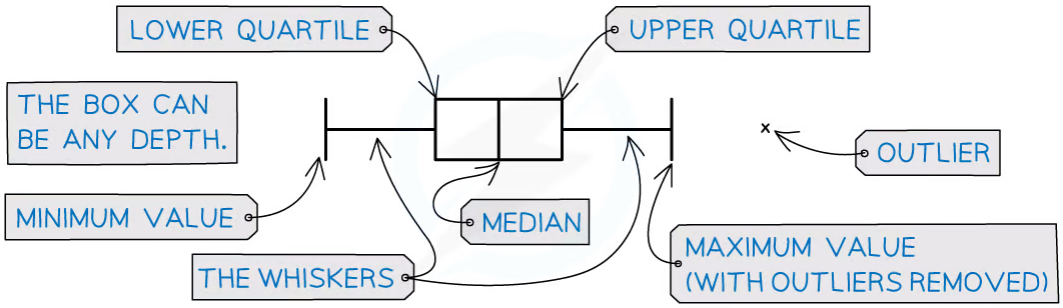


# Box Plots

Univariate data is data that is in **one variable.**

## What is a box plot (box and whisker diagram)?

- A box plot is a graph that clearly shows key statistics from a data set
  - It shows the **median, quartiles, minimum** and **maximum values** and **outliers**
  - It does not show any other individual data items
- The middle 50% of the data will be represented by the box section of the graph and the lower and upper 25% of the data will be represented by each of the whiskers
- Any **outliers** are represented with a **cross** on the **outside of the whiskers**
  - If there is an outlier then the whisker will end at the value before the outlier
- Only one axis is used when graphing a box plot
- It is still important to make sure the axis has a clear, even scale and is labelled with units



Copyright © Save My Exams. All Rights Reserved

## What are box plots useful for?

- Box plots can clearly show the shape of the distribution
  - If a box plot is symmetrical about the median then the data could be **normally distributed**
- Box plots are often used for **comparing two sets of data**
  - Two box plots will be drawn next to each other using the same axis
  - They are useful for **comparing data** because it is easy to see the main shape of the distribution of the data from a box plot
    - You can easily compare the medians and interquartile ranges

### Exam Tip

- In an exam you can use your GDC to draw a box plot if you have the raw data
  - You calculator's box plot can also include outliers so this is a good way to check

www.thrivingscholars.com

## Worked Example

The distances, in metres, travelled by 15 snails in a one-minute period are recorded and shown below:

0.5, 0.7, 1.0, 1.1, 1.2, 1.2, 1.2, 1.3, 1.4, 1.4, 1.4, 1.4, 1.5, 1.5, 1.5

a)

i)

Find the values of $Q_1$, $Q_2$ and $Q_3$.

ii)

Find the interquartile range.

iii)

Identify any outliers.

Using GDC

$Q_1 = 1.1$ m  $Q_2 = 1.3$ m  $Q_3 = 1.4$ m

$IQR = Q_3 - Q_1 = 1.4 - 1.1$

$IQR = 0.3$ m

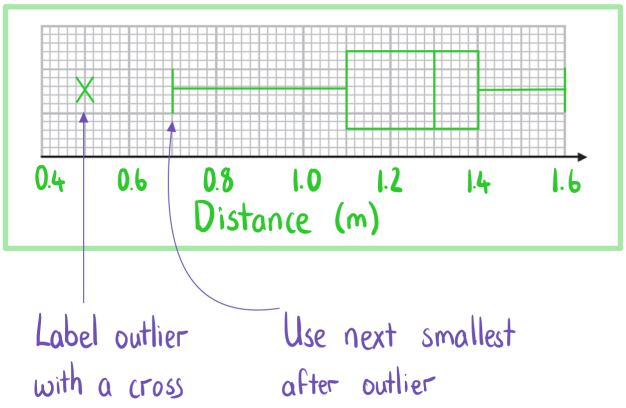
$Q_1 - 1.5 \times IQR = 1.1 - 1.5 \times 0.3 = 0.65$

$Q_3 + 1.5 \times IQR = 1.4 + 1.5 \times 0.3 = 1.85$

0.5 m is an outlier

b)

Draw a box plot for the data.



www.thrivingscholars.com

# Cumulative Frequency Graphs

## What is cumulative frequency?

- The cumulative frequency of $x$ is the running total of the frequencies for the values that are less than or equal to $x$
- For grouped data you use the upper boundary of a class interval to find the cumulative frequency of that class

## What is a cumulative frequency graph?

- A cumulative frequency graph is used with data that has been organised into a **grouped frequency** table
- Some coordinates are plotted
  - The $x$-coordinates are the **upper boundaries** of the class intervals
  - The $y$-coordinates are the **cumulative frequencies** of that class interval
- The coordinates are then joined together by hand using a **smooth increasing curve**

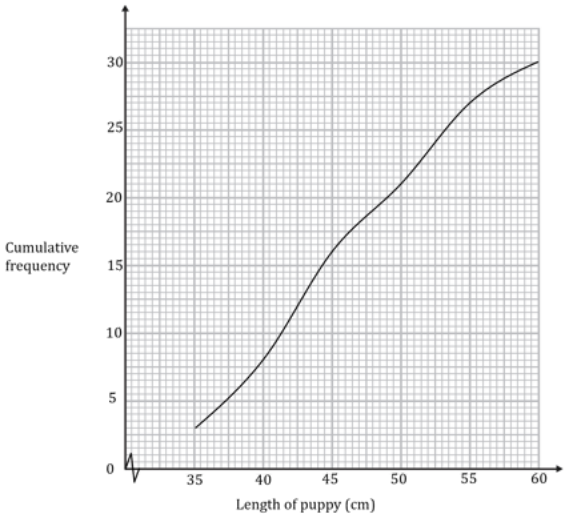
## What are cumulative frequency graphs useful for?

- They can be used to **estimate** statistical measures
  - Draw a **horizontal line** from the $y$-axis to the curve
    - For the median: draw the line at 50% of the total frequency
    - For the lower quartile: draw the line at 25% of the total frequency
    - For the upper quartile: draw the line at 75% of the total frequency
    - For the $p^{th}$ percentile: draw the line at $p$% of the total frequency
  - Draw a **vertical line** down from the curve to the $x$-axis
  - This **$x$-value** is the relevant statistical measure
- They can used to estimate the number of values that are bigger/small than a given value
  - Draw a **vertical line** from the given value on the $x$-axis to the curve
  - Draw a **horizontal line** from the curve to the $y$-axis
  - This value is an estimate for how many values are less than or equal to the given value
    - To estimate the number that is greater than the value subtract this number from the total frequency
  - They can be used to **estimate** the **interquartile range** $\text{IQR} = Q_3 - Q_1$
  - They can be used to construct a **box plot** for grouped data

www.thrivingscholars.com

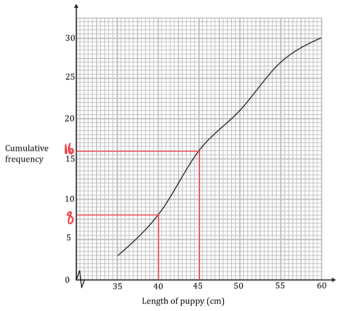
## Worked Example

The cumulative frequency graph below shows the lengths in cm, $l$, of 30 puppies in a training group.




a)

Given that the interval $40 \leq l < 45$ was used when collecting data, find the frequency of this class.

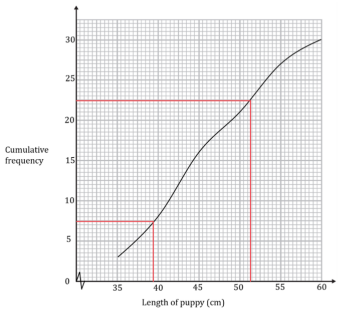



16 - 8

Frequency = 8

b)

Use the graph to find an estimate for the interquartile range of the lengths.




$\frac{1}{4} \times 30 = 7.5$ $Q_1 = 39.5$

$\frac{3}{4} \times 30 = 22.5$ $Q_3 = 51.4$

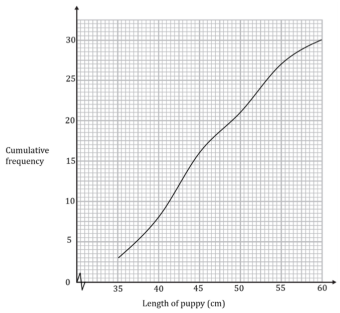
$IQR = Q_3 - Q_1 = 51.4 - 39.5$

IQR = 11.9 cm

c)

Estimate the percentage of puppies with length more than 51 cm.

www.thrivingscholars.com




$30 - 22 = 8$ puppies

longer than 51 cm

$\frac{8}{30} \times 100\% = 26.666...\%$

**26.7 % (3sf)**

www.thrivingscholars.com

# Histograms

## What is a (frequency) histogram?

- A frequency histogram clearly shows the frequency of class intervals
  - The classes will have **equal class intervals**
  - The **frequency** will be on the *y*-axis
  - The bar for a class interval will begin at the lower boundary and end at the upper boundary
- A frequency histogram is **similar to a bar chart**
  - A **bar chart** is used for **qualitative or discrete data** and **has gaps** between the bars
  - A **frequency histogram** is used for **continuous data** and **has no gaps** between bars

## What are (frequency) histograms useful for?

- They show the **modal class** clearly
- They show the shape of the distribution
  - It is important the class intervals are of equal width
- They can show whether the variable can be modelled by a **normal distribution**
  - If the shape is symmetrical and bell-shaped

www.thrivingscholars.com

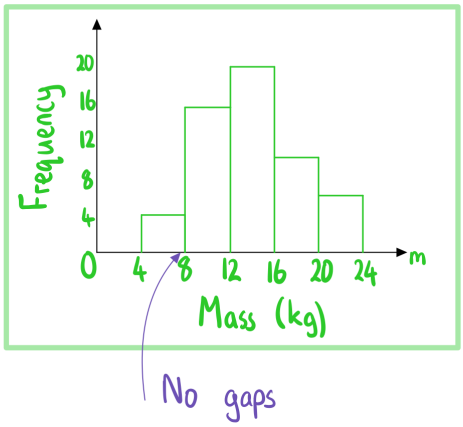
## Worked Example

The table below and its corresponding histogram show the mass, in kg, of some new born bottlenose dolphins.

| Mass, $m$ kg | Frequency |
|---|---|
| $4 \leq m < 8$ | 4 |
| $8 \leq m < 12$ | 15 |
| $12 \leq m < 16$ | 19 |
| $16 \leq m < 20$ | 10 |
| $20 \leq m < 24$ | 6 |

a)

Draw a frequency histogram to represent the data.




b)

Write down the modal class.

Modal class = class with highest frequency

Modal class = $12 \leq m < 16$

www.thrivingscholars.com





# Interpreting Data

## How do I interpret statistical measures?

- The **mode** is useful for **qualitative data**
  - It is not as useful for quantitative data as there is not always a unique mode
- The **mean includes all values**
  - It is affected by outliers
  - A smaller/larger mean is preferable depending on the scenario
    - A smaller mean time for completing a puzzle is better
    - A bigger mean score on a test is better
- The **median is not affected by outliers**
  - It does not use all the values
- The r**ange gives the full spread** of the all of the data
  - It is affected by outliers
- The **interquartile range gives the spread of the middle 50%** about the median and is not affected by outliers
  - It does not use all the values
  - A bigger IQR means the data is more spread out about the median
  - A smaller IQR means the data is more centred about the median
- The **standard deviation** and **variance** use all the values to give a measure of the **average spread** of the data about the mean
  - They are affected by outliers
  - A bigger standard deviation means the data is more spread out about the mean
  - A smaller standard deviation means the data is more centred about the mean

## How do I choose which diagram to use to represent data?

- **Box plots**
  - Can be used with ungrouped **univariate** data
  - Shows the range, interquartile range and quartiles clearly
  - Very useful for comparing data patterns quickly
- **Cumulative frequency graphs**
  - Can be used with continuous grouped univariate data
  - Shows the running total of the frequencies that fall below the upper bound of each class
- **Histograms**
  - Can be used with continuous grouped univariate data
  - Used with equal class intervals
  - Shows the frequencies of the group
- **Scatter diagrams**
  - Can be used with ungrouped **bivariate** data
  - Shows the graphical relationship between the variables

## How do I compare two or more data sets?

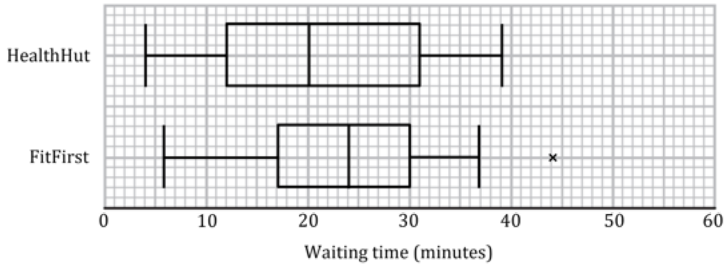
- Compare a **measure of central tendency**
  - If the data **contains outliers – use the median**

www.thrivingscholars.com

- If the data is **roughly symmetrical – use the mean**
- Compare a **measure of dispersion**
  - If the data **contains outliers – use the interquartile range**
  - If the data is **roughly symmetrical – use the standard deviation**
- Consider whether it is better to have a smaller or bigger average
  - This will depend on the context
    - A smaller average time for completing a puzzle is better
    - A bigger average score on a test is better
- Consider whether it is better to have a smaller or bigger spread
  - Usually a smaller spread means it is more consistent
- Always relate the **comparisons to the context** and consider reasons
  - Consider the **sampling technique** and the **data collection** method

## Worked Example

The box plots below show the waiting times for the two doctor surgeries, HealthHut and FitFirst.




Compare the two distributions of waiting times in context.

Compare :

- a measure of central tendency
- a measure of dispersion

HealthHut's median waiting time is smaller than FitFirst's (20 < 24). On average patients get seen quicker at HealthHut.

FitFirst's interquartile range is smaller than HealthHut's (13 < 19). There is less variability of waiting times at FitFirst.

www.thrivingscholars.com





# Scatter Diagrams

## What does bivariate data mean?

- **Bivariate data** is data which is collected on **two variables** and looks at how one of the factors affects the other
  - Each data value from one variable will be **paired** with a data value from the other variable
  - The two variables are often related, but do not have to be

## What is a scatter diagram?

- A **scatter diagram** is a way of graphing bivariate data
  - One variable will be on the $x$-axis and the other will be on the y-axis
  - The variable that can be **controlled** in the data collection is known as the **independent** or **explanatory variable** and is plotted on the $x$-axis
  - The variable that is **measured** or discovered in the data collection is known as the **dependent** or **response variable** and is plotted on the y-axis
- Scatter diagrams can contain **outliers** that do not follow the trend of the data

### Exam Tip

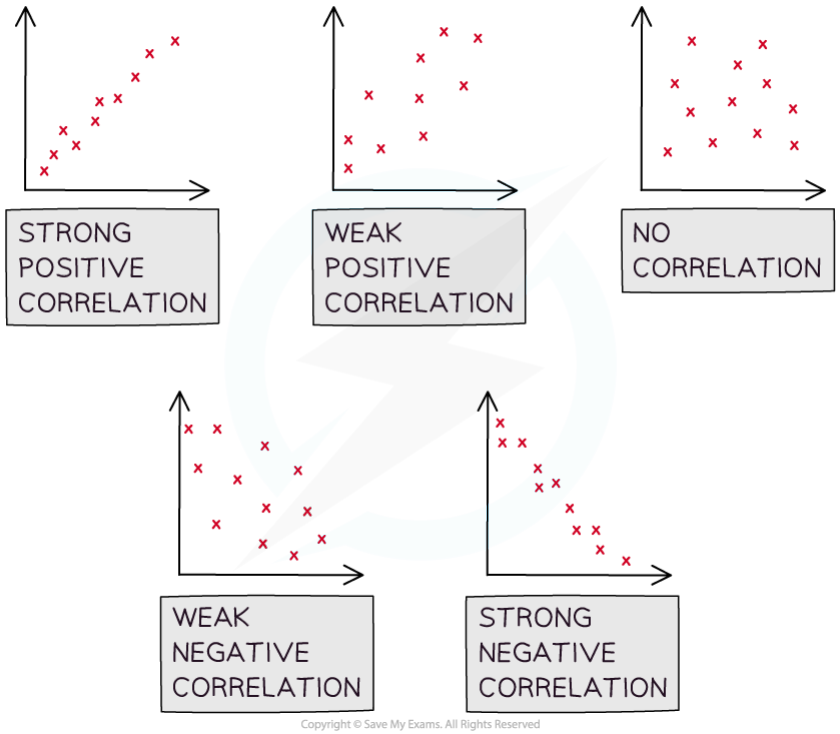
- If you use scatter diagrams in your Internal Assessment then be aware that finding outliers for bivariate data is different to finding outliers for univariate data
  - $(x, y)$ could be an outlier for the bivariate data even if $x$ and $y$ are not outliers for their separate univariate data

www.thrivingscholars.com

# Correlation

## What is correlation?

- **Correlation** is how the **two variables change in relation to each other**
  - Correlation could be the result of a **causal relationship** but this is not always the case
- **Linear correlation** is when the changes are proportional to each other
- **Perfect linear correlation** means that the bivariate data will all lie on a straight line on a scatter diagram
- When describing correlation mention
  - The type of the correlation
    - **Positive correlation** is when an **increase** in one variable results in the other variable **increasing**
    - **Negative correlation** is when an **increase** in one variable results in the other variable **decreasing**
    - **No linear correlation** is when the data points don't appear to follow a trend
  - The strength of the correlation
    - **Strong linear correlation** is when the data points lie **close** to a **straight line**
    - **Weak linear correlation** is when the data points are **not close** to a **straight line**
- If there is **strong linear correlation** you can draw a l**ine of best fit** (by eye)
  - The line of best fit will pass through the mean point $(\overline{x}, \overline{y})$
  - If you are asked to draw a line of best fit
    - Plot the mean point
    - Draw a line going through it that follows the trend of the data



Copyright © Save My Exams. All Rights Reserved

## What is the difference between correlation and causation?

- It is important to be aware that just because correlation exists, it does not mean that the change in one of the variables is **causing** the change in the other variable

www.thrivingscholars.com



    - **Correlation does not imply causation!**
- If a change in one variable **causes** a change in the other then the two variables are said to have a **causal relationship**
    - Observing correlation between two variables does **not always** mean that there is a causal relationship
        - There could be **underlying factors** which is causing the correlation
    - Look at the two variables in question and consider the context of the question to decide if there could be a causal relationship
        - If the two variables are temperature and number of ice creams sold at a park then it is likely to be a causal relationship
        - Correlation may exist between global temperatures and the number of monkeys kept as pets in the UK but they are unlikely to have a causal relationship

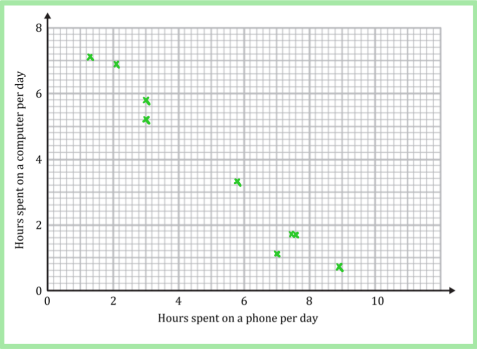
www.thrivingscholars.com

## Worked Example

A teacher is interested in the relationship between the number of hours her students spend on a phone per day and the number of hours they spend on a computer. She takes a sample of nine students and records the results in the table below.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Hours spent on a phone per day** | 7.6 | 7.0 | 8.9 | 3.0 | 3.0 | 7.5 | 2.1 | 1.3 | 5.8 |
| **Hours spent on a computer per day** | 1.7 | 1.1 | 0.7 | 5.8 | 5.2 | 1.7 | 6.9 | 7.1 | 3.3 |

a)
Draw a scatter diagram for the data.



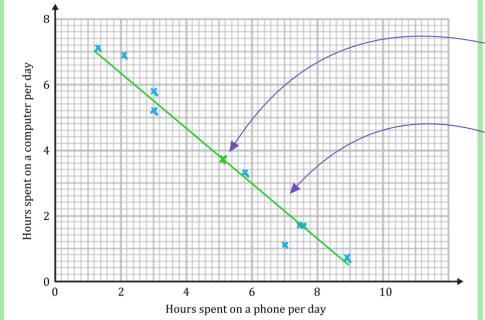

b)
Describe the correlation.

Strong negative linear correlation

c)
Draw a line of best fit.

www.thrivingscholars.com

Mean point $(\bar{x}, \bar{y}) = (5.133..., 3.722...)$




Plot the mean point

Draw it by eye

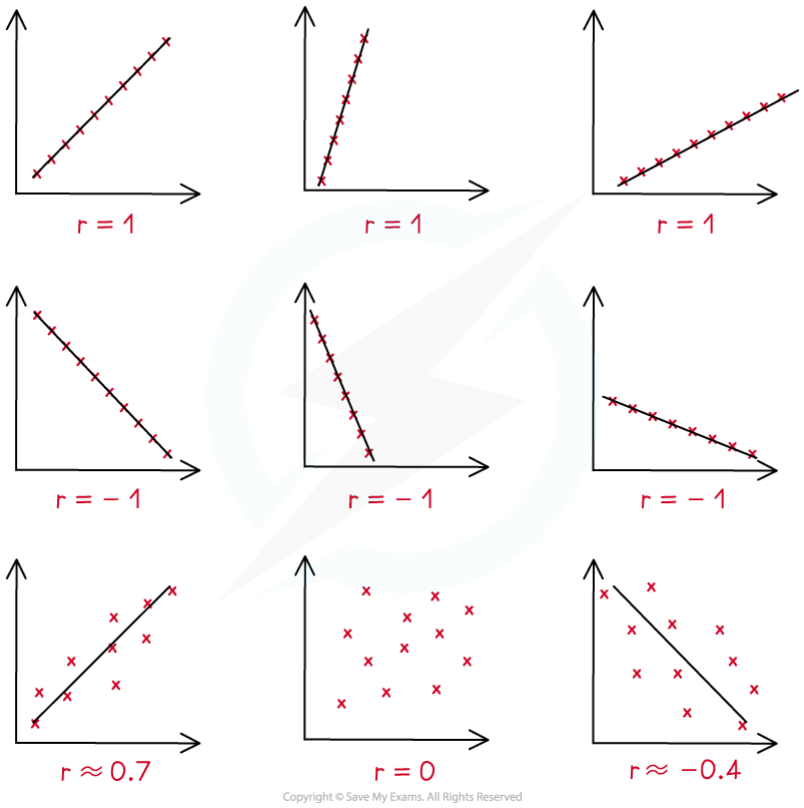
www.thrivingscholars.com



# PMCC

## What is Pearson's product-moment correlation coefficient?

- Pearson's product-moment correlation coefficient (PMCC) is a way of giving a numerical value to a **linear relationship** of bivariate data
- The PMCC of a sample is denoted by the letter $r$
  - $r$ can take any value such that $-1 \leq r \leq 1$
  - A **positive value** of $r$ describes **positive correlation**
  - A **negative value** of $r$ describes **negative correlation**
  - $r = 0$ means there is **no linear correlation**
  - $r = 1$ means **perfect positive linear** correlation
  - $r = -1$ means **perfect negative linear** correlation
  - The closer to 1 or -1 the stronger the correlation




Copyright © Save My Exams. All Rights Reserved

## How do I calculate Pearson's product-moment correlation coefficient (PMCC)?

- You will be expected to use the statistics mode on your GDC to calculate the PMCC
- The formula can be useful to deepen your understanding

$$r = \frac{S_{xy}}{S_x S_y}$$

www.thrivingscholars.com



- $S_{xy} = \sum_{i=1}^{n} x_i y_i - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$ is linked to the **covariance**
- $S_x = \sqrt{\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2}$ and $S_y = \sqrt{\sum_{i=1}^{n} y_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right)^2}$ are linked to the **variances**

  - You **do not need to learn this** as using your GDC will be expected

## When does the PMCC suggest there is a linear relationship?

- **Critical values** of *r* indicate when the PMCC would suggest there is a linear relationship
  - In your exam you will be given critical values where appropriate
  - Critical values will depend on the size of the sample
- If the **absolute value** of the **PMCC** is **bigger** than the **critical value** then this suggests a linear model is appropriate

www.thrivingscholars.com

# Linear Regression

## What is linear regression?

- If **strong linear correlation** exists on a scatter diagram then the data can be modelled by a **linear model**
  - Drawing lines of best fit by eye is not the best method as it can be difficult to judge the best position for the line
- The **least squares regression line** is the line of best fit that minimises the **sum of the squares** of the gap between the line and each data value
- It can be calculated by either looking at:
  - **vertical distances** between the line and the data values
    - This is the **regression line of *y* on *x***
  - **horizontal distances** between the line and the data values
    - This is the **regression line of *x* on *y***

## How do I find the regression line of *y* on *x*?

- The **regression line of y on x** is written in the form $y = ax + b$
- *a* is the **gradient** of the line
  - It represents the change in *y* for each individual unit change in *x*
    - If *a* is **positive** this means *y* **increases** by *a* for a unit increase in *x*
    - If *a* is **negative** this means *y* **decreases** by |a| for a unit increase in *x*
- *b* is the ***y* – intercept**
  - It shows the value of *y* when *x* is zero
- You are expected to use your **GDC** to find the equation of the regression line
  - Enter the bivariate data and choose the **model "*ax* + *b*"**
  - Remember the **mean point** $(\overline{x}, \overline{y})$ will lie on the regression line

## How do I find the regression line of *x* on *y*?

- The **regression line of x on y** is written in the form $x = cy + d$
- c is the **gradient** of the line
  - It represents the change in *x* for each individual unit change in *y*
    - If c is **positive** this means *x* **increases** by c for a unit increase in *y*
    - If c is **negative** this means *x* **decreases** by |c| for a unit increase in *y*
- *d* is the ***x* – intercept**
  - It shows the value of *x* when *y* is zero
- You are expected to use your **GDC** to find the equation of the regression line
  - It is found the same way as the regression line of *y* on *x* but with the two data sets **switched around**
  - Remember the **mean point** $(\overline{x}, \overline{y})$ will lie on the regression line

## How do I use a regression line?

- The regression line can be used to decide what type of correlation there is if there is no scatter diagram
  - If the gradient is **positive** then the data set has **positive correlation**
  - If the gradient is **negative** then the data set has **negative correlation**
- The regression line can also be used to **predict** the value of a **dependent variable** from an **independent variable**

www.thrivingscholars.com



- The equation for the *y* on *x* line should only be used to make predictions for *y*
  - Using a *y* on *x* line to predict *x* is not always reliable
- The equation for the *x* on *y* line should only be used to make predictions for *x*
  - Using an *x* on *y* line to predict *y* is not always reliable
- Making a prediction within the range of the given data is called **interpolation**
  - This is usually reliable
  - The stronger the correlation the more reliable the prediction
- Making a prediction outside of the range of the given data is called **extrapolation**
  - This is much less reliable
- The prediction will be more reliable if the number of data values in the original sample set is bigger

- The *y* on *x* and *x* on *y* regression lines intersect at the mean point $(\overline{x}, \overline{y})$

### Exam Tip

- Once you calculate the values of *a* and *b* store then in your GDC
  - This means you can use the full display values rather than the rounded values when using the linear regression equation to predict values
  - This avoids rounding errors

www.thrivingscholars.com

## Worked Example

The table below shows the scores of eight students for a maths test and an English test.

| **Maths ($x$)** | 7 | 18 | 37 | 52 | 61 | 68 | 75 | 82 |
|---|---|---|---|---|---|---|---|---|
| **English ($y$)** | 5 | 3 | 9 | 12 | 17 | 41 | 49 | 97 |

a)
Write down the value of Pearson's product-moment correlation coefficient, $r$.

Enter data into GDC.

$r = 0.79433...$

$r = 0.794$ (3 sf)

b)
Write down the equation of the regression line of $y$ on $x$, giving your answer in the form $y = ax + b$ where $a$ and $b$ are constants to be found.

a is the coefficient of $x$ $\quad a = 0.943579...$

b is the constant term $\quad b = -18.05398...$

$y = 0.944x - 18.1$

c)
Write down the equation of the regression line of $x$ on $y$, giving your answer in the form $x = cy + d$ where $c$ and $d$ are constants to be found.

Swap the two sets of data

c is the coefficient of $y$ $\quad c = 0.668700...$

d is the constant term $\quad d = 30.52410...$

$x = 0.669y + 30.5$

d)
Use the appropriate regression line to predict the score on the maths test of a student who got a score of 63 on the English test.

www.thrivingscholars.com

$y = 63$ so use $x$ on $y$ line

$x = (0.668700\ldots) \times 63 + (30.52410\ldots) = 72.652\ldots$

Maths score 72.7

www.thrivingscholars.com



# 4.3 Probability

## 4.3.1 Probability & Types of Events

### Probability Basics

### What key words and terminology are used with probability?

- An **experiment** is a repeatable activity that has a result that can be observed or recorded
  - **Trials** are what we call the repeats of the experiment
- An **outcome** is a possible result of a trial
- An **event** is an outcome or a collection of outcomes
  - Events are usually denoted with capital letters: *A*, *B*, etc
  - $n(A)$ is the number of outcomes that are included in event *A*
  - An event can have one or more than one outcome
- A **sample space** is the set of all possible outcomes of an experiment
  - This is denoted by *U*
  - $n(U)$ is the total number of outcomes
  - It can be represented as a **list** or a **table**

### How do I calculate basic probabilities?

- If all outcomes are **equally likely** then probability for each outcome is the same
  - Probability for each outcome is $\frac{1}{n(U)}$
- **Theoretical probability** of an event can be calculated without using an experiment by dividing the number of outcomes of that event by the total number of outcomes

$$P(A) = \frac{n(A)}{n(U)}$$

  - This is given in the **formula booklet**
  - Identifying all possible outcomes either as a list or a table can help
- **Experimental probability** (also known as **relative frequency**) of an outcome can be calculated using results from an experiment by dividing its frequency by the number of trials
  - **Relative frequency** of an outcome is $\frac{\text{Frequency of that outcome from the trials}}{\text{Total number of trials } (n)}$

### How do I calculate the expected number of occurrences of an outcome?

- **Theoretical probability** can be used to calculate the **expected number of occurrences** of an outcome from *n* trials
- If the probability of an outcome is *p* and there are *n* trials then:
  - The expected number of occurrences is ***np***
  - This **does not mean** that there will **exactly *np* occurrences**
  - If the experiment is repeated multiple times then we expect the number of occurrences to average out to be *np*

### What is the complement of an event?

- The probabilities of all the outcomes **add up to 1**

www.thrivingscholars.com



- Complementary events are when there are **two events** and **exactly one** of them will occur
  - One event has to occur but both events can not occur at the same time
- The **complement of event *A*** is the event where event ***A* does not happen**
  - This can be thought of as **not *A***
  - This is denoted $A'$

$$\mathrm{P}(A) + \mathrm{P}(A') = 1$$

    - This is in the **formula booklet**
    - It is commonly written as $\mathrm{P}(A') = 1 - \mathrm{P}(A)$

## What are different types of combined events?

- The **intersection** of two events ($A$ and $B$) is the event where **both *A* and *B*** occur
  - This can be thought of as ***A* and *B***
  - This is denoted as $A \cap B$
- The **union** of two events ($A$ and $B$) is the event where ***A* or *B* or both occur**
  - This can be thought of as ***A* or *B***
  - This is denoted $A \cup B$
- The event where $A$ occurs given that event $B$ has occurred is called **conditional probability**
  - This can be thought as ***A* given *B***
  - This is denoted $A \mid B$

## How do I find the probability of combined events?

- The probability of $A$ **or** $B$ (or both) occurring can be found using the formula

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

    - This is given in the **formula booklet**
    - You subtract the probability of $A$ and $B$ both occurring because it has been included twice (once in $\mathrm{P}(A)$ and once in $\mathrm{P}(B)$ )
- The probability of $A$ **and** $B$ occurring can be found using the formula

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A)$$

    - A rearranged version is given in the **formula booklet**
    - Basically you multiply the probability of $A$ by the probability of $B$ then happening



Exam Tip

- In an exam drawing a Venn diagram or tree diagram can help even if the question does not ask you to

www.thrivingscholars.com

## Worked Example

Dave has two fair spinners, *A* and *B*. Spinner *A* has three sides numbered 1, 4, 9 and spinner *B* has four sides numbered 2, 3, 5, 7. Dave spins both spinners and forms a two-digit number by using the spinner *A* for the first digit and spinner *B* for the second digit.

$T$ is the event that the two-digit number is a multiple of 3.

a)
List all the possible two-digit numbers.

A two-way table would be a systematic way to list all the outcomes

| | 2 | 3 | 5 | 7 |
|---|---|---|---|---|
| 1 | 12 | 13 | 15 | 17 |
| 4 | 42 | 43 | 45 | 47 |
| 9 | 92 | 93 | 95 | 97 |

b)
Find $\mathrm{P}(T)$.

$P(T) = \frac{n(T)}{n(U)}$ ← Number of multiples of 3 / ← Total number of outcomes

{12, 15, 42, 45, 93} are the multiples of 3

$P(T) = \frac{5}{12}$

c)
Find $\mathrm{P}(T')$.

$P(T) + P(T') = 1 \Rightarrow P(T') = 1 - P(T)$

$P(T') = 1 - \frac{5}{12}$

$P(T') = \frac{7}{12}$

www.thrivingscholars.com

# Independent & Mutually Exclusive Events

## What are mutually exclusive events?

- Two events are **mutually exclusive** if they **cannot both occur**
  - For example: when rolling a dice the events "getting a prime number" and "getting a 6" are mutually exclusive
- If *A* and *B* are mutually exclusive events then:
  - $\mathrm{P}(A \cap B) = 0$

## What are independent events?

- Two events are **independent** if **one occurring does not affect the probability of the other occurring**
  - For example: when flipping a coin twice the events “getting a tails on the first flip” and “getting a tails on the second flip” are independent
- If *A* and *B* are independent events then:
  - $\mathrm{P}(A \mid B) = \mathrm{P}(A)$ and $\mathrm{P}(B \mid A) = \mathrm{P}(B)$
- If *A* and *B* are independent events then:
  - $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$
    - This is given in the **formula booklet**
    - This is a useful formula to test whether two events are statistically independent

## How do I find the probability of combined mutually exclusive events?

- If *A* and *B* are **mutually exclusive** events then

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$$

    - This is given in the **formula booklet**
    - This occurs because $\mathrm{P}(A \cap B) = 0$
- For any two events *A* and *B* the events $A \cap B$ and $A \cap B'$ are **mutually exclusive** and *A* is the **union** of these two events
  - $\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B')$
    - This works for any two events *A* and *B*

www.thrivingscholars.com

## Worked Example

a)

A student is chosen at random from a class. The probability that they have a dog is 0.8, the probability they have a cat is 0.6 and the probability that they have a cat or a dog is 0.9.

Find the probability that the student has both a dog and a cat.

Let D be event "has a dog" and C be "has a cat"

$P(D \cup C) = P(D) + P(C) - P(D \cap C)$

$0.9 = 0.8 + 0.6 - P(D \cap C)$

$P(D \cap C) = 0.5$

b)

Two events, $Q$ and $R$, are such that $P(Q) = 0.8$ and $P(Q \cap R) = 0.1$.

Given that $Q$ and $R$ are independent, find $P(R)$.

Q and R independent $\Rightarrow P(Q \cap R) = P(Q)P(R)$

$0.1 = 0.8 \times P(R) \quad \therefore P(R) = \frac{0.1}{0.8}$

$P(R) = 0.125$ or $\frac{1}{8}$

c)

Two events, $S$ and $T$, are such that $P(S) = 2P(T)$.

Given that $S$ and $T$ are mutually exclusive and that $P(S \cup T) = 0.6$ find $P(S)$ and $P(T)$.

S and T mutually exclusive $\Rightarrow P(S \cup T) = P(S) + P(T)$

$0.6 = P(S) + P(T)$

$P(S) = 2P(T)$

$0.6 = 2P(T) + P(T)$

$0.6 = 3P(T)$

$P(T) = 0.2$ and $P(S) = 0.4$

www.thrivingscholars.com





# Conditional Probability

## What is conditional probability?

- **Conditional probability** is where the probability of an **event** happening can vary depending on the outcome of a prior event
- The event *A* happening **given that** event *B* has happened is denoted *A*|*B*
- A common example of conditional probability involves selecting multiple objects from a bag **without replacement**
  - The probability of selecting a certain item changes depending on what was selected before
    - This is because the total number of items will change as they are not replaced once they have been selected

## How do I calculate conditional probabilities?

- Some conditional probabilities can be calculated by using counting outcomes
  - Probabilities without replacement can be calculated like this
  - For example: There are 10 balls in a bag, 6 of them are red, two of them are selected without replacement
    - To find the probability that the second ball selected is red given that the first one is red count how many balls are left:
    - A red one has already been selected so there are 9 balls left and 5 are red so the probability is $\frac{5}{9}$
- You can use sample space diagrams to find the probability of *A* given *B*:
  - reduce your sample space to just include outcomes for event *B*
  - find the proportion that also contains outcomes for event *A*
- There is a formula for conditional probability that you should use
  - $\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$
  - This is given in the **formula booklet**
  - This can be rearranged to give $\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A \mid B)$
  - By symmetry you can also write $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A)$

## How do I tell if two events are independent using conditional probabilities?

- If $A$ and $B$ are two events then they are independent if:
  - $\mathrm{P}(A \mid B) = \mathrm{P}(A) = \mathrm{P}(A \mid B')$
- Equally you can still use $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$ to test for independence
  - This is given in the **formula booklet**

www.thrivingscholars.com

## Worked Example

Let $R$ be the event that it is raining in Weatherville and $T$ be the event that there is a thunderstorm in Weatherville.

It is known that $\mathrm{P}(T) = 0.035$, $\mathrm{P}(T \cap R) = 0.03$ and $\mathrm{P}(T \mid R) = 0.15$.

a)
Find the probability that it is raining in Weatherville.

Formula booklet

| | |
|---|---|
| Conditional probability | $\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ |

$P(T|R) = \frac{P(T \cap R)}{P(R)}$

Substitute the values in

$0.15 = \frac{0.03}{P(R)}$

$P(R) = \frac{0.03}{0.15}$

$P(R) = 0.2$

b)
State whether the events $R$ and $T$ are independent. Give a reason for your answer.

If R and T are independent then $P(T|R) = P(T)$

$P(T|R) = 0.15$ and $P(T) = 0.035$

R and T are not independent as $P(T|R) \neq P(T)$

www.thrivingscholars.com





# Bayes' Theorem

## What is Bayes' Theorem

- **Bayes' Theorem** allows you **switch the order** of conditional probabilities
  - If you know $\mathrm{P}(B)$, $\mathrm{P}(B')$ and $\mathrm{P}(A \mid B)$ then Bayes' Theorem allows you to find $\mathrm{P}(B \mid A)$
- Essentially if you have a **tree diagram** you will already know the conditional probabilities of the **second branches**
  - Bayes' Theorem allows you to find the **conditional probabilities** if you **switch the order** of the events
- For any two events *A* and *B* Bayes' Theorem states:

$$\mathrm{P}(B \mid A) = \frac{\mathrm{P}(B)\mathrm{P}(A \mid B)}{\mathrm{P}(B)\mathrm{P}(A \mid B) + \mathrm{P}(B')\mathrm{P}(A \mid B')}$$

  - This is given in the **formula booklet**
  - This formula is derived using the formulae:
    - $\mathrm{P}(B \mid A) = \dfrac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$
    - $\mathrm{P}(A) = \mathrm{P}(B \cap A) + \mathrm{P}(B' \cap A)$
    - $\mathrm{P}(B \cap A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ and $\mathrm{P}(B' \cap A) = \mathrm{P}(B')\mathrm{P}(A \mid B')$
- Bayes' Theorem can be **extended** to **mutually exclusive events** $B_1, B_2, \ldots, B_n$ and any other event *A*
  - In your exam you will have a **maximum of three** mutually exclusive events

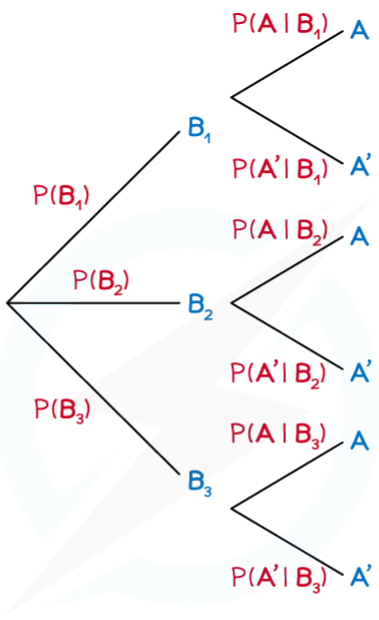
$$\mathrm{P}(B_i \mid A) = \frac{\mathrm{P}(B_i)\mathrm{P}(A \mid B_i)}{\mathrm{P}(B_1)\mathrm{P}(A \mid B_1) + \mathrm{P}(B_2)\mathrm{P}(A \mid B_2) + \mathrm{P}(B_3)\mathrm{P}(A \mid B_3)}$$

    - This is given in the **formula booklet**

## How do I calculate conditional probabilities using Bayes' Theorem?

- Start by **drawing a tree diagram**
  - Label $B_1$ & $B_2$ (& $B_3$ if necessary) on the **first set** of branches
  - Label ***A*** **&** ***A'*** on the **second set** of branches
- The questions will give you enough information to **label the probabilities** on this tree
- **Identify the probabilities needed** to use Bayes' Theorem
  - The probabilities will come in pairs: $\mathrm{P}(B_i)$ and $\mathrm{P}(A \mid B_i)$

www.thrivingscholars.com




$$P(B_1 | A) = \frac{P(B_1)\,P(A | B_1)}{P(B_1)\,P(A | B_1) + P(B_2)\,P(A | B_2) + P(B_3)\,P(A | B_3)}$$

SWAP $B_1$ for $B_1$, $B_2$ or $B_3$ DEPENDING ON THE PROBABILITY NEEDED

THE DENOMINATOR DOES NOT CHANGE

Copyright © Save My Exams. All Rights Reserved

## Exam Tip

- In an exam you are less likely to make a mistake when using the formula if you draw a tree diagram first

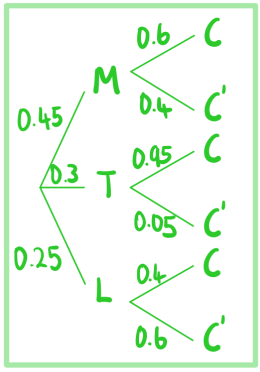
www.thrivingscholars.com

## Worked Example

Lucy is doing a quiz. For each question there's a 45% chance that it is about music, 30% chance that it is about TV and 25% chance that it is about literature. The probability that Lucy answers a question correctly is 0.6 for music, 0.95 for TV and 0.4 for literature.

a)
Draw a tree diagram to represent this information.

Let M be the event that the question is on music
T be the event that the question is on TV
L be the event that the question is on literature
C be the event that Lucy answers correctly




$P(C'|M) = 1 - P(C|M)$
$= 1 - 0.6$

$P(C'|T) = 1 - P(C|T)$
$= 1 - 0.95$

$P(C'|L) = 1 - P(C|L)$
$= 1 - 0.4$

b)
Given that Lucy answered a question correctly, find the probability that it was about TV.

Formula booklet

| | |
|---|---|
| Bayes' theorem | $P(B_i \mid A) = \frac{P(B_i)P(A \mid B_i)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + P(B_3)P(A \mid B_3)}$ |

Let $A = C$ $B_1 = M$ $B_2 = T$ $B_3 = L$

$$P(T|C) = \frac{P(T)P(C|T)}{P(M)P(C|M) + P(T)P(C|T) + P(L)P(C|L)}$$

$$= \frac{0.3 \times 0.95}{0.45 \times 0.6 + 0.3 \times 0.95 + 0.25 \times 0.4}$$

$$P(T|C) = \frac{57}{131} = 0.435 \text{ (3sf)}$$

www.thrivingscholars.com



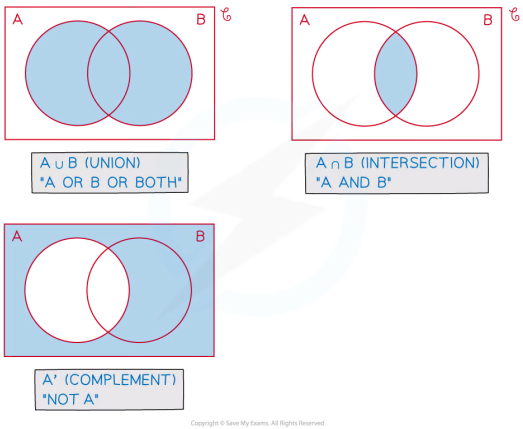
# Venn Diagrams

## What is a Venn diagram?

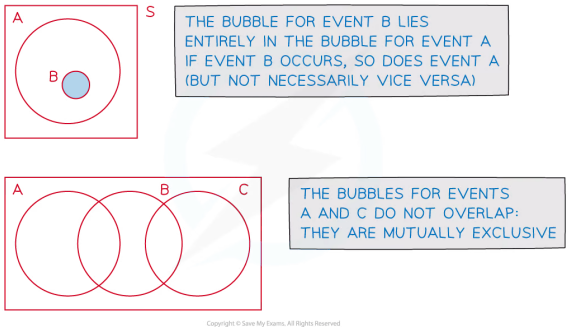
- A Venn diagram is a way to illustrate **events** from an **experiment** and are particularly useful when there is an overlap between possible **outcomes**
- A Venn diagram consists of
  - a **rectangle** representing the **sample space (*U*)**
    - The rectangle is labelled *U*
    - Some mathematicians instead use *S* or $\xi$
  - a **circle** for each **event**
    - Circles may or may not overlap depending on which **outcomes** are shared between **events**
- The numbers in the circles represent either the **frequency** of that event or the **probability** of that event
  - If the **frequencies** are used then they should **add up to the total frequency**
  - If the **probabilities** are used then they should **add up to 1**

## What do the different regions mean on a Venn diagram?

- $A'$ is represented by the regions that are **not in** the *A* circle
- $A \cap B$ is represented by the region where the *A* and *B* circles **overlap**
- $A \cup B$ is represented by the regions that **are in** *A* or *B* or both
- Venn diagrams show '**AND**' and '**OR**' statements easily
- Venn diagrams also instantly show **mutually exclusive** events as these circles will **not overlap**
- **Independent** events can not be instantly seen
  - You need to use probabilities to deduce if two events are independent

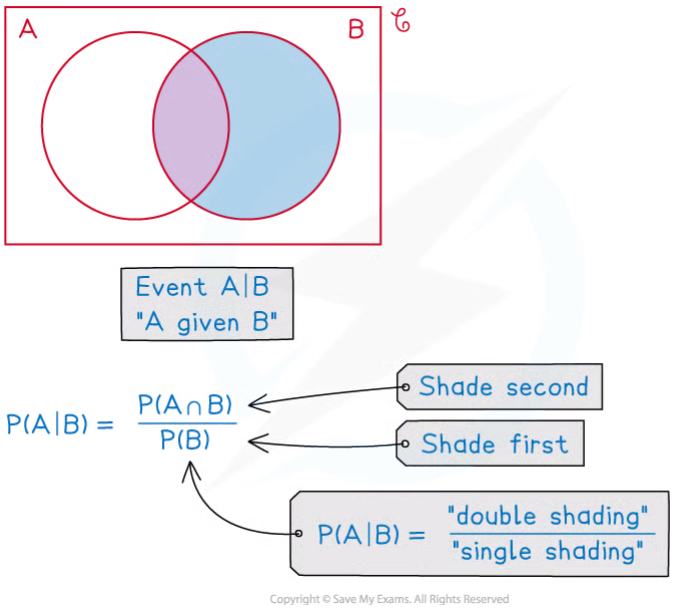


www.thrivingscholars.com




## How do I solve probability problems involving Venn diagrams?

- Draw, or add to a given Venn diagram, filling in as many values as possible from the information provided in the question
- It is usually helpful to work from the centre outwards
  - Fill in **intersections** (overlaps) first
- If two events are independent you can use the formula
  - $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$
- To find the conditional probability $\mathrm{P}(A \mid B)$
  - Add together the frequencies/probabilities in the *B* circle
    - This is your denominator
  - Out of those frequencies/probabilities add together the ones that are also in the *A* circle
    - This is your numerator
  - Evaluate the fraction



www.thrivingscholars.com

### Exam Tip

- If you struggle to fill in a Venn diagram in an exam:
  - Label the missing parts using algebra
  - Form equations using known facts such as:
    - the sum of the probabilities should be 1
    - $P(A \cap B) = P(A)P(B)$ if A and B are independent events

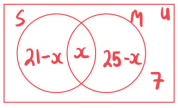
www.thrivingscholars.com

## Worked Example

40 people are asked if they have sugar and/or milk in their coffee. 21 people have sugar, 25 people have milk and 7 people have neither.

a)
Draw a Venn diagram to represent the information.

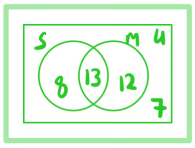
Find the centre first




Total should be 40

$(21-x)+x+(25-x)+7=40$

$53-x=40 \quad \therefore x=13$




b)
One of the 40 people are randomly selected, find the probability that they have sugar but not milk with their coffee.

S and not M is the part of S circle that does not include M

$P(S \cap M') = \frac{8}{40}$ Remember to write as a fraction of the total

$P(S \cap M') = \frac{1}{5}$

c)
Given that a person who has sugar is selected at random, find the probability that they have milk with their coffee.

Given that sugar has been selected we only want the S circle as our total.

Out of the S circle 13 also have milk

$P(M|S) = \frac{13}{21}$

www.thrivingscholars.com

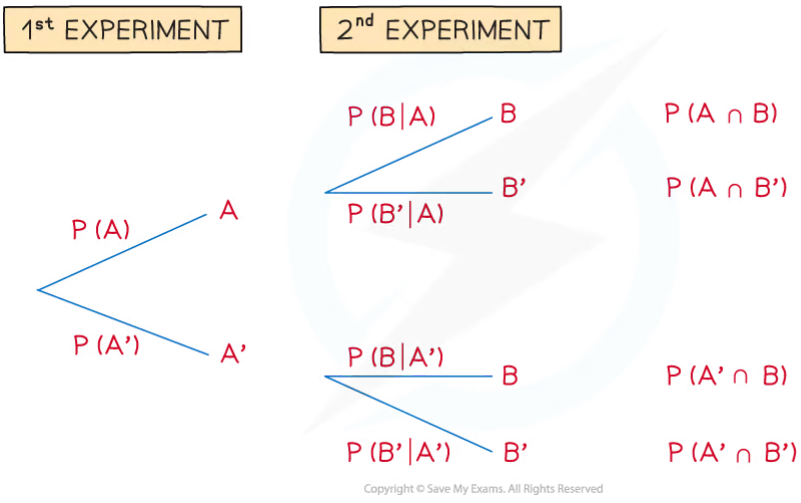
# Tree Diagrams

## What is a tree diagram?

- A **tree diagram** is another way to show the outcomes of combined events
  - They are very useful for intersections of events
- The events on the branches must be **mutually exclusive**
  - Usually they are an event and its complement
- The probabilities on the second sets of branches **can depend** on the outcome of the first event
  - These are **conditional probabilities**
- When selecting the items from a bag:
  - The second set of branches will be the **same** as the first if the items **are replaced**
  - The second set of branches will be the **different** to the first if the items **are not replaced**

## How are probabilities calculated using a tree diagram?

- To find the probability that two events happen together you **multiply** the corresponding probabilities on their branches
  - It is helpful to find the probability of all combined outcomes once you have drawn the tree
- To find the probability of an event you can:
  - **add together** the probabilities of the **combined outcomes** that are part of that event
    - For example: $\mathrm{P}(A \cup B) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B') + \mathrm{P}(A' \cap B)$
  - **subtract** the probabilities of the combined outcomes that are not part of that event from 1
    - For example: $\mathrm{P}(A \cup B) = 1 - \mathrm{P}(A' \cap B')$



Copyright © Save My Exams. All Rights Reserved

## Do I have to use a tree diagram?

- If there are **multiple events** or trials then a tree diagram can get big
- You can break down the problem by using the words **AND/OR/NOT** to help you find probabilities without a tree
- You can speed up the process by only drawing parts of the tree that you are interested in

www.thrivingscholars.com



## Which events do I put on the first branch?

- If the events *A* and *B* are **independent** then the **order does not matter**
- If the events *A* and *B* are **not independent** then the **order does matter**
  - If you have the probability of ***A* given *B*** then put ***B* on the first set** of branches
  - If you have the probability of ***B* given *A*** then put ***A* on the first set** of branches

### Exam Tip

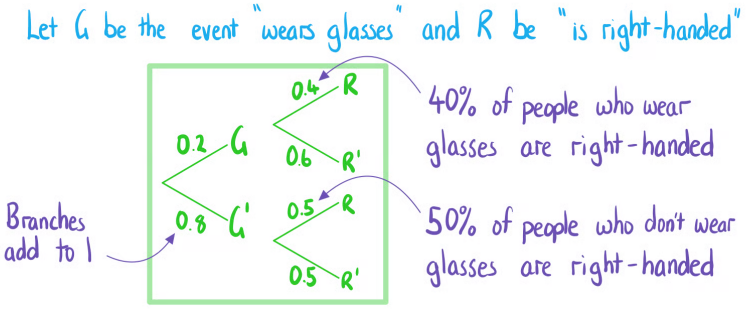
- In an exam do not waste time drawing a full tree diagram for scenarios with lots of events unless the question asks you to
  - Only draw the parts that you are interested in

www.thrivingscholars.com

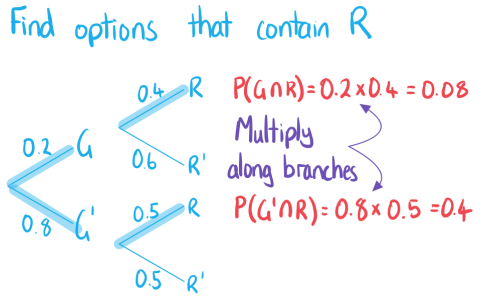
## Worked Example

20% of people in a company wear glasses. 40% of people in the company who wear glasses are right-handed. 50% of people in the company who don't wear glasses are right-handed.

a)
Draw a tree diagram to represent the information.




b)
One of the people in the company are randomly selected, find the probability that they are right-handed.




$P(R) = P(G \cap R) + P(G \cap R') = 0.08 + 0.4$

$P(R) = 0.48$

c)
Given that a person who is right-handed is selected at random, find the probability that they wear glasses.

$$P(G|R) = \frac{P(G \cap R)}{P(R)} = \frac{0.08}{0.48}$$

$$P(G|R) = \frac{1}{6}$$

www.thrivingscholars.com



# 4.4 Probability Distributions

## 4.4.1 Discrete Probability Distributions

### Discrete Probability Distributions

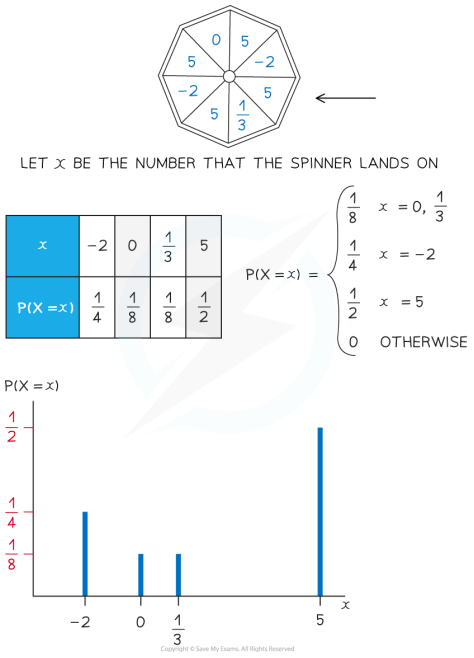
### What is a discrete random variable?

- A **random variable** is a variable whose value depends on the outcome of a **random event**
  - The value of the random variable is not known until the event is carried out (this is what is meant by 'random' in this case)
- **Random variables** are denoted using **upper case letters** ($X$, $Y$, etc)
- **Particular outcomes** of the event are denoted using **lower case letters** ($x$, $y$, etc)
- $\mathrm{P}(X=x)$ means "the probability of the random variable $X$ taking the value $x$"
- A **discrete** random variable (often abbreviated to DRV) can only take **certain values** within a set
  - Discrete random variables **usually count** something
  - Discrete random variables usually can only take a finite number of values but it is possible that it can take an infinite number of values (see the examples below)
- **Examples** of discrete random variables include:
  - The number of times a coin lands on heads when flipped 20 times
    - this has a finite number of outcomes: {0,1,2,…,20}
  - The number of emails a manager receives within an hour
    - this has an infinite number of outcomes: {1,2,3,…}
  - The number of times a dice is rolled until it lands on a 6
    - this has an infinite number of outcomes: {1,2,3,…}
  - The number that a dice lands on when rolled once
    - this has a finite number of outcomes: {1,2,3,4,5,6}

### What is a probability distribution of a discrete random variable?

- A **discrete probability distribution** fully describes **all the values** that a discrete random variable can take along with their **associated probabilities**
  - This can be given in a **table**
  - Or it can be given as a **function** (called a discrete probability distribution function or "pdf")
  - They can be represented by **vertical line graphs** (the possible values for along the horizontal axis and the probability on the vertical axis)
- The **sum of the probabilities** of **all the values** of a discrete random variable is **1**
  - This is usually written $\sum \mathrm{P}(X=x)=1$
- A **discrete uniform distribution** is one where the random variable takes a finite number of values each with an **equal probability**
  - If there are n values then the probability of each one is $\frac{1}{n}$

www.thrivingscholars.com




## How do I calculate probabilities using a discrete probability distribution?

- First **draw a table** to represent the probability distribution
  - If it is given as a function then find each probability
  - If any probabilities are unknown then use algebra to represent them
- **Form an equation** using $\sum \mathrm{P}(X=x)=1$
  - Add together all the probabilities and make the sum equal to 1
- To find $\mathrm{P}(X=k)$
  - If $k$ is a possible value of the random variable $X$ then $\mathrm{P}(X=k)$ will be given in the table
  - If $k$ is not a possible value then $\mathrm{P}(X=k)=0$
- To find $\mathrm{P}(X \leq k)$
  - Identify all possible values, $x_i$, that $X$ can take which satisfy $x_i \leq k$
  - Add together all their corresponding probabilities
  - $\mathrm{P}(X \leq k) = \sum_{x_i \leq k} \mathrm{P}(X=x_i)$
  - Some mathematicians use the notation $\mathrm{F}(x)$ to represent the cumulative distribution
    - $\mathrm{F}(x)=\mathrm{P}(X \leq x)$
- Using a similar method you can find $\mathrm{P}(X<k)$, $\mathrm{P}(X>k)$ and $\mathrm{P}(X \geq k)$
- As all the probabilities add up to 1 you can form the following equivalent equations:
  - $\mathrm{P}(X<k)+\mathrm{P}(X=k)+\mathrm{P}(X>k)=1$
  - $\mathrm{P}(X>k)=1-\mathrm{P}(X \leq k)$
  - $\mathrm{P}(X \geq k)=1-\mathrm{P}(X<k)$

## How do I know which inequality to use?

- $\mathrm{P}(X \leq k)$ would be used for phrases such as:
  - At most , no greater than , etc
- $\mathrm{P}(X<k)$ would be used for phrases such as:

www.thrivingscholars.com



- Fewer than
- $P(X \geq k)$ would be used for phrases such as:
  - At least , no fewer than , etc
- $P(X > k)$ would be used for phrases such as:
  - Greater than , etc

## Worked Example

The probability distribution of the discrete random variable $X$ is given by the function

$$P(X = x) = \begin{cases} kx^2 & x = -3, -1, 2, 4 \\ 0 & \text{otherwise.} \end{cases}$$

a)

Show that $k = \frac{1}{30}$.

Construct a table

| $x$ | $-3$ | $-1$ | $2$ | $4$ |
|---|---|---|---|---|
| $P(X=x)$ | $9k$ | $k$ | $4k$ | $16k$ |

Substitute in the values of $x$
e.g. $P(X=-3) = k(-3)^2 = 9k$

The probabilities add up to 1

$9k + k + 4k + 16k = 1$

$30k = 1$

$k = \frac{1}{30}$

b)

Calculate $P(X \leq 3)$.

Substitute $k$ into the probabilities

| $x$ | $-3$ | $-1$ | $2$ | $4$ |
|---|---|---|---|---|
| $P(X=x)$ | $\frac{3}{10}$ | $\frac{1}{30}$ | $\frac{2}{15}$ | $\frac{8}{15}$ |

$X \leq 3 : X = -3, -1, 2$

$P(X \leq 3) = P(X=-3) + P(X=-1) + P(X=2)$

$= \frac{3}{10} + \frac{1}{30} + \frac{2}{15}$

$P(X \leq 3) = \frac{7}{15}$

www.thrivingscholars.com





# Expected Values E(X)

## What does E(X) mean and how do I calculate E(X)?

- **E(X)** means the **expected value** or the **mean** of a **random variable X**
  - The expected value does not need to be an obtainable value of *X*
  - For example: the expected value number of times a coin will land on tails when flipped 5 times is 2.5
- For a **discrete** random variable, it is calculated by:
  - **Multiplying each value** of $X$ with its corresponding **probability**
  - **Adding** all these terms together

$$\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$$

    - This is given in the **formula booklet**
- Look out for **symmetrical** distributions (where the values of X are symmetrical and their probabilities are symmetrical) as the mean of these is the same as the median
  - For example: if X can take the values 1, 5, 9 with probabilities 0.3, 0.4, 0.3 respectively then by symmetry the mean would be 5

## How can I decide if a game is fair?

- Let *X* be the random variable that represents the **gain/loss** of a player in a game
  - *X* will be **negative** if there is a **loss**
- Normally the expected gain or loss is calculated by **subtracting** the **cost to play** the game from the **expected value** of the **prize**
- If E(*X*) is **positive** then it means the player can **expect to make a gain**
- If E(*X*) is **negative** then it means the player can **expect to make a loss**
- The game is called **fair** if the **expected gain is 0**
  - E(*X*) = 0

www.thrivingscholars.com

## Worked Example

Daphne pays $5 to play a game where she wins a prize of $1, $5, $10 or $100. The random variable $W$ represents the amount she wins and has the probability distribution shown in the following table:

| $w$ | 1 | 5 | 10 | 100 |
|---|---|---|---|---|
| $P(W=w)$ | 0.35 | 0.5 | 0.05 | 0.01 |

a)

Calculate the expected value of Daphne's prize.

Formula booklet

| | |
|---|---|
| Expected value of a discrete random variable $X$ | $E(X)=\sum x\,P(X=x)$ |

$E(W) = \sum w P(W=w)$

$= 1 \times 0.35 + 5 \times 0.5 + 10 \times 0.05 + 100 \times 0.01$

Expected value = $4.35

b)

Determine whether the game is fair.

A game is fair is expected gain/loss is 0

Prize - cost

$4.35 - 5 = -0.65$

Expected loss is $0.65 so game is not fair

www.thrivingscholars.com

# Variance Var(X)

## What does Var(X) mean and how do I calculate Var(X)?

- **Var(X)** means the variance of a **random variable X**
  - The **standard deviation** is the **square root** of the variance
    - This provides a **measure of the spread** of the outcomes of *X*
  - The variance and standard deviation can **never be negative**
- The variance of X is the **mean of the squared difference** between X and the mean

$$\mathrm{Var}(X) = \mathrm{E}(X - \mu)^2$$

  - This is given in the **formula booklet**
- This formula can be rearranged into the more useful form:

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

  - This is given in the **formula booklet**
    - Compare this formula to the formula for the **variance of a set of data**
  - This formula works for both **discrete** and **continuous** X

## How do I calculate E(X²) for discrete X?

- **E(X²)** means the **expected value** or the **mean** of the **random variable** defined as **X²**
- For a **discrete** random variable, it is calculated by:
  - **Squaring** each value of X to get the values of $X^2$
  - **Multiplying each value** of $X^2$ which its corresponding **probability**
  - **Adding** all these terms together
    - $\mathrm{E}(X^2) = \sum x^2 \mathrm{P}(X = x)$
    - This is given in the formula booklet as part of the formula for Var(*X*)
    - $\mathrm{Var}(X) = \sum x^2 \mathrm{P}(X = x) - \mu^2$
- **E(f(X))** can be found in a similar way

## Is E(X²) equal to E(X)²?

- **Definitely not!**
  - They are only equal if X can only take one value
- E(X²) is the **mean of the values of X²**
- E(X)² is the **square of the mean of the values of X**
- To see the difference
  - Imagine a random variable X that can only take 1 and -1 with equal chance
  - E(X) = 0 so **E(X)² = 0**
  - The square values are 1 and 1 so **E(X²) = 1**

### Exam Tip

- In an exam you can enter the probability distribution into your GDC using the statistics mode
  - Enter the possible values as the data
  - Enter the probabilities as the frequencies
- You can then calculate the mean and variance just like you would with data

www.thrivingscholars.com

## Worked Example

The score on a game is represented by the random variable $S$ defined below.

| $s$ | 0 | 1 | 2 | 10 |
|---|---|---|---|---|
| $P(S=s)$ | 0.4 | 0.3 | 0.25 | 0.05 |

Calculate $\text{Var}(S)$.

Calculate E(S)

Formula booklet

| Expected value of a discrete random variable $X$ | $E(X)=\sum x\,P(X=x)$ |
|---|---|

$E(S) = \sum_S P(S=s) = 0\times0.4 + 1\times0.3 + 2\times0.25 + 10\times0.05 = 1.3$

Calculate $E(S^2)$

$E(S^2) = \sum_S^2 P(S=s) = 0^2\times0.4 + 1^2\times0.3 + 2^2\times0.25 + 10^2\times0.05 = 6.3$

Calculate Var(S)

Formula booklet

| Variance | $\text{Var}(X)=E(X-\mu)^2=E(X^2)-[E(X)]^2$ |
|---|---|

$\text{Var}(S) = E(S^2) - [E(S)]^2 = 6.3 - 1.3^2$

$\text{Var}(S) = 4.61$

www.thrivingscholars.com



# Transformation of a Single Variable

## How do I calculate the expected value and variance of a transformation of X?

- Suppose $X$ is **transformed** by the function $f$ to form a new variable $T = f(X)$
  - This means the function $f$ is applied to all possible values of $X$
- Create a **new probability distribution table**
  - The top row contains the values $t_i = f(x_i)$
  - The bottom row still contains the values $\mathrm{P}(X = x_i)$ which are unchanged as:
    - $\mathrm{P}(X = x_i) = \mathrm{P}(f(X) = f(x_i)) = \mathrm{P}(T = t_i)$
    - Some values of $T$ may be equal so you can add their probabilities together
- The **mean** is calculated in the same way
  - $\mathrm{E}(T) = \sum t\mathrm{P}(X = x)$
- The **variance** is calculated using the same formula
  - $\mathrm{Var}(T) = \mathrm{E}(T^2) - [\mathrm{E}(T)]^2$

## Are there any shortcuts?

- There are formulae which can be used if the transformation is **linear**
  - $T = aX + b$ where $a$ and $b$ are constants
- If the transformation is **not linear** then there are **no shortcuts**
  - You will have to first find the probability distribution of $T$

## What are the formulae for E($aX \pm b$) and Var($aX \pm b$)?

- If $a$ and $b$ are constants then the following formulae are true:
  - **E($aX \pm b$) = $a$E($X$) ± b**
  - **Var($aX \pm b$) = $a^2$ Var($X$)**
    - These are given in the **formula booklet**
- This is the same as linear transformations of data
  - The mean is affected by multiplication and addition/subtraction
  - The variance is affected by multiplication but not addition/subtraction
- Remember division can be written as a multiplication
  - $\frac{X}{a} = \frac{1}{a}X$

www.thrivingscholars.com

## Worked Example

$X$ is a random variable such that $\mathrm{E}(X) = 5$ and $\mathrm{Var}(X) = 4$.

Find the value of:

(i)
$\mathrm{E}(3X + 5)$
(ii)
$\mathrm{Var}(3X + 5)$
(iii)
$\mathrm{Var}(2 - X)$.

Formula booklet

| | |
|---|---|
| Linear transformation of a single random variable | $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$<br>$\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$ |

$E(3X+5) = 3E(X) + 5 = 3(5) + 5$ **$E(3X+5) = 20$**

$Var(3X+5) = 3^2 Var(X) = 9(4)$ **$Var(3X+5) = 36$**

$Var(2-X) = (-1)^2 Var(X) = 1(4)$ **$Var(2-X) = 4$**

www.thrivingscholars.com

# 4.5 Binomial Distribution

## 4.5.1 The Binomial Distribution

### Properties of Binomial Distribution

### What is a binomial distribution?

- A binomial distribution is a **discrete probability distribution**
- A **discrete random variable** $X$ follows a **binomial distribution** if it **counts the number of successes** when an experiment satisfies the following conditions:
  - There are a **fixed finite number of trials (*n*)**
  - The outcome of each trial is **independent** of the outcomes of the other trials
  - There are **exactly two outcomes** of each trial (**success or failure**)
  - The **probability of success is constant (*p*)**
- If $X$ follows a binomial distribution then it is denoted $X \sim \mathrm{B}(n, p)$
  - $n$ is the **number of trials**
  - $p$ is the **probability of success**
- The **probability of failure is 1 – *p*** which is sometimes denoted as ***q***
- The formula for the probability of ***r* successful trials** is given by:
  - $\mathrm{P}(X = r) = {}^{n}C_{r} \times p^{r}(1-p)^{n-r}$ for $r = 0, 1, 2, \ldots, n$
    - ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$ where $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$
  - You will be expected to use the distribution function on your **GDC to calculate probabilities** with the binomial distribution

### What are the important properties of a binomial distribution?

- The **expected number (mean)** of successful trials is

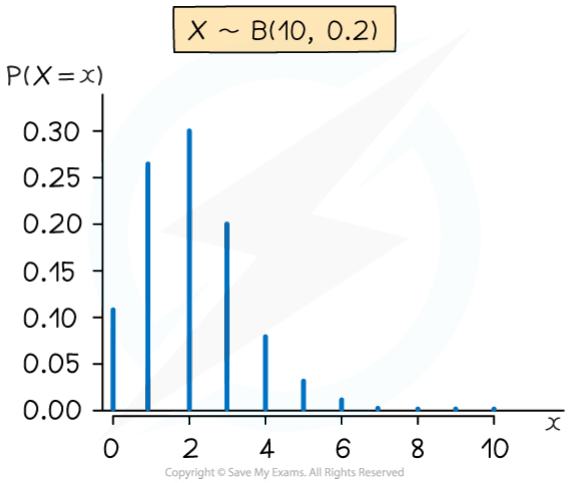
$$\mathrm{E}(X) = np$$

  - You are given this in the **formula booklet**
- The **variance** of the number of successful trials is

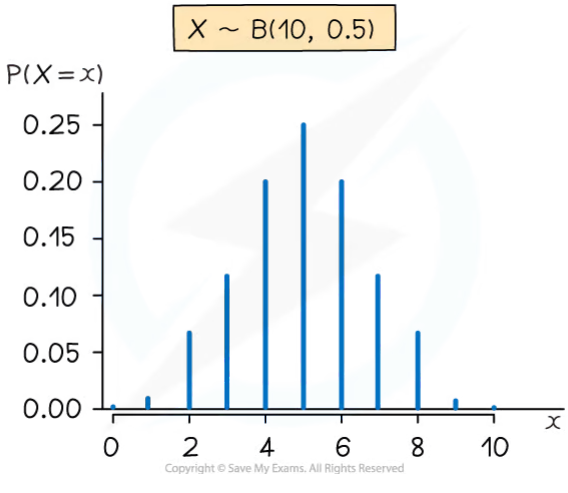
$$\mathrm{Var}(X) = np(1-p)$$

  - You are given this in the **formula booklet**
  - Square root to get the **standard deviation**
- The distribution can be represented visually using a vertical line graph
  - If $p$ is **close to 0** then the graph has a **tail to the right**
  - If $p$ is **close to 1** then the graph has a **tail to the left**
  - If $p$ is **close to 0.5** then the graph is **roughly symmetrical**
  - **If $p = 0.5$** then the graph is **symmetrical**

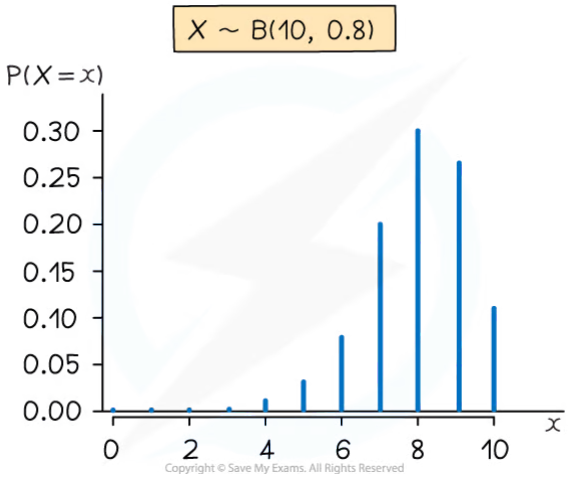
www.thrivingscholars.com



Copyright © Save My Exams. All Rights Reserved



Copyright © Save My Exams. All Rights Reserved



Copyright © Save My Exams. All Rights Reserved

www.thrivingscholars.com



# Modelling with Binomial Distribution

## How do I set up a binomial model?

- **Identify** what a **trial** is in the scenario
  - For example: rolling a dice, flipping a coin, checking hair colour
- **Identify** what the **successful outcome** is in the scenario
  - For example: rolling a 6, landing on tails, having black hair
- **Identify** the **parameters**
  - *n* is the number of trials and *p* is the probability of success in each trial
- Make sure you **clearly state** what your **random variable** is
  - For example, let be the number of students in a class of 30 with black hair

## What can be modelled using a binomial distribution?

- Anything that satisfies the **four conditions**
- For example: let $T$ be the number of times a fair coin lands on tails when flipped 20 times:
  - A trial is flipping a coin: There are 20 trials so ***n* = 20**
  - We can assume each coin flip does not affect subsequent coin flips: they are **independent**
  - A success is when the coin lands on tails: **Two outcomes** - tails or not tails (heads)
  - The coin is fair: The probability of tails is constant with ***p* = 0.5**
- Sometimes it might **seem like there are more than two outcomes**
  - For example: let $Y$ be the number of yellow cars that are in a car park full of 100 cars
    - Although there are more than two possible colours of cars, here the trial is whether a car is yellow so there are two outcomes (yellow or not yellow)
    - $Y$ would still need to fulfil the other conditions in order to follow a binomial distribution
- Sometimes a **sample may be taken from a population**
  - For example: 30% of people in a city have blue eyes, a sample of 30 people from the city is taken and $X$ is the number of them with blue eyes
    - As long as the population is large and the sample is random then it can be assumed that each person has a 30% chance of having blue eyes

## What can not be modelled using a binomial distribution?

- Anything where the number of trials is **not fixed** or is **infinite**
  - The number of emails received in an hour
  - The number of times a coin is flipped until it lands on heads
- Anything where the outcome of **one trial affects** the outcome of the **other trials**
  - The number of caramels that a person eats when they eat 5 sweets from a bag containing 6 caramels and 4 marshmallows
    - If you eat a caramel for your first sweet then there are less caramels left in the bag when you choose your second sweet
  - Anything where there are **more than two outcomes** of a trial
    - A person's shoe size
    - The number a dice lands on when rolled
  - Anything where the **probability of success changes**
    - The number of times that a person can swim a length of a swimming pool in under a minute when swimming 50 lengths

www.thrivingscholars.com



- The probability of swimming a lap in under a minute will decrease as the person gets tired
- The probability is **not constant**

### Exam Tip

- An exam question might involve different types of distributions so make it clear which distribution is being used for each variable

www.thrivingscholars.com

## Worked Example

It is known that 8% of a large population are immune to a particular virus. Mark takes a sample of 50 people from this population. Mark uses a binomial model for the number of people in his sample that are immune to the virus.

a)
State the distribution that Mark uses.

A trial is checking if a person is immune to the virus
A success is if the person is immune.
Let X be the number of people in the sample immune to the virus

$X \sim B(50, 0.08)$

Number of people in sample

Probability of being immune to the virus

b)
State two assumptions that Mark must make in order to use a binomial model.

Mark needs to assume that:

- each person in the population has an 8% chance of being immune
- the sample is random and the people are independent a person being immune does not affect the immunity of others

For example:
If all 50 came from the same family then they would not be independent

c)
Calculated the expected number of people in the sample that are immune to the virus.

Formula booklet

| Binomial distribution $X \sim \mathrm{B}(n, p)$ | |
|---|---|
| Mean | $\mathrm{E}(X) = np$ |

$E(X) = 50 \times 0.08$

4 people

www.thrivingscholars.com





# Calculating Binomial Probabilities

Throughout this section we will use the random variable $X \sim \mathrm{B}(n, p)$. For binomial, the probability of $X$ taking a non-integer or negative value is always zero. Therefore any values of $X$ mentioned in this section will be assumed to be non-negative integers.

## How do I calculate P($X = x$): the probability of a single value for a binomial distribution?

- You should have a **GDC** that can calculate **binomial probabilities**
- You want to use the "**Binomial Probability Distribution**" function
  - This is sometimes shortened to BPD, Binomial PD or Binomial Pdf
- You will need to enter:
  - The '*x*' value - the value of *x* for which you want to find $\mathrm{P}(X = x)$
  - The '*n*' value - the **number of trials**
  - The '*p*' value - the **probability of success**
- Some calculators will give you the option of **listing the probabilities** for **multiple values of *x* at once**
- There is a formula that you can use but you are expected to be able to use the distribution function on your GDC
  - $\mathrm{P}(X = x) = {}^{n}\mathrm{C}_{x} \times p^{x}(1-p)^{n-x}$
    - ${}^{n}\mathrm{C}_{x} = \dfrac{n!}{r!(n-r)!}$

## How do I calculate P($a \le X \le b$): the cumulative probabilities for a binomial distribution?

- You should have a **GDC** that can calculate **cumulative binomial probabilities**
  - Most calculators will find $\mathrm{P}(a \le X \le b)$
  - Some calculators can only find $\mathrm{P}(X \le b)$
    - The identities below will help in this case
- You should use the "**Binomial Cumulative Distribution**" function
  - This is sometimes shortened to BCD, Binomial CD or Binomial Cdf
- You will need to enter:
  - The lower value - this is the **value *a***
    - This can be zero in the case $\mathrm{P}(X \le b)$
  - The upper value - this is the **value *b***
    - This can be *n* in the case $\mathrm{P}(X \ge a)$
  - The '*n*' value - the **number of trials**
  - The '*p*' value - the **probability of success**

## How do I find probabilities if my GDC only calculates P($X \le x$)?

- To calculate P($X \le x$) just enter *x* into the cumulative distribution function
- To calculate P($X < x$) use:
  - $\mathrm{P}(X < x) = \mathrm{P}(X \le x - 1)$ which works when *X* is a binomial random variable
    - P($X < 5$) = P($X \le 4$)

www.thrivingscholars.com



- To calculate P($X > x$) use:
  - $\mathrm{P}(X > x) = 1 - \mathrm{P}(X \leq x)$ which works for any random variable $X$
    - P($X > 5$) = 1 - P($X \leq 5$)
- To calculate P($X \geq x$) use:
  - $\mathrm{P}(X \geq x) = 1 - \mathrm{P}(X \leq x - 1)$ which works when $X$ is a binomial random variable
    - P($X \geq 5$) = 1 - P($X \leq 4$)
- To calculate P($a \leq X \leq b$) use:
  - $\mathrm{P}(a \leq X \leq b) = \mathrm{P}(X \leq b) - \mathrm{P}(X \leq a - 1)$ which works when $X$ is a binomial random variable
    - P($5 \leq X \leq 9$) = P($X \leq 9$) - P($X \leq 4$)

## What if an inequality does not have the equals sign (strict inequality)?

- For a binomial distribution (as it is discrete) you could **rewrite all strict inequalities** (< and >) as **weak inequalities** (≤ and ≥) by using the identities for a binomial distribution
  - $\mathrm{P}(X < x) = \mathrm{P}(X \leq x - 1)$ and $\mathrm{P}(X > x) = \mathrm{P}(X \geq x + 1)$
  - For example: P($X < 5$) = P($X \leq 4$) and P($X > 5$) = P($X \geq 6$)
- It helps to think about the **range of integers** you want
  - Identify the smallest and biggest integers in the range
- If your range has no minimum or maximum then use 0 or $n$
  - $\mathrm{P}(X \leq b) = \mathrm{P}(0 \leq X \leq b)$
  - $\mathrm{P}(X \geq a) = \mathrm{P}(a \leq X \leq n)$
- $\mathrm{P}(a < X \leq b) = \mathrm{P}(a + 1 \leq X \leq b)$
  - P($5 < X \leq 9$) = P($6 \leq X \leq 9$)
- $\mathrm{P}(a \leq X < b) = \mathrm{P}(a \leq X \leq b - 1)$
  - P($5 \leq X < 9$) = P($5 \leq X \leq 8$)
- $\mathrm{P}(a < X < b) = \mathrm{P}(a + 1 \leq X \leq b - 1)$
  - P($5 < X < 9$) = P($6 \leq X \leq 8$)

### Exam Tip

- If the question is in context then write down the inequality as well as the final answer
  - This means you still might gain a mark even if you accidentally type the wrong numbers into your GDC

www.thrivingscholars.com

## Worked Example

The random variable $X \sim \mathrm{B}(40, 0.35)$. Find:

i)
$\mathrm{P}(X = 10)$.

Identify n and p  n = 40  p = 0.35

Use binomial probability distribution on GDC

P(X = 10) = 0.057056...

**P(X = 10) = 0.057 (3sf)**

ii)
$\mathrm{P}(X \leq 10)$.

Identify upper and lower values

P(X ≤ 10) = P(0 ≤ X ≤ 10)

Use binomial cumulative distribution on GDC

P(X ≤ 10) = 0.121491...

**P(X ≤ 10) = 0.121 (3sf)**

iii)
$\mathrm{P}(8 < X < 15)$.

Identify upper and lower values

P(8 < X < 15) = P(9 ≤ X ≤ 14)

Use binomial cumulative distribution on GDC

P(9 ≤ X ≤ 14) = 0.541827...

**P(8<X<15) = 0.542 (3sf)**

www.thrivingscholars.com

# 4.6 Normal Distribution

## 4.6.1 The Normal Distribution

### Properties of Normal Distribution

The binomial distribution is an example of a discrete probability distribution. The normal distribution is an example of a **continuous** probability distribution.

### What is a continuous random variable?

- A continuous random variable (often abbreviated to CRV) is a random variable that can take **any value** within a range of infinite values
  - Continuous random variables **usually measure** something
  - For example, height, weight, time, etc

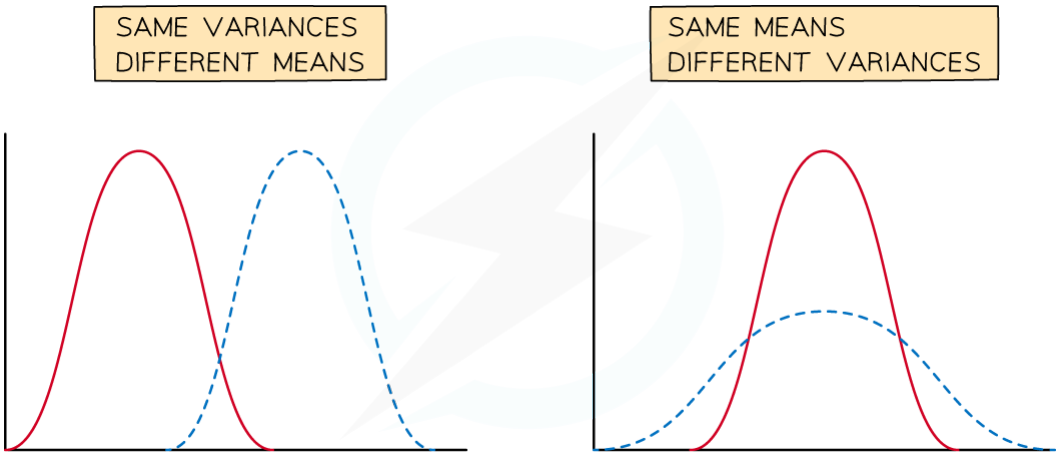
### What is a continuous probability distribution?

- A continuous probability distribution is a probability distribution in which the random variable $X$ is continuous
- The probability of $X$ being a **particular value is always zero**
  - $\mathrm{P}(X = k) = 0$ for any value $k$
  - Instead we define the **probability density function** $\mathrm{f}(x)$ for a specific value
    - This is a function that describes the **relative likelihood** that the random variable would be close to that value
  - We talk about the **probability** of $X$ being within a **certain range**
- A continuous probability distribution can be represented by a continuous graph (the values for $X$ along the horizontal axis and probability **density** on the vertical axis)
- The **area under the graph** between the points $x = a$ and $x = b$ is equal to $\mathrm{P}(a \leq X \leq b)$
  - The **total area under the graph equals 1**
- As $\mathrm{P}(X = k) = 0$ for any value $k$, it does not matter if we use strict or weak inequalities
  - $\mathrm{P}(X \leq k) = \mathrm{P}(X < k)$ for any value $k$ when $X$ is a **continuous random variable**

### What is a normal distribution?

- A normal distribution is a **continuous probability distribution**
- The **continuous random variable** $X$ can follow a normal distribution if:
  - The distribution is **symmetrical**
  - The distribution is **bell-shaped**
- If $X$ follows a normal distribution then it is denoted $X \sim \mathrm{N}(\mu, \sigma^2)$
  - $\mu$ is the **mean**
  - $\sigma^2$ is the **variance**
  - $\sigma$ is the **standard deviation**
- If the **mean** changes then the graph is **translated horizontally**
- If the **variance** increases then the graph is **widened horizontally** and **made taller vertically** to maintain the same area
  - A **small variance** leads to a **tall** curve with a **narrow** centre
  - A **large variance** leads to a **short** curve with a **wide** centre

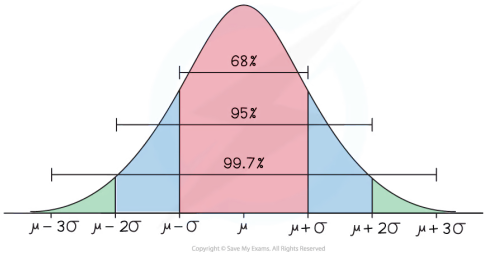
www.thrivingscholars.com




Copyright © Save My Exams. All Rights Reserved

## What are the important properties of a normal distribution?

- The **mean** is $\mu$
- The **variance** is $\sigma^2$
  - If you need the **standard deviation** remember to square root this
- The normal distribution is symmetrical about
  - Mean = Median = Mode = $\mu$
- There are the results:
  - Approximately **two-thirds (68%)** of the data lies within **one standard deviation** of the mean ($\mu \pm \sigma$)
  - Approximately **95%** of the data lies within **two standard deviations** of the mean ($\mu \pm 2\sigma$)
  - Nearly **all of the data (99.7%)** lies within **three standard deviations** of the mean ($\mu \pm 3\sigma$)




Copyright © Save My Exams. All Rights Reserved

www.thrivingscholars.com

# Modelling with Normal Distribution

## What can be modelled using a normal distribution?

- A lot of real-life continuous variables can be modelled by a normal distribution provided that the population is large enough and that the variable is **symmetrical** with **one mode**
- For a normal distribution $X$ can take any real value, however values far from the mean (more than 4 standard deviations away from the mean) have a probability density of **practically zero**
  - This fact allows us to model variables that are not defined for all real values such as height and weight

## What can not be modelled using a normal distribution?

- Variables which have **more than one mode** or **no mode**
  - For example: the number given by a random number generator
- Variables which are **not symmetrical**
  - For example: how long a human lives for

### Exam Tip

- An exam question might involve different types of distributions so make it clear which distribution is being used for each variable

### Worked Example

The random variable $S$ represents the speeds (mph) of a certain species of cheetahs when they run. The variable is modelled using $\mathrm{N}(40,\ 100)$.

a)
Write down the mean and standard deviation of the running speeds of cheetahs.

$\mu = 40$ and $\sigma^2 = 100$

↑ Square root to get standard deviation

Mean $\mu = 40$
Standard deviation $\sigma = 10$

b)
State two assumptions that have been made in order to use this model.

We assume that the distribution of the speeds is
- symmetrical
- bell-shaped

www.thrivingscholars.com





# Calculating Normal Probabilities

Throughout this section we will use the random variable $X \sim \mathrm{N}(\mu, \sigma^2)$. For $X$ distributed normally, $X$ can take any real number. Therefore any values mentioned in this section will be assumed to be real numbers.

## How do I find probabilities using a normal distribution?

- The **area under a normal curve** between the points $x = a$ and $x = b$ is equal to the **probability** $\mathrm{P}(a < X < b)$
  - Remember for a normal distribution you do not need to worry about whether the inequality is strict (< or >) or weak (≤ or ≥)
    - $\mathrm{P}(a < X < b) = \mathrm{P}(a \leq X \leq b)$
- You will be **expected to use** distribution functions on your **GDC** to find the probabilities when working with a normal distribution

## How do I calculate P($X = x$): the probability of a single value for a normal distribution?

- The probability of a **single value** is **always zero** for a normal distribution
  - You can picture this as the area of a single line is zero
- $\mathrm{P}(X = x) = 0$
- Your GDC is likely to have a "**Normal Probability Density**" function
  - This is sometimes shortened to NPD, Normal PD or Normal Pdf
  - **IGNORE THIS FUNCTION** for this course!
  - This calculates the **probability density function** at a point **NOT the probability**

## How do I calculate P($a < X < b$): the probability of a range of values for a normal distribution?

- You need a **GDC** that can calculate **cumulative normal probabilities**
- You want to use the "**Normal Cumulative Distribution**" function
  - This is sometimes shortened to NCD, Normal CD or Normal Cdf
- You will need to enter:
  - The 'lower bound' - this is the value $a$
  - The 'upper bound' - this is the value $b$
  - The '$\mu$' value - this is the mean
  - The '$\sigma$' value - this is the standard deviation
- **Check the order carefully** as some calculators ask for standard deviation before mean
  - Remember it is the standard deviation
    - so if you have the **variance** then **square root it**
- **Always sketch** a quick diagram to visualise which area you are looking for

## How do I calculate P($X > a$) or P($X < b$) for a normal distribution?

- You will still use the "**Normal Cumulative Distribution**" function
- $\mathrm{P}(X > a)$ can be estimated using an **upper bound that is sufficiently bigger** than the **mean**
  - Using a value that is more than 4 standard deviations **bigger than the mean** is quite accurate
  - Or an easier option is just to input lots of 9's for the upper bound (**99999999...** or $\mathbf{10^{99}}$)

www.thrivingscholars.com

- $P(X < b)$ can be estimated using a **lower bound that is sufficiently smaller** than the **mean**
  - Using a value that is more than 4 standard deviations **smaller than the mean** is quite accurate
  - Or an easier option is just to input lots of 9's for the lower bound with a negative sign (**-99999999...** or $-10^{99}$)

## Are there any useful identities?

- $P(X < \mu) = P(X > \mu) = 0.5$
- As $P(X = a) = 0$ you can use:
  - $P(X < a) + P(X > a) = 1$
  - $P(X > a) = 1 - P(X < a)$
  - $P(a < X < b) = P(X < b) - P(X < a)$
- These are useful when:
  - The mean and/or standard deviation are unknown
  - You only have a diagram
  - You are working with the **inverse distribution**

### Exam Tip

- Check carefully whether you have entered the standard deviation or variance into your GDC

www.thrivingscholars.com

## Worked Example

The random variable $Y \sim N(20, 5^2)$. Calculate:

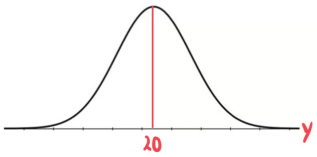
i)

$P(Y = 20)$.

Identify $\mu$ and $\sigma$

$\mu = 20$ $\sigma^2 = 5^2$ so $\sigma = 5$

Sketch!

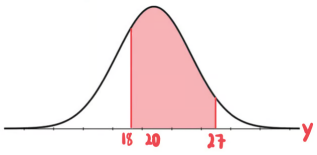



$P(Y = 20) = 0$

ii)

$P(18 \leq Y < 27)$.

Sketch!




Using GDC

Lower = 18

Upper = 27

We can use $\leq$ or $<$

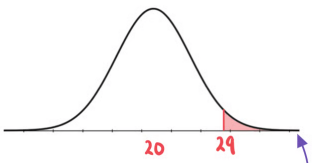
$P(18 < Y < 27) = 0.574665...$

0.575 (3sf)

iii)

$P(Y > 29)$

Sketch!




No upper bound so choose a big number

Using GDC

Lower = 29

Upper = 99999

$P(Y > 29) = 0.035930...$

0.0359 (3sf)

www.thrivingscholars.com

# Inverse Normal Distribution

## Given the value of P($X < a$) how do I find the value of $a$?

- Your **GDC** will have a function called "**Inverse Normal Distribution**"
  - Some calculators call this InvN
- Given that $\mathrm{P}(X < a) = p$ you will need to enter:
  - The 'area' - this is the value $p$
    - Some calculators might ask for the 'tail' - this is the left tail as you know the area to the left of $a$
  - The '$\mu$' value - this is the mean
  - The '$\sigma$' value - this is the standard deviation

## Given the value of P($X > a$) how do I find the value of $a$?

- Given $\mathrm{P}(X > a) = p$
- Use $\mathrm{P}(X < a) = 1 - \mathrm{P}(X > a)$ to rewrite this as
  - $\mathrm{P}(X < a) = 1 - p$
- Then use the **method for P($X < a$)** to find $a$
- If your calculator does have the **tail option** (left, right or centre) then you can use the "Inverse Normal Distribution" function straightaway by:
  - Selecting 'right' for the tail
  - Entering the area as '$p$'

### Exam Tip

- Always check your **answer makes sense**
  - If P($X < a$) is **less than 0.5** then $a$ should be **smaller than the mean**
  - If P($X < a$) is **more than 0.5** then $a$ should be **bigger than the mean**
  - A sketch will help you see this

www.thrivingscholars.com

## Worked Example

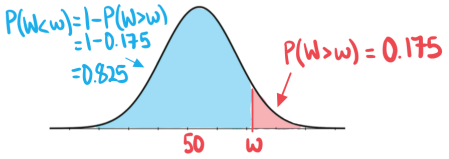
The random variable $W \sim \mathrm{N}(50, 36)$.

Find the value of $w$ such that $\mathrm{P}(W > w) = 0.175$.

Identify $\mu$ and $\sigma$

$\mu = 50$ $\sigma^2 = 36$ so $\sigma = 6$

Sketch!




P(W>w) is less than 0.5

so w is bigger than the mean

Area from left is 0.825

Use Inverse Normal Distribution function on GDC

$w = 55.6075...$

$w = 55.6$ (3sf)

www.thrivingscholars.com





# Standard Normal Distribution

## What is the standard normal distribution?

- The **standard normal distribution** is a normal distribution where the **mean is 0** and the **standard deviation is 1**
  - It is denoted by $Z$
  - $Z \sim \mathrm{N}(0,\ 1^2)$

## Why is the standard normal distribution important?

- Any **normal distribution curve** can be transformed to the standard normal distribution curve by a **horizontal translation** and a **horizontal stretch**
- Therefore we have the relationship:
  - $Z = \dfrac{X - \mu}{\sigma}$
  - Where $X \sim \mathrm{N}(\mu,\ \sigma^2)$ and $Z \sim \mathrm{N}(0,\ 1^2)$
- Probabilities are related by:
  - $\mathrm{P}(a < X < b) = \mathrm{P}\left(\dfrac{a - \mu}{\sigma} < Z < \dfrac{b - \mu}{\sigma}\right)$
  - This will be useful when the mean or variance is unknown
- Some mathematicians use the function $\Phi(z)$ to represent $\mathrm{P}(Z < z)$

# z-values

## What are *z*-values (standardised values)?

- For a normal distribution $X \sim \mathrm{N}(\mu,\ \sigma^2)$ the *z*-value (standardised value) of an *x*-value tells you how many standard deviations it is away from the mean
  - If $z = 1$ then that means the *x*-value is 1 standard deviation bigger than the mean
  - If $z = -1$ then that means the *x*-value is 1 standard deviation smaller than the mean
- If the *x*-value is **more than the mean** then its corresponding *z*-value will be **positive**
- If the *x*-value is **less than the mean** then its corresponding *z*-value will be **negative**
- The *z*-value can be calculated using the formula:
  - $z = \dfrac{x - \mu}{\sigma}$
  - This is given in the **formula booklet**
- *z*-values can be used to compare values from different distributions

www.thrivingscholars.com

# Finding Sigma and Mu

## How do I find the mean (μ) or the standard deviation (σ) if one of them is unknown?

- If the **mean** or **standard deviation** of $X \sim \mathrm{N}(\mu, \sigma^2)$ is **unknown** then you will need to use the **standard normal distribution**
- You will need to use the formula
  - $z = \dfrac{x - \mu}{\sigma}$ or its rearranged form $x = \mu + \sigma z$
- You will be given a **probability for a specific value** of
  - $\mathrm{P}(X < x) = p$ or $\mathrm{P}(X > x) = p$
- To find the unknown parameter:
- **STEP 1**: **Sketch** the normal curve
  - Label the known value and the mean
- **STEP 2**: **Find** the ***z*-value** for the given value of ***x***
  - Use the **Inverse Normal Distribution** to find the value of $z$ such that $\mathrm{P}(Z < z) = p$ or $\mathrm{P}(Z > z) = p$
  - Make sure the direction of the inequality for $Z$ is consistent with the inequality for $X$
  - Try to **use lots of decimal places** for the *z*-value or **store your answer** to **avoid rounding errors**
    - You should use at least one extra decimal place within your working than your intended degree of accuracy for your answer
- **STEP 3**: **Substitute** the known values into $z = \dfrac{x - \mu}{\sigma}$ or $x = \mu + \sigma z$
  - You will be given and one of the parameters ($\mu$ or $\sigma$) in the question
  - You will have calculated *z* in STEP 2
- **STEP 4**: **Solve** the equation

## How do I find the mean (μ) and the standard deviation (σ) if both of them are unknown?

- If **both** of them are **unknown** then you will be given two probabilities for two specific values of ***x***
- The process is the same as above
  - You will now be able to **calculate two *z*-values**
  - You can form **two equations** (rearranging to the form $x = \mu + \sigma z$ is helpful)
  - You now have to **solve the two equations simultaneously** (you can use your calculator to do this)
  - Be careful not to mix up which z-value goes with which value of *x*

www.thrivingscholars.com

## Worked Example

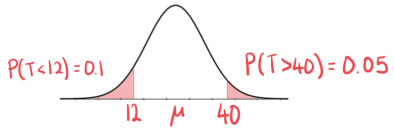
It is known that the times, in minutes, taken by students at a school to eat their lunch can be modelled using a normal distribution with mean $\mu$ minutes and standard deviation $\sigma$ minutes.

Given that 10% of students at the school take less than 12 minutes to eat their lunch and 5% of the students take more than 40 minutes to eat their lunch, find the mean and standard deviation of the time taken by the students at the school.

Let $T \sim N(\mu, \sigma^2)$ be the time taken to eat lunch

STEP 1

Sketch the information




STEP 2

Find the corresponding z-values using inverse normal on GDC

$Z \sim N(0, 1^2)$

$P(Z < z_1) = 0.1 \Rightarrow z_1 = -1.2815...$

$P(Z > z_2) = 0.05 \Rightarrow P(Z < z_2) = 0.95 \Rightarrow z_2 = 1.6448...$

STEP 3

Form equations using $z = \frac{x-\mu}{\sigma}$ or $x = \mu + \sigma z$

$12 = \mu - (1.2815...)\sigma$

$40 = \mu + (1.6448...)\sigma$

STEP 4

Solve equations using GDC

$\mu = 24.26...$ $\sigma = 9.568...$

Mean = 24.3 mins (3sf)

Standard deviation = 9.57 mins (3sf)

www.thrivingscholars.com

# 4.7 Further Probability Distributions

## 4.7.1 Probability Density Function

### Calculating Probabilities using PDF

A **continuous random variable** can take *any* value in an interval so is typically used when continuous quantities are involved (time, distance, weight, etc)

### What is a probability density function (p.d.f.)?

- For a continuous random variable, a function can be used to model probabilities
  - This function is called a **probability density function** (p.d.f.), denoted by f(x)
- For f(x) to represent a p.d.f. the following conditions must apply
  - $f(x) \geq 0$ for **all** values of x
  - The **area** under the graph of $y = f(x)$ must **total** 1
- In most problems, the **domain** of x is restricted to an interval, $a \leq X \leq b$ say, with all values of x outside of the interval having f(x)=0

### How do I find probabilities using a probability density function (p.d.f.)?

- The probability that the continuous random variable X lies in the interval $a \leq X \leq b$, where X has the probability density function f(x), is given by

$$\mathrm{P}(a \leq X \leq b) = \int_a^b f(x)\,\mathrm{d}x$$

  - $P(a \leq X \leq b) = P(a < X < b)$
    - For **any** continuous random variable (including the normal distribution) $P(X = n) = 0$
    - One way to think of this is that a = b in the integral above
- For **linear** functions it can be easier to find the probability using the area of geometric shapes
  - Rectangles: $A = bh$
  - Triangles: $A = \frac{1}{2}(bh)$
  - Trapezoids: $A = \frac{1}{2}(a+b)h$

### How do I determine whether a function is a pdf?

- Some questions may ask for justification of the use of a given function for a probability density function
  - In such cases check that the function meets the two conditions
    - $f(x) \geq 0$ for **all** values of x
    - **total area** under the graph is 1

### How do I use a pdf to find probabilities?

**STEP 1**

Identify the **probability density function,** f(x) - this may be given as a **graph,** an **equation** or as a **piecewise function**

e.g. $f(x) = \begin{cases} 0.02x & 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$

www.thrivingscholars.com



Identify the **limits** of X for a particular problem
Remember that $P(a \leq X \leq b) = P(a < X < b)$

**STEP 2**
Sketch, or use your GDC to draw, the graph of $y = f(x)$
Look for basic shapes (rectangles, triangles and trapezoids) as finding these areas is easier without using integration
Look for symmetry in the graph that may make the problem easier
Break the area required into two or more parts if it makes the problem easier

**STEP 3**
Find the area(s) required using basic shapes or integration and answer the question

- Trickier problems may involve finding a limit of the integral given its value
    - i.e. Find one of the boundaries in the domain of X, given the probability
        - e.g. Find the value of a given that $P(0 \leq X \leq a) = 0.09$

www.thrivingscholars.com

## Worked Example

The continuous random variable, X, has probability density function.

$$f(x) = \begin{cases} 0.08x & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

a)

Show that f(x) can represent a probability density function.

There are two conditions for a function to be a p.d.f.

$$\int_0^5 0{\cdot}08x \, dx = \left[ \frac{0{\cdot}08x^2}{2} \right]_0^5$$

$$= \left[ 0{\cdot}04x^2 \right]_0^5$$

$$= 0{\cdot}04 \times 5^2 - 0 = 1$$

$$\therefore \int_{-\infty}^{\infty} f(x) \, dx = 0$$

Also, $f(x) \geq 0$ for all values of $x$

so f(x) meets both conditions to represent a probability density function.

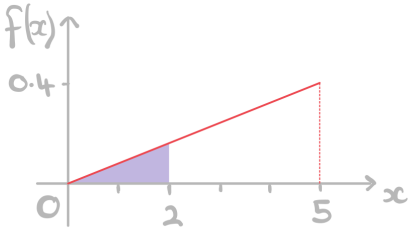
b)

Find, both geometrically and using integration, P(0 ≤ X ≤ 2).

www.thrivingscholars.com

STEP 1: Identify the p.d.f. - given piecewise in the question
Identify the limits of X
$f(x) = 0.08x$
$P(0 \leq X \leq 2)$: Lower limit 0
Upper limit 2

STEP 2: Sketch, or use GDC to draw, $y = f(x)$




The probability required is the area of a triangle

STEP 3: Find the required area(s) and answer the question
Geometrically

$$A = \frac{1}{2} \times 2 \times (0.08 \times 2) = 0.16$$
$\uparrow f(2)$

Using integration

$$A = \int_0^2 0.08x \, dx$$

$$A = \left[ 0.04x^2 \right]_0^2 = 0.16$$

$$\therefore P(0 \leq X \leq 2) = 0.16$$

c)
Write down P(X = 3.2).

$$P(X = 3.2) = 0$$

$P(X = n) = 0$ for all values of $n$

www.thrivingscholars.com

# Median & Mode of a CRV

## What is meant by the median of a continuous random variable?

- The **median,** $m$, of a **continuous random variable,** $X$, with **probability density function** $f(x)$ is defined as the value of $X$ such that

$$\mathrm{P}(X < m) = \mathrm{P}(X > m) = 0.5$$

- Since $\mathrm{P}(X = m) = 0$ this can also be written as $\mathrm{P}(X \leq m) = \mathrm{P}(X \geq m) = 0.5$
- **IF** the p.d.f. is **symmetrical** (i.e. the graph of $y = f(x)$ is symmetrical) then the **median** will be **halfway** between the **lower** and **upper** limits of $x$
  - In such cases the graph of $y = f(x)$ has **axis** of **symmetry** in the line x = m

## How do I find the median of a continuous random variable?

- The **median,** $m$, of a continuous random variable, $X$, with probability density function $f(x)$ is defined as the value of $X$ such that

$$\int_{-\infty}^{m} f(x)\,\mathrm{d}x = \frac{1}{2}$$

**or**

$$\int_{m}^{\infty} f(x)\,\mathrm{d}x = \frac{1}{2}$$

- The **equation** that should be used will depend on the **information** in the **question**
  - If the **graph** of $y = f(x)$ is **symmetrical,** symmetry may be used to **deduce** the **median**
  - This may often be the case if $f(x)$ is **linear** and the **area under the graph** is a basic **shape** such as a **rectangle**

## How do I find the median of a continuous random variable with a piecewise p.d.f.?

- For **piecewise functions,** the **location** of the **median** will determine **which equation** to use in order to find it
  - For example
    - if $f(x) = \begin{cases} \frac{1}{5}x & 0 \leq x \leq 2 \\ \frac{2}{15}(5-x) & 2 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$
    - then $\int_0^2 \frac{1}{5}x\,\mathrm{d}x = 0.4$ so the median must lie in the interval $2 \leq x \leq 5$
    - so to find the median, $m$, solve $\int_2^m \frac{2}{15}(5-x)\,\mathrm{d}x = 0.1$

      ('0.4 of the area' already used for $0 \leq x \leq 2$)
    - Use a GDC to plot the function and evalutae integral(s)

## What is meant by the mode of a continuous random variable?

www.thrivingscholars.com



- The **mode** of a **continuous random variable,** $X$, with **probability density function** $f(x)$ is the **value** of $x$ that produces the **greatest value** of $f(x)$

## How do I find the mode of a continuous random variable?

- This will depend on the **type** of **function** $f(x)$; the easiest way to find the **mode** is by considering the **shape** of the **graph** of $y = f(x)$
- If the **graph** is a **curve** with a **maximum point,** the **mode** can be **found** by **differentiating** and **solving** $f'(x) = 0$
    - If there is **more than one solution** to $f'(x) = 0$ then **further work** may be needed in deducing the mode
        - There could be **more than one** mode
        - Look for **valid values** of $x$ from the **domain** of the p.d.f.
        - Use the **second derivative** ($f''(x)$) to **deduce** the **nature** of each **stationary point**
        - **Check** the **values** of $f(x)$ at the **lower** and **upper limits** of $x$, one of these could be the **maximum value** $f(x)$ reaches
- If the graph of $y = f(x)$ is **symmetrical,** symmetry may be used to deduce the mode
    - For a **symmetrical** p.d.f., **median = mode = mean**

www.thrivingscholars.com

## Worked Example

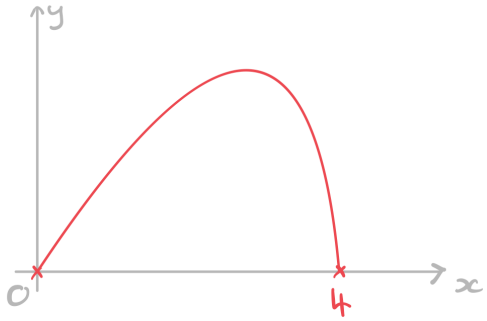
The continuous random variable $X$ has probability function $f(x)$ defined as

$$f(x) = \frac{1}{64}(16x - x^3) \quad 0 \le x \le 4$$

a)

Find the median of X, giving your answer to three significant figures.

Sketch the graph of $y = f(x)$ using your GDC to help




For the median, solve "$\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$"

$$\frac{1}{64}\int_0^m (16x - x^3)\,dx = \frac{1}{2}$$

$f(x) = 0$ for $x < 0$

$$\left[8x^2 - \frac{1}{4}x^4\right]_0^m = 32$$

$$8m^2 - \frac{1}{4}m^4 = 32$$

$$m^4 - 32m^2 + 128 = 0$$

This is a 'hidden quadratic' in $m^2$.

Using a GDC,

$m = \pm 5.226\,251\ldots$

or $m = \pm 2.164\,784\ldots$

Only one of these four values lies in the range $0 \le x \le 4$

$\therefore$ Median, $m = 2.16$ (3 s.f.)

b)

Find the *exact* value of the mode of X.

www.thrivingscholars.com



Differentiate, solving $f'(x)=0$ to find the mode

$f'(x) = \frac{1}{64}(16-3x^2)$

$16-3x^2 = 0$

Using a GDC (ensure you get exact answers)

$x = \pm\frac{4\sqrt{3}}{3}$

Clearly from sketch of graph, $x = \frac{4}{3}\sqrt{3}$ is a (local) maximum

Also, $x = -\frac{4\sqrt{3}}{3}$ does not lie in the interval $0 \leq x \leq 4$

$\therefore$ Mode $= \frac{4}{3}\sqrt{3}$

www.thrivingscholars.com

# Mean & Variance of a CRV

## What are the mean and variance of a continuous random variable?

- E($X$) is the **expected value,** or **mean,** of the **continuous random variable** $X$
  - E($X$) can also be denoted by $\mu$
- Var($X$) is the **variance** of the continuous random variable $X$
  - Var($X$) can also be denoted by $\sigma^2$
  - The **standard deviation,** $\sigma$, is the **square root** of the **variance**

## How do I find the mean and variance of a continuous random variable?

- The mean is given by

$$\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$$

  - This is given in the **formula booklet**
  - If the graph of $y = f(x)$ has **axis** of **symmetry,** $x = a$, then $\mathrm{E}(X) = a$
- The **variance** is given by

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

where $\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x$

  - This is given in the **formula booklet**
  - Another version of the variance is given in the **formula booklet**

$$\mathrm{Var}(x) = \int_{-\infty}^{\infty} (x-\mu)^2\, f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x - \mu^2$$

  - but the first version above is usually more practical for solving problems
- Be careful about confusing $\mathrm{E}(X^2)$ and $[\mathrm{E}(X)]^2$
  - $\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x$ "mean of the squares"
  - $[\mathrm{E}(X)]^2 = \left[\int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x\right]^2$ "mean of the squares"

## How do I find the mean and variance of a linear transformation of a continuous random variable?

- For the **continuous random variable,** $X$, with **mean** E($X$) and **variance** Var($X$) then

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$

and

$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

### Exam Tip

- Using your **GDC** to draw the graph of $y = f(x)$ can **highlight** any **symmetrical** properties which **reduce** the **work** involved in finding the **mean** and **variance**

www.thrivingscholars.com

## Worked Example

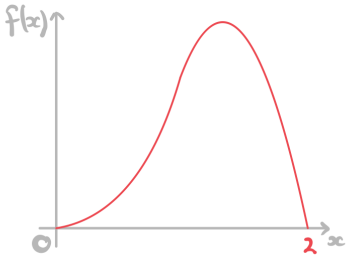
A continuous random variable, $X$, is modelled by the probability distribution function, $f(x)$, such that

$$f(x) = \begin{cases} 1.5x^2(1-0.5x) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a)

Find the mean of $X$.

Use your GDC to sketch the graph of $y = f(x)$




No symmetry!

$\therefore \mu = \int_0^2 x\left[1.5x^2(1-0.5x)\right]\,dx$ Evaluate using your GDC

$\mu = 1.2$

To do without GDC.

$\mu = \int_0^2 (1.5x^3 - 0.75x^4)\,dx$

$\mu = \left[0.375x^4 - 0.15x^5\right]_0^2$

$\mu = 6 - 4.8 = 1.2$

b)

Find standard deviation of $X$.

www.thrivingscholars.com

$\sigma = \sqrt{\text{Var}(X)}$ $\quad \text{Var}(X) = E(X^2) - [E(X)]^2$

$$E(X^2) = \int_0^2 x^2 [1.5x^2(1 - 0.5x)]\, dx$$

Using GDC, $E(X^2) = 1.6$

$\therefore \sigma = \sqrt{1.6 - (1.2)^2} = \sqrt{0.16} = 0.4$

($E(X^2)$ ↑, $E(X)$ ↖)

$$\boxed{\sigma = 0.4}$$

www.thrivingscholars.com

www.thrivingscholars.com

# IB Maths DP

## 5. Calculus

**CONTENTS**


5.1 Differentiation
- 5.1.1 Introduction to Differentiation
- 5.1.2 Applications of Differentiation

5.2 Further Differentiation
- 5.2.1 Differentiating Special Functions
- 5.2.2 Techniques of Differentiation
- 5.2.3 Higher Order Derivatives
- 5.2.4 Further Applications of Differentiation
- 5.2.5 Concavity & Points of Inflection
- 5.2.6 Derivatives & Graphs

5.3 Integration
- 5.3.1 Introduction to Integration
- 5.3.2 Applications of Integration

5.4 Further Integration
- 5.4.1 Integrating Special Functions
- 5.4.2 Techniques of Integration
- 5.4.3 Definite Integrals
- 5.4.4 Further Applications of Integration

5.5 Optimisation
- 5.5.1 Modelling with Differentiation

5.6 Kinematics
- 5.6.1 Kinematics Toolkit
- 5.6.2 Calculus for Kinematics

5.7 Basic Limits & Continuity
- 5.7.1 Basic Limits & Continuity

5.8 Advanced Differentiation
- 5.8.1 First Principles Differentiation
- 5.8.2 Applications of Chain Rule
- 5.8.3 Implicit Differentiation
- 5.8.4 Differentiating Further Functions

5.9 Advanced Integration
- 5.9.1 Integrating Further Functions

www.thrivingscholars.com

5.9.2 Further Techniques of Integration
5.9.3 Integrating with Partial Fractions
5.9.4 Advanced Applications of Integration
5.9.5 Modelling with Volumes of Revolution
5.10 Differential Equations
5.10.1 Numerical Solutions to Differential Equations
5.10.2 Analytical Solutions to Differential Equations
5.10.3 Modelling with Differential Equations
5.11 MacLaurin Series
5.11.1 Maclaurin Series
5.11.2 Maclaurin Series from Differential Equations
5.12 Further Limits (inc l'Hôpital's Rule)
5.12.1 Further Limits

www.thrivingscholars.com

# 5.1 Differentiation

## 5.1.1 Introduction to Differentiation

### Introduction to Derivatives

- Before introducing a **derivative**, an understanding of a **limit** is helpful

### What is a limit?

- The **limit** of a **function** is the value a function (of $x$) approaches as $x$ approaches a particular value from either side
  - Limits are of interest when the function is undefined at a particular value
  - For example, the function $f(x) = \frac{x^4 - 1}{x - 1}$ will approach a limit as $x$ approaches 1 from both below and above but is undefined at $x = 1$ as this would involve dividing by zero

### What might I be asked about limits?

- You may be asked to predict or estimate limits from a table of function values or from the graph of $y = f(x)$
- You may be asked to use your GDC to plot the graph and use values from it to estimate a limit

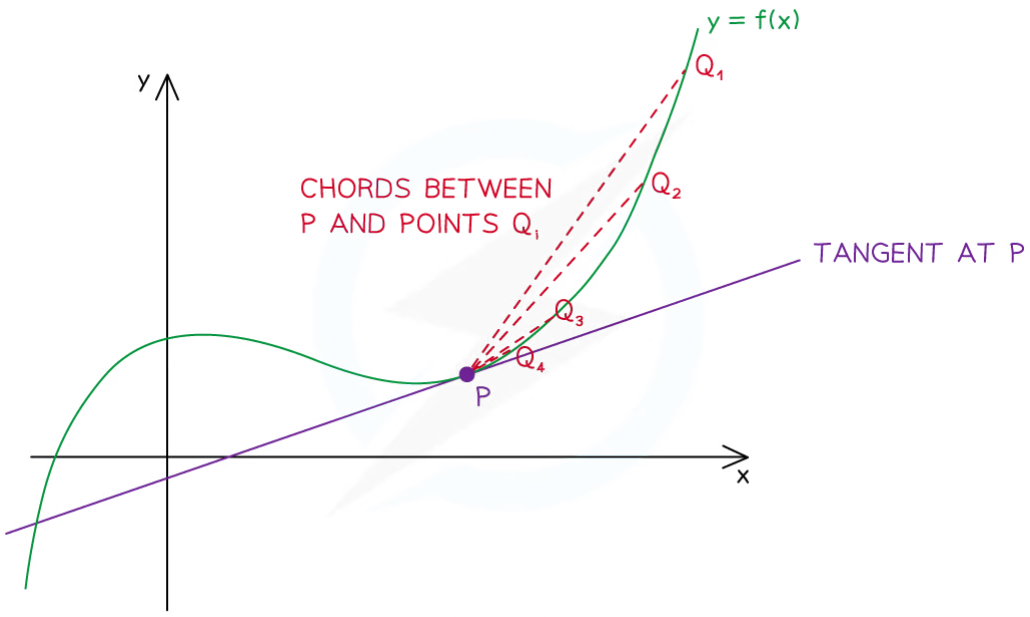
### What is a derivative?

- **Calculus** is about **rates** of **change**
  - the way a car's position on a road changes is its speed (velocity)
  - the way the car's speed changes is its acceleration
- The **gradient** (rate of change) of a (non-linear) **function** varies with $x$
- The **derivative** of a function is a function that relates the **gradient** to the value of $x$
- The derivative is also called the **gradient function**

### How are limits and derivatives linked?

- Consider the point $P$ on the graph of $y = f(x)$ as shown below
  - $[PQ_i]$ is a series of chords

www.thrivingscholars.com



Copyright © Save My Exams. All Rights Reserved

- The **gradient** of the **function** $f(x)$ at the point $P$ is **equal** to the **gradient** of the **tangent** at point $P$
- The **gradient** of the **tangent** at point $P$ is the **limit** of the **gradient** of the chords $[PQ_i]$ as point $Q$ 'slides' down the curve and gets ever closer to point $P$
- The **gradient** of the function changes as $x$ changes
- The **derivative** is the function that calculates the gradient from the value $x$

## What is the notation for derivatives?

- For the function $y = f(x)$, the **derivative**, with respect to $x$, would be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$$

- Different variables may be used
  - e.g. If $V = f(s)$ then $\frac{\mathrm{d}V}{\mathrm{d}s} = f'(s)$

## What might I be asked about derivatives?

- You may be asked to use the graphing features of your GDC to find the **gradients** of a **function** at different values of $x$
- From a **series** of **gradient** values, you may be asked to suggest an **expression** for the **derivative** (gradient function) of a function

www.thrivingscholars.com

## Worked Example

The graph of $y = f(x)$ where $f(x) = x^3 - 2$ passes through the points $P(2, 6)$, $A(2.3, 10.167)$, $B(2.1, 7.261)$ and $C(2.05, 6.615125)$.

a)

Find the gradient of the chords $[PA]$, $[PB]$ and $[PC]$.

Gradient of a line (chord) is "$\frac{y_2 - y_1}{x_2 - x_1}$"

[PA]: $\frac{10.167 - 6}{2.3 - 2} = 13.89$

[PB]: $\frac{7.261 - 6}{2.1 - 2} = 12.61$

[PC]: $\frac{6.615125 - 6}{2.05 - 2} = 12.3$

Gradient of chords are: [PA] 13.89
[PB] 12.61
[PC] 12.3025

b)

Estimate the gradient of the tangent to the curve at the point $P$.

There will be a limit the gradient of the chord reaches as the difference in the x-coordinates approaches zero.

Estimate of gradient of tangent at $x = 2$ is 12

c)

Use your GDC to find the gradient of the tangent at the pont $P$.

Using GDC, plot $y = x^3 - 2$,
draw a tangent at $x = 2$
GDC can tell you either/both of the equation of the tangent and $\frac{dy}{dx}$

GDC gradient is 12

www.thrivingscholars.com

# Differentiating Powers of x

## What is differentiation?

- **Differentiation** is the process of finding an expression of the **derivative** (**gradient function**) from the expression of a function

## How do I differentiate powers of x?

- **Powers** of $x$ are **differentiated** according to the following formula:
  - If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ where $n \in \mathbb{Q}$
  - This is given in the **formula booklet**
- If the power of $x$ is **multiplied** by a **constant** then the derivative is also multiplied by that constant
  - If $f(x) = ax^n$ then $f'(x) = anx^{n-1}$ where $n \in \mathbb{Q}$ and $a$ is a constant
- The **alternative notation** (to $f'(x)$) is to use $\frac{dy}{dx}$
  - If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$
    - e.g. If $y = -4x^{\frac{1}{2}}$ then $\frac{dy}{dx} = -4 \times \frac{1}{2} \times x^{\frac{1}{2}-1} = -2x^{-\frac{1}{2}}$
- Don't forget these **two** special cases:
  - If $f(x) = ax$ then $f'(x) = a$
    - e.g. If $y = 6x$ then $\frac{dy}{dx} = 6$
  - If $f(x) = a$ then $f'(x) = 0$
    - e.g. If $y = 5$ then $\frac{dy}{dx} = 0$
  - These allow you to differentiate **linear terms** in $x$ and **constants**
- Functions involving **roots** will need to be rewritten as **fractional powers** of $x$ first
  - e.g. If $f(x) = 2\sqrt{x}$ then rewrite as $f(x) = 2x^{\frac{1}{2}}$ and differentiate
- Functions involving **fractions** with **denominators** in terms of $x$ will need to be rewritten as **negative powers** of $x$ first
  - e.g. If $f(x) = \frac{4}{x}$ then rewrite as $f(x) = 4x^{-1}$ and differentiate

## How do I differentiate sums and differences of powers of x?

- The formulae for differentiating powers of $x$ apply to **all rational** powers so it is possible to differentiate any expression that is a **sum** or **difference** of **powers** of $x$
  - e.g. If $f(x) = 5x^4 - 3x^{\frac{2}{3}} + 4$ then

$$f'(x) = 5 \times 4x^{4-1} - 3 \times \frac{2}{3}x^{\frac{2}{3}-1} + 0$$

$$f'(x) = 20x^3 - 2x^{-\frac{1}{3}}$$

www.thrivingscholars.com

- **Products** and **quotients cannot** be differentiated in this way so would need **expanding/simplifying** first
  - e.g. If $f(x) = (2x - 3)(x^2 - 4)$ then expand to $f(x) = 2x^3 - 3x^2 - 8x + 12$ which **is** a **sum/difference** of powers of $x$ and can be differentiated

## What might I be asked to do once I've found the derivative (gradient function)?

- The **derivative** can be used to find the **gradient** of a **function** at any point
  - The gradient of a function at a point is equal to the **gradient** of the **tangent** to the curve at that point
  - A question may refer to the gradient of the tangent
  - A GDC can be used to help find gradients or verify answers
- Take extra care when differentiating negative and fractional powers of $x$
  - $\frac{1}{2} - 1 = -\frac{1}{2}$ but $\frac{1}{3} - 1 = -\frac{2}{3}$

### Exam Tip

- A common mistake is not simplifying expressions before differentiating
  - The derivative of $(x^2 + 3)(x^3 - 2x + 1)$ can **not** be found by multiplying the derivatives of $(x^2 + 3)$ and $(x^3 - 2x + 1)$

www.thrivingscholars.com

## Worked Example

The function $f(x)$ is given by

$$f(x) = 2x^3 + \frac{4}{\sqrt{x}}$$

where $x > 0$.

a)
Find the derivative of $f(x)$.

Rewrite f(x) so every term is a power of x

$f(x) = 2x^3 + 4x^{-\frac{1}{2}}$

Differentiate by applying the formula

$f'(x) = 6x^2 - 2x^{-\frac{3}{2}}$

$ax^{n-1}$

take care with negatives

$-\frac{1}{2} - 1 = -\frac{3}{2}$

$\therefore f'(x) = 6x^2 - 2x^{-\frac{3}{2}}$

b)
Find the gradient of the tangent to the curve $y = f(x)$ at the point where $x = 4$.

$f'(4) = 6(4)^2 - \frac{2}{(4)^{\frac{3}{2}}} = 96 - \frac{2}{(2)^3} = 96 - \frac{1}{4} = 95.75$

$\therefore$ The gradient of the tangent to the curve $y = f(x)$ when $x = 4$ is 95.75

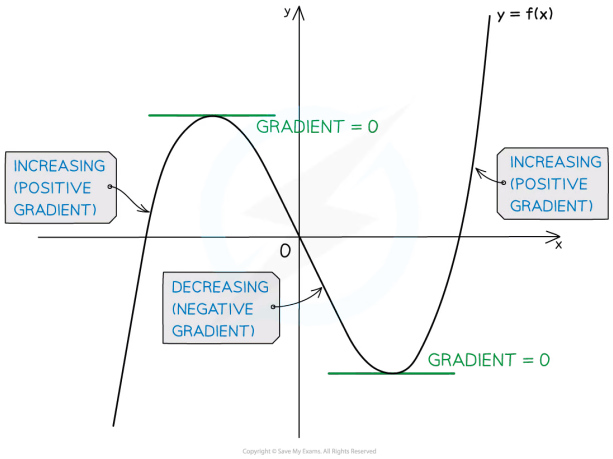
www.thrivingscholars.com



# Increasing & Decreasing Functions

## What are increasing and decreasing functions?

- A function, $f(x)$, is **increasing** if $\boldsymbol{f'(x) > 0}$
  - This means the **value** of the **function** ('output') **increases** as $x$ **increases**
- A function, $f(x)$, is **decreasing** if $\boldsymbol{f'(x) < 0}$
  - This means the **value** of the **function** ('output') **decreases** as $x$ **increases**
- A function, $f(x)$, is **stationary** if $\boldsymbol{f'(x) = 0}$




## How do I find where functions are increasing, decreasing or stationary?

- To identify the **intervals** on which a function is increasing or decreasing

**STEP 1**
Find the derivative **f'(x)**

**STEP 2**
Solve the inequalities
$\boldsymbol{f'(x) > 0}$ (for increasing intervals) and/or
$\boldsymbol{f'(x) < 0}$ (for decreasing intervals)

- Most functions are a combination of **increasing, decreasing** and **stationary**
  - a range of values of $x$ (**interval**) is given where a function satisfies each condition
  - e.g. The function $f(x) = x^2$ has **derivative** $f'(x) = 2x$ so
    - $f(x)$ is **decreasing** for $x < 0$
    - $f(x)$ is **stationary** at $x = 0$
    - $f(x)$ is **increasing** for $x > 0$

www.thrivingscholars.com

## Worked Example

$f(x) = x^2 - x - 2$

a)
Determine whether $f(x)$ is increasing or decreasing at the points where $x = 0$ and $x = 3$.

Differentiate

$f'(x) = 2x - 1$

At $x = 0$, $f'(0) = 2 \times 0 - 1 = -1 < 0$ $\therefore$ decreasing

At $x = 3$, $f'(3) = 2 \times 3 - 1 = 6 > 0$ $\therefore$ increasing

$\therefore$ At $x = 0$, $f(x)$ is decreasing

At $x = 3$, $f(x)$ is increasing

b)
Find the values of $x$ for which $f(x)$ is an increasing function.

$f(x)$ is increasing when $f'(x) > 0$

$f'(x) > 0$

$2x - 1 > 0$

$x > \frac{1}{2}$

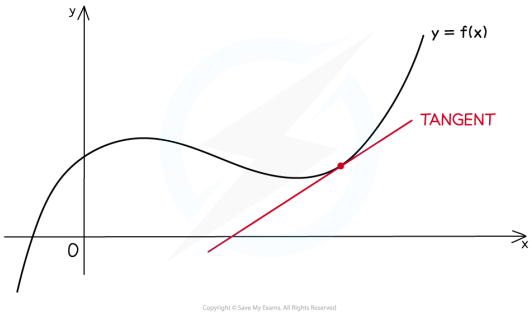
$\therefore$ $f(x)$ is increasing for $x > \frac{1}{2}$

www.thrivingscholars.com

# Tangents & Normals

## What is a tangent?

- At any point on a curve (the graph of a (non-linear) **function**), the **tangent** is the straight line that passes through that point and has the same **gradient** as the curve at that point



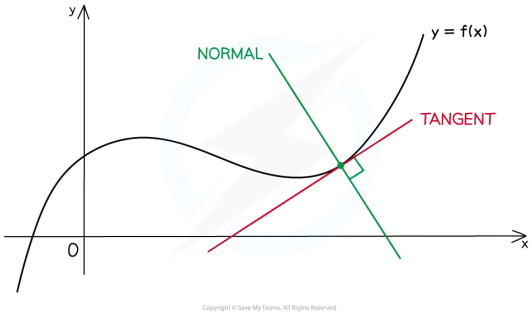

## How do I find the equation of a tangent?

- The **equation** of the **tangent** to the function $y = f(x)$ at the point $(x_1, y_1)$ is

$$y - y_1 = f'(x_1)(x - x_1)$$

## What is a normal?

- At any point on a curve (the graph of a (non-linear) **function**), the **normal** is the straight line that passes through that point and is **perpendicular** to the **tangent** at that point




## How do I find the equation of a normal?

- The **equation** of the **normal** to the function $y = f(x)$ at the point $(x_1, y_1)$ is

$$y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$$

www.thrivingscholars.com

### Exam Tip

- The equations of a tangent and a normal are not in the formula booklet
  - However both can be derived from the equation of a straight line $y - y_1 = m(x - x_1)$
  - This is given in the formula booklet

www.thrivingscholars.com

## Worked Example

The function $f(x)$ is defined by

$$f(x) = 2x^4 + \frac{3}{x^2} \qquad x \neq 0$$

a)

Find an equation for the tangent to the curve $y = f(x)$ at the point where $x = 1$, giving your answer in the form $y = mx + c$.

First find f'(x) by differentiating

$f(x) = 2x^4 + 3x^{-2}$ Rewrite as powers of x

$f'(x) = 8x^3 - 6x^{-3}$

For a tangent, "$y - y_1 = f(a)(x - x_1)$"

At $x = 1$, $y = 2(1)^4 + \frac{3}{(1)^2} = 5$

$f'(1) = 8(1)^3 - \frac{6}{(1)^3} = 2$

$\therefore\ y - 5 = 2(x - 1)$

Tangent at $x = 1$, is $y = 2x + 3$

b)

Find an equation for the normal to the curve $y = f(x)$ at the point where $x = 1$, giving your answer in the form $ax + by + d = 0$, where $a$, $b$ and $d$ are integers.

For a normal, "$y - y_1 = \frac{-1}{f'(a)}(x - x_1)$"

Using results from part a):

$y - 5 = \frac{-1}{2}(x - 1)$

$y = -\frac{1}{2}x + \frac{11}{2}$

$2y = -x + 11$

$\therefore$ Equation of normal is $x + 2y - 11 = 0$

www.thrivingscholars.com

# 5.2 Further Differentiation

## 5.2.1 Differentiating Special Functions

### Differentiating Trig Functions

### How do I differentiate sin, cos and tan?

- The derivative of $y = \sin x$ is $\frac{dy}{dx} = \cos x$
- The derivative of $y = \cos x$ is $\frac{dy}{dx} = -\sin x$
- The derivative of $y = \tan x$ is $\frac{dy}{dx} = \sec^2 x$
  - This result can be derived using **quotient rule**
- For the **linear** function $ax + b$, where $a$ and $b$ are constants,
  - the derivative of $y = \sin(ax + b)$ is $\frac{dy}{dx} = a\cos(ax + b)$
  - the derivative of $y = \cos(ax + b)$ is $\frac{dy}{dx} = -a\sin(ax + b)$
  - **the derivative of $y = \tan(ax + b)$ is $\frac{dy}{dx} = a\sec^2(ax + b)$**
- For the **general** function $f(x)$,
  - the derivative of $y = \sin(f(x))$ is $\frac{dy}{dx} = f'(x)\cos(f(x))$
  - the derivative of $y = \cos(f(x))$ is $\frac{dy}{dx} = -f'(x)\sin(f(x))$
  - the derivative of $y = \tan(f(x))$ is $\frac{dy}{dx} = f'(x)\sec^2(f(x))$
- These last three results can be derived using the **chain rule**
- For calculus with trigonometric functions angles must be measured in **radians**
  - Ensure you know how to change the angle mode on your GDC

**Exam Tip**

- As soon as you see a question involving differentiation and trigonometry put your GDC into radians mode

www.thrivingscholars.com

## Worked Example

a)

Find $f'(x)$ for the functions

i. $f(x) = \sin x$

ii. $f(x) = \cos(5x + 1)$

i. $f'(x) = \cos x$

ii. $f'(x) = -5\sin(5x+1)$ (Linear function $ax+b$)

b) A curve has equation $y = \tan\left(6x^2 - \frac{\pi}{4}\right)$.

Find the gradient of the tangent to the curve at the point where $x = \frac{\sqrt{\pi}}{2}$.

Give your answer as an exact value.

This is of the form $y = \tan(f(x))$

so $\frac{dy}{dx} = f'(x)\sec^2(f(x))$

$f(x) = 6x^2 - \frac{\pi}{4}$

$\therefore f'(x) = 12x$

$\therefore \frac{dy}{dx} = 12x\sec^2\left(6x^2 - \frac{\pi}{4}\right)$

At $x = \frac{\sqrt{\pi}}{2}$, $\frac{dy}{dx} = 12\left(\frac{\sqrt{\pi}}{2}\right)\sec^2\left[6\left(\frac{\sqrt{\pi}}{2}\right)^2 - \frac{\pi}{4}\right]$

$= \frac{6\sqrt{\pi}}{\cos^2\left(\frac{5\pi}{4}\right)}$ $\left(\sec^2 x = \frac{1}{\cos^2 x}\right)$

$\therefore \frac{dy}{dx} = 12\sqrt{\pi}$ at $x = \frac{\sqrt{\pi}}{2}$

www.thrivingscholars.com



# Differentiating e^x & lnx

## How do I differentiate exponentials and logarithms?

- The derivative of $y = e^x$ is $\frac{dy}{dx} = e^x$ where $x \in \mathbb{R}$
- The derivative of $y = \ln x$ is $\frac{dy}{dx} = \frac{1}{x}$ where $x > 0$
- For the **linear** function $ax + b$, where $a$ and $b$ are constants,
  - the derivative of $y = e^{(ax+b)}$ is $\frac{dy}{dx} = ae^{(ax+b)}$
  - the derivative of $y = \ln(ax + b)$ is $\frac{dy}{dx} = \frac{a}{(ax+b)}$
    - in the special case $b = 0$, $\frac{dy}{dx} = \frac{1}{x}$ ($a$'s cancel)
- For the **general** function $f(x)$,
  - the derivative of $y = e^{f(x)}$ is $\frac{dy}{dx} = f'(x)e^{f(x)}$
  - the derivative of $y = \ln(f(x))$ is $\frac{dy}{dx} = f'(x)\ln(f(x))$
- The last two sets of results can be derived using the **chain rule**

### Exam Tip

- Remember to avoid the common mistakes:
  - the derivative of $\ln kx$ with respect to $x$ is $\frac{1}{x}$, NOT $\frac{k}{x}$
  - the derivative of $e^{kx}$ with respect to $x$ is $ke^{kx}$, NOT $kxe^{kx-1}$

www.thrivingscholars.com

## Worked Example

A curve has the equation $y = e^{-3x+1} + 2\ln 5x$.

Find the gradient of the curve at the point where $x = 2$ gving your answer in the form $y = a + be^c$, where $a$, $b$ and $c$ are integers to be found.

$y = e^{-3x+1} + 2(\ln 5x)$

$\therefore \frac{dy}{dx} = -3e^{-3x+1} + 2\left(\frac{1}{x}\right)$

"$y = e^{ax+b}$, $\frac{dy}{dx} = ae^{ax+b}$"

"$y = \ln(ax+b)$, special case $b=0$, $\frac{dy}{dx} = \frac{1}{x}$"

At $x = 2$, $\frac{dy}{dx} = -3e^{-3(2)+1} + \frac{2}{2} = -3e^{-5} + 1$

$\therefore$ Gradient at $x=2$ is $1-3e^{-5}$

i.e. $a=1$, $b=-3$, $c=-5$

Your GDC may be able to find gradients but probably not in the exact form required. It is still helpful to check approximate answers though.

www.thrivingscholars.com





# Chain Rule

## What is the chain rule?

- The **chain rule** states if $y$ is a function of $u$ and $u$ is a function of $x$ then

$$y = f(u(x))$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

  - This is given in the **formula booklet**

- In **function notation** this could be written

$$y = f(g(x))$$

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

## How do I know when to use the chain rule?

- The chain rule is used when we are trying to differentiate **composite functions**
  - "function of a function"
  - these can be identified as the variable (usually $x$) does not 'appear alone'
    - $\sin x$ – **not** a composite function, $x$ 'appears alone'
    - $\sin(3x+2)$ **is a composite function;** $x$ is tripled and has 2 added to it before the sine function is applied

## How do I use the chain rule?

**STEP 1**

Identify the two functions

Rewrite $y$ as a function of $u$; $y = f(u)$

Write $u$ as a function of $x$; $u = g(x)$

**STEP 2**

Differentiate $y$ with respect to $u$ to get $\frac{dy}{du}$

Differentiate $u$ with respect to $x$ to get $\frac{du}{dx}$

**STEP 3**

Obtain $\frac{dy}{dx}$ by applying the formula $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ and substitute $u$ back in for $g(x)$

- In trickier problems **chain rule** may have to be applied **more than once**

## Are there any standard results for using chain rule?

- There are **five** general results that can be useful

www.thrivingscholars.com



- If $y = (f(x))^n$ then $\frac{dy}{dx} = nf'(x)f(x)^{n-1}$
- If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$
- If $y = \ln(f(x))$ then $\frac{dy}{dx} = f'(x)\ln(f(x))$
- If $y = \sin(f(x))$ then $\frac{dy}{dx} = f'(x)\cos(f(x))$
- If $y = \cos(f(x))$ then $\frac{dy}{dx} = f'(x)\sin(f(x))$



## Exam Tip

- You should aim to be able to spot and carry out the chain rule mentally (rather than use substitution)
  - every time you use it, say it to yourself in your head
    *"differentiate the first function ignoring the second, then multiply by the derivative of the second function"*

www.thrivingscholars.com

## Worked Example

a)

Find the derivative of $y=(x^2-5x+7)^7$.

STEP 1 Identify the two functions and rewrite

$y = u^7$ i.e. $f(u)=u^7$

$u = x^2-5x+7$ i.e. $g(x)=x^2-5x+7$

STEP 2 Find $\frac{dy}{du}$ and $\frac{du}{dx}$.

$\frac{dy}{du} = 7u^6$ $\frac{du}{dx} = 2x-5$

STEP 3 Apply chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Chain rule is in the formula booklet

$\frac{dy}{dx} = 7u^6(2x-5)$

and substitute u back for g(x)

$$\frac{dy}{dx} = 7(2x-5)(x^2-5x+7)^6$$

b)

Find the derivative of $y=\sin(e^{2x})$.

$y = \sin(e^{2x})$ "... differentiate sin □, ignore $e^{2x}$..."

$\frac{dy}{dx} = \cos(e^{2x}) \times 2e^{2x}$ "... multiply by derivative of $e^{2x}$..."

↑ "$y=e^{ax+b}$, $\frac{dy}{dx}=ae^{ax+b}$" or by applying chain rule again

$$\therefore \frac{dy}{dx} = 2e^{2x}\cos(e^{2x})$$

www.thrivingscholars.com

# Product Rule

## What is the product rule?

- The **product rule** states if $y$ is the product of two functions $u(x)$ and $v(x)$ then

$$y = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written as

$$y = f(x)g(x)$$

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$$

- **'Dash notation'** may be used as a **shorter** way of writing the rule

$$y = uv$$

$$y' = uv' + vu'$$

- Final answers should match the notation used throughout the question

## How do I know when to use the product rule?

- The **product rule** is used when we are trying to **differentiate** the **product** of **two functions**
  - these can easily be confused with composite functions (see **chain rule)**
    - $\sin(\cos x)$ is a composite function, "sin of cos of $x$"
    - $\sin x \cos x$ is a product, "sin $x$ times cos $x$"

## How do I use the product rule?

- Make it clear what $u$, $v$, $u'$ and $v'$ are
  - arranging them in a square can help
    - opposite diagonals match up

**STEP 1**

Identify the two functions, $u$ and $v$

Differentiate both $u$ and $v$ with respect to $x$ to find $u'$ and $v'$

**STEP 2**

Obtain $\frac{dy}{dx}$ by applying the product rule formula $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Simplify the answer if straightforward to do so or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding $u'$ and $v'$

www.thrivingscholars.com

## Exam Tip

- Use $u$, $v$, $u'$ and $v'$ for the elements of product rule
  - lay them out in a 'square' (imagine a 2×2 grid)
  - those that are paired together are then on opposite diagonals ($u$ and $v'$, $v$ and $u'$)
- For trickier functions chain rule may be reuqired inside product rule
  - i.e. chain rule may be needed to differentiate $u$ and $v$

## Worked Example

a) Find the derivative of $y = e^x \sin x$.

$y = e^x \sin x$

STEP 1 Identify functions and differentiate

$u = e^x$ $\quad$ $v = \sin x$

$u' = e^x$ $\quad$ $v' = \cos x$

↑ arranging u, v, u', v' in a square makes product rule 'diagonal pairs'

STEP 2 Apply product rule: '$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$'

(As it is given in the formula booklet)

$y' = e^x \cos x + e^x \sin x$

$\therefore \frac{dy}{dx} = e^x(\cos x + \sin x)$

It is straightforward to take a factor of $e^x$ out

b) Find the derivative of $y = 5x^2 \cos 3x^2$.

$y = 5x^2 \cos 3x^2$

STEP 1 $u = 5x^2$ $\quad$ $v = \cos 3x^2$

$u' = 10x$ $\quad$ $v' = -\sin 3x^2 \times 6x$ ← chain rule

$v' = -6x \sin 3x^2$

STEP 2 $y' = -30x^3 \sin 3x^2 + 10x \cos 3x^2$

$\therefore \frac{dy}{dx} = 10x(\cos 3x^2 - 3x^2 \sin 3x^2)$

www.thrivingscholars.com

# Quotient Rule

## What is the quotient rule?

- The **quotient rule** states if $y$ is the quotient $\frac{u(x)}{v(x)}$ then

$$y = \frac{u}{v}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written

$$y = \frac{f(x)}{g(x)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

- As with product rule, **'dash notation'** may be used

$$y = \frac{u}{v}$$

$$y' = \frac{vu' - uv'}{v^2}$$

- Final answers should match the notation used throughout the question

## How do I know when to use the quotient rule?

- The **quotient rule** is used when trying to differentiate a fraction where **both** the **numerator** and **denominator** are **functions** of $x$
  - if the **numerator** is a **constant**, **negative powers** can be used
  - if the **denominator** is a **constant**, treat it as a **factor** of the expression

## How do I use the quotient rule?

- Make it clear what $u$, $v$, $u'$ and $v'$ are
  - arranging them in a square can help
    - opposite diagonals match up (like they do for product rule)

**STEP 1**

Identify the two functions, $u$ and $v$

Differentiate both $u$ and $v$ with respect to $x$ to find $u'$ and $v'$

**STEP 2**

Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the quotient rule formula $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

www.thrivingscholars.com

Be careful using the formula – because of the **minus sign** in the **numerator**, the **order** of the functions is important
Simplify the answer if straightforward or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding $u'$ and $v'$,

### Exam Tip

- Use $u$, $v$, $u'$ and $v'$ for the elements of quotient rule
  - lay them out in a 'square' (imagine a 2×2 grid)
  - those that are paired together are then on opposite diagonals ($v$ and $u'$, $u$ and $v'$)
- Look out for functions of the form $y = f(x)(g(x))^{-1}$
  - These can be differentiated using a combination of **chain rule** and **product rule**
    (it would be good practice to try!)
  - ... but it can also be seen as a quotient rule question in disguise
  - ... and vice versa!
    - A quotient could be seen as a product by rewriting the denominator as $(g(x))^{-1}$

www.thrivingscholars.com

## Worked Example

Differentiate $f(x) = \dfrac{\cos 2x}{3x+2}$ with respect to $x$.

STEP 1 Identify u and v, differentiate

$u = \cos 2x$ $\quad$ $v = 3x+2$

$u' = -2\sin 2x$ $\quad$ $v' = 3$

$u' = -2\sin 2x$: chain rule

opposite diagonals match up

STEP 2 Apply quotient rule: $\left(\dfrac{u}{v}\right)' = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

(As it is given in the formula booklet)

$$f'(x) = \frac{(3x+2)(-2\sin 2x) - (\cos 2x)(3)}{(3x+2)^2}$$

$$\therefore f'(x) = \frac{-2(3x+2)\sin 2x - 3\cos 2x}{(3x+2)^2}$$

(Nothing obvious/easy to simplify and question does not specify a particular form)

www.thrivingscholars.com





# Second Order Derivatives

## What is the second order derivative of a function?

- If you **differentiate** the **derivative** of a **function** (i.e. differentiate the function a second time) you get the **second order derivative** of the function
- There are two forms of **notation** for the **second order derivative**
  - $y = \mathrm{f}(x)$
  - $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)$ (First order derivative)
  - $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{f}''(x)$ (Second order derivative)
- Note the position of the superscript 2's
  - **d**ifferentiating twice (so $\mathbf{d^2}$) with respect to $x$ twice (so $\boldsymbol{x^2}$)
- The **second order derivative** can be referred to simply as the **second derivative**
  - Similarly, the **first order derivative** can be just the **first derivative**
- A **first order derivative** is the **rate** of **change** of a function
  - a **second order derivative** is the **rate** of **change** of the **rate** of **change** of a function
    - i.e. the **rate** of **change** of the function's **gradient**
- **Second order derivatives** can be used to
  - test for local minimum and maximum points
  - help determine the nature of stationary points
  - help determine the concavity of a function
  - graph derivatives

## How do I find a second order derivative of a function?

- By **differentiating twice!**
- This may involve
  - rewriting **fractions, roots,** etc as **negative** and/or **fractional powers**
  - differentiating **trigonometric** functions, **exponentials** and **logarithms**
  - using **chain rule**
  - using **product** or **quotient** rule



### Exam Tip

- Negative and/or fractional powers can cause problems when finding second derivatives so work carefully through each term

www.thrivingscholars.com

## Worked Example

Given that $f(x) = 4 - \sqrt{x} + \dfrac{3}{\sqrt{x}}$

a)

Find $f'(x)$ and $f''(x)$.

a) $f(x) = 4 - x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$ ← REWRITE AS POWERS OF x

$f'(x) = -(\frac{1}{2})x^{\frac{1}{2}-1} + 3(-\frac{1}{2})x^{-\frac{1}{2}-1}$

$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$ ← DIFFERENTIATE ONCE TO FIND $f'(x)$

$f''(x) = -\frac{1}{2}(-\frac{1}{2})x^{-\frac{1}{2}-1} - \frac{3}{2}(-\frac{3}{2})x^{-\frac{3}{2}-1}$

$f''(x) = \frac{1}{4}x^{\frac{3}{2}} + \frac{9}{4}x^{\frac{5}{2}}$ ← DIFFERENTIATE A SECOND TIME TO FIND $f''(x)$

b)

Evaluate $f''(3)$.

Give your answer in the form $a\sqrt{b}$, where $b$ is an integer and $a$ is a rational number.

b) $f''(x) = \dfrac{1}{4x\sqrt{x}} + \dfrac{9}{4x^2\sqrt{x}}$ ← $x^{\frac{3}{2}} = x\sqrt{x}$ $\quad x^{\frac{5}{2}} = x^2\sqrt{x}$

$f''(3) = \dfrac{1}{12\sqrt{3}} + \dfrac{9}{36\sqrt{3}}$

$= \dfrac{12}{36\sqrt{3}} = \dfrac{1}{3\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{9}$

$f''(3) = \frac{1}{9}\sqrt{3}$ ← RATIONALISE DENOMINATOR

www.thrivingscholars.com

# Higher Order Derivatives

## What is meant by higher order derivatives of a function?

- Many functions can be **differentiated** numerous times
  - The third, fourth, fifth, etc derivatives of a function are generally called **higher order derivatives**
- It may not be possible, or practical to (algebraically) differentiate complicated functions more than once or twice
- **Polynomials** will, eventually, have higher order derivatives of zero
  - Since powers of x reduce by 1 each time

## What is the notation for higher order derivatives?

- The notation for higher order derivatives follows the logic from the first and second derivatives

$$f^{(n)}(x) \text{ or } \frac{\mathrm{d}^n y}{\mathrm{d}x^n}$$

except the 'dash' (prime) notation is replaced with numbers as this would become cumbersome after the first few

  - e.g. the fifth derivative would be

$$f^{(5)}(x) \text{ or } \frac{\mathrm{d}^5 y}{\mathrm{d}x^5}$$

## How do I find a higher order derivative of a function?

- By differentiating as many times as required!
- This may involve
  - rewriting fractions, roots, etc as negative and/or fractional powers
  - differentiating trigonometric functions, exponentials and logarithms
  - using chain rule
  - using product or quotient rule

### Exam Tip

- If you are required to evaluate a higher order derivative at a specific point your GDC can help
  - Typically a GDC will only work out the first and second derivative directly from the original function
  - But, if you wanted the fourth derivative, say, you only need differentiate twice algebraically, then call this the 'original' function on your GDC

www.thrivingscholars.com

## Worked Example

It is given that $f(x) = \sin 2x$.

a)

Show that $f^4(x) = 16f(x)$.

$f^4(x)$ is the FOURTH derivative

$f(x) = \sin 2x$

$f'(x) = 2\cos 2x$ ('sin → cos', chain rule)

$f''(x) = -4\sin 2x$ ('cos → -sin', chain rule)

$f^3(x) = -8\cos 2x$ You should notice a pattern by now ...

$f^4(x) = 16\sin 2x$

$\therefore f^4(x) = 16\sin 2x = 16f(x)$ as required

b)

Without further working, write down an expression for $f^8(x)$.

We can see from part (a)

- the coefficient of each derivative is a power of 2
- $\sin 2x$ ($f(x)$) is involved in every even derivative
- $\sin 2x$ is positive in every other even derivative

$\therefore f^8(x) = 256\sin 2x$

www.thrivingscholars.com





# Stationary Points & Turning Points

**What is the difference between a stationary point and a turning point?**

- A **stationary point** is a point at which the **gradient function** is equal to zero
  - The **tangent** to the **curve** of the **function** is **horizontal**
- A **turning point** exhibits this property, but in addition the **function changes** from **increasing** to **decreasing,** or **vice versa**
  - The curve '**turns**' from '**going upwards**' to '**going downwards**' or **vice versa**
  - **Turning points** will either be **(local) minimum** or **maximum** points
- A **point** of **inflection** ***could*** also be a **stationary point** but is **not** a turning point

**How do I find stationary points and turning points?**

- For the function $y = f(x)$, **stationary points** can be found using the following process

**STEP 1**

Find the **gradient function,** $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$

**STEP 2**

Solve the equation $f'(x) = 0$ to find the $x$-coordiante(s) of any stationary points

**STEP 3**

If the $y$-coordaintes of the stationary points are also required then substitute the $x$-coordinate(s) into $f(x)$

- A GDC will solve $f'(x) = 0$ and most will find the coordinates of turning points (minimum and maximum points) in graphing mode

www.thrivingscholars.com

# Testing for Local Minimum & Maximum Points

**What are local minimum and maximum points?**

- Local **minimum** and **maximum** points are two types of **stationary** point
  - The **gradient function** (derivative) at such points equals zero
  - i.e. $f'(x) = 0$
- A **local minimum** point, $(x,\ f(x))$ will be the lowest value of $f(x)$ in the **local** vicinity of the value of $x$
  - The function may reach a **lower** value further afield
- Similarly, a **local maximum** point, $(x,\ f(x))$ will be the lowest value of $f(x)$ in the **local** vicinity of the value of $x$
  - The function may reach a **greater** value further afield
- The graphs of many functions **tend** to **infinity** for **large** values of $x$ (and/or **minus infinity** for **large negative** values of $x$)
- The **nature** of a stationary point refers to whether it is a **local minimum** point, a **local maximum** point or a **point of inflection**
- A **global** minimum point would represent the **lowest** value of $f(x)$ for **all values** of $x$
  - similar for a **global** maximum point

**How do I find the nature of a stationary point?**

- The **nature** of a **stationary point** can be determined using the **first derivative** but it is *usually* quicker and easier to use the **second derivative**
  - only in cases when the second derivative is **zero** is the first derivative method needed
- For the function $f(x)$ …

**STEP 1**

Find $f'(x)$ and solve $f'(x) = 0$ to find the $x$-coordinates of any stationary points

**STEP 2** (Second derivative)

Find $f''(x)$ and evaluate it at each of the stationary points found in **STEP 1**

**STEP 3** (Second derivative)

- If $f''(x) = 0$ then the nature of the stationary point **cannot** be determined; use the **first derivative** method (**STEP 4**)
- If $f''(x) > 0$ then the curve of the graph of $y = f(x)$ is **concave up** and the stationary point is a **local minimum** point
- If $f''(x) < 0$ then the curve of the graph of $y = f(x)$ is **concave down** and the stationary point is a **local maximum** point

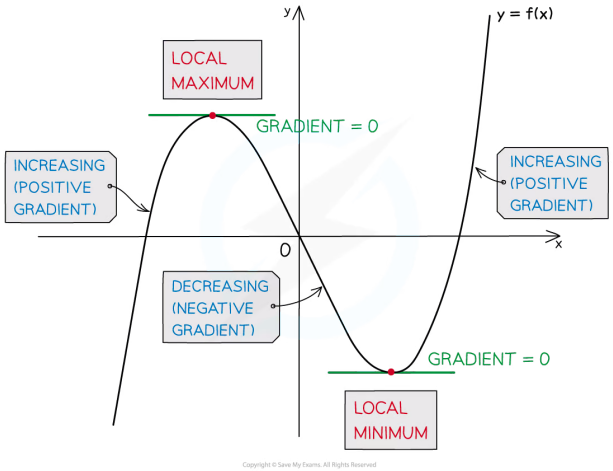
**STEP 4** (First derivative)

Find the sign of the first derivative just either side of the stationary point; i.e. evaluate $f'(x-h)$ and $f'(x+h)$ for small $h$

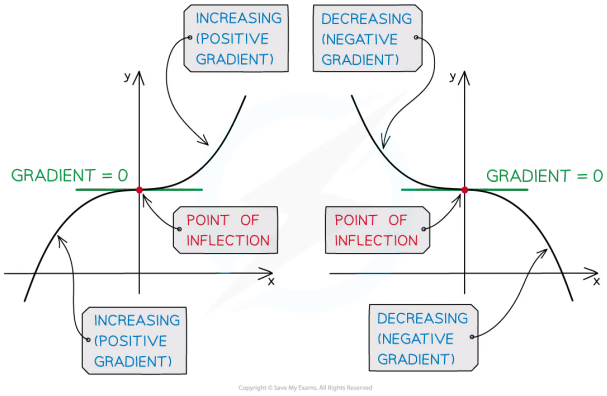
- A **local minimum point** changes the function from **decreasing** to **increasing**
  - the **gradient** changes from **negative** to **positive**
  - $f'(x-h) < 0,\ \ f'(x) = 0,\ \ f'(x+h) > 0$
- A **local maximum point** changes the **function** from **increasing** to **decreasing**
  - the **gradient** changes from **positive** to **negative**

www.thrivingscholars.com

- $f'(x-h)>0,\ f'(x)=0,\ f'(x+h)<0$




- A **stationary point** of **inflection** results from the function **either increasing or decreasing** on **both sides** of the stationary point
  - the **gradient** does **not change** sign
  - $f'(x-h)>0,\ f'(x+h)>0$ or $f'(x-h)<0,\ f'(x+h)<0$
  - a **point** of **inflection** does **not** necessarily have $f'(x)=0$
    - this method will only find those that do - and are often called **horizontal points** of **inflection**




## Exam Tip

- Exam questions may use the phrase "classify turning points" instead of "find the nature of turning points"
- Using your GDC to sketch the curve is a valid test for the nature of a stationary point in an exam unless the question says "show that..." or asks for an algebraic method
- Even if required to show a full algebraic solution you can still use your GDC to tell you what you're aiming for and to check your work

www.thrivingscholars.com



## Worked Example

Find the coordinates and the nature of any stationary points on the graph of $y = f(x)$ where $f(x) = 2x^3 - 3x^2 - 36x + 25$.

At stationary points, $f'(x) = 0$

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$

$6(x^2 - x - 6) = 0$

$(x - 3)(x + 2) = 0$

$x = 3$, $y = f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 25 = -56$

$x = -2$, $y = f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 25 = 69$

Using the second derivative to determine their nature

$f''(x) = 12x - 6 = 6(2x - 1)$

$f''(3) = 6(2 \times 3 - 1) = 30 > 0$

$\therefore x = 3$ is a local minimum point

$f''(-2) = 6(2 \times -2 - 1) = -30 < 0$

$\therefore x = -2$ is a local maximum point

(Note: In this case, both stationary points are turning points)

**Turning points are:**
**(3, -56) local minimum point**
**(-2, 69) local maximum point**

Use a GDC to graph $y = f(x)$ and the max/min solving feature to check the answers.

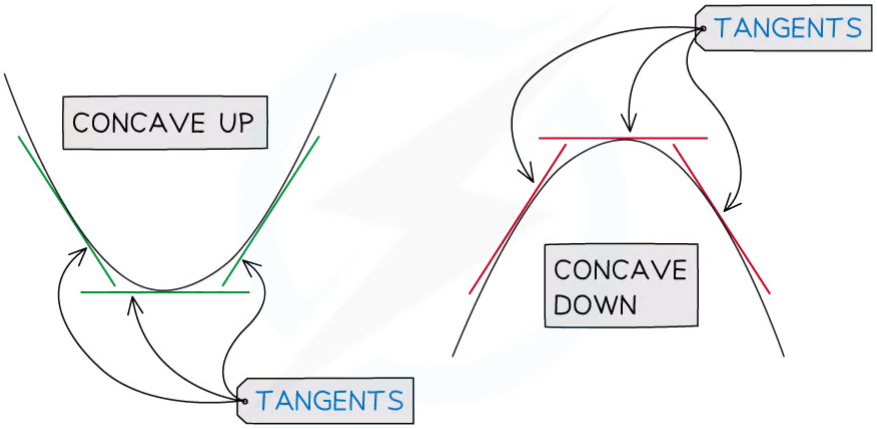
www.thrivingscholars.com



# Concavity of a Function

**What is concavity?**

- **Concavity** is the way in which a **curve** (or surface) **bends**
- Mathematically,
  - a curve is **CONCAVE DOWN** if $f''(x) \leq 0$ for all values of $x$ in an interval
  - a curve is **CONCAVE UP** if $f''(x) \geq 0$ for all values of $x$ in an interval
- **CONCAVE DOWN** is often called **concave**
- **CONCAVE UP** is often called **convex**




Copyright © Save My Exams. All Rights Reserved

## Exam Tip

- In an exam an easy way to remember the difference is:
  - Concave **down** is the shape of (the mouth of) a sad smiley ☹
  - **Concave up** is the shape of (the mouth of) a happy smiley ☺

www.thrivingscholars.com

## Worked Example

The function $f(x)$ is given by $f(x) = x^3 - 3x + 2$.

a)

Determine whether the curve of the graph of $y = f(x)$ is concave down or concave up at the points where $x = -2$ and $x = 2$.

$f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$

$f''(x) = 6x$

$f''(-2) = 6 \times -2 = -12 < 0$ (concave down)

$f''(2) = 6 \times 2 = 12 > 0$ (concave up)

At $x = -2$, $y = f(x)$ is concave down

At $x = 2$, $y = f(x)$ is concave up

Use your GDC to plot the graph of $y = f(x)$ and to help see if your answers are sensible

b)

Find the values of $x$ for which the curve of the graph $y = f(x)$ of is concave up.

$f''(x) = 6x$ from part (a)

Concave up is $f''(x) > 0$

$6x > 0$ when $x > 0$

$\therefore$ $y = f(x)$ is concave up for $x > 0$

Use your GDC to check your answer

www.thrivingscholars.com

# Points of Inflection

**What is a point of inflection?**

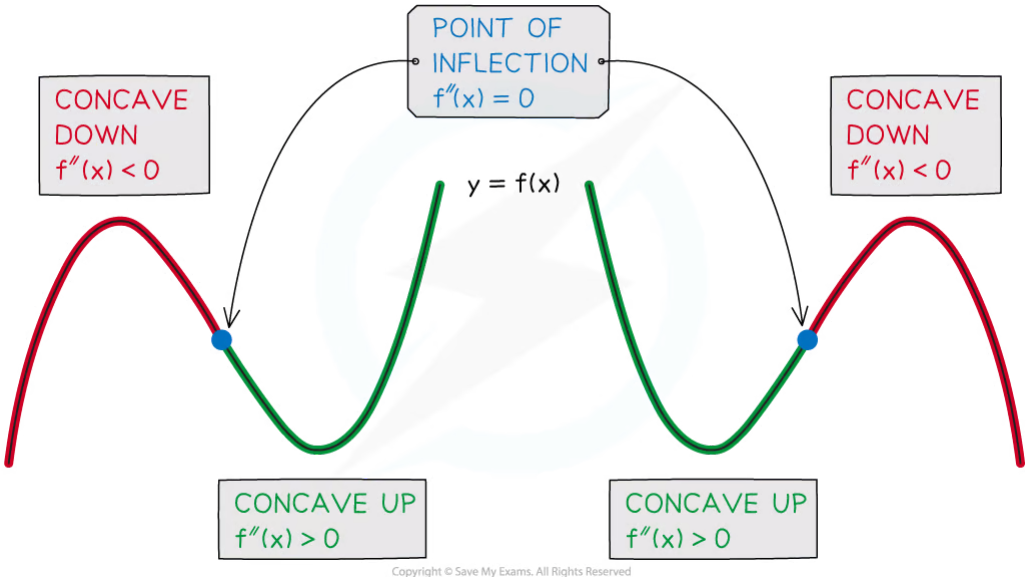
- A point at which the curve of the graph of $y = f(x)$ changes **concavity** is a **point** of **inflection**
- The alternative spelling, **inflexion,** may sometimes be used

**What are the conditions for a point of inflection?**

- A point of inflection requires **BOTH** of the following two conditions to hold
  - the **second derivative** is zero
    - $f''(x) = 0$

**AND**

  - the graph of $y = f(x)$ changes **concavity**
    - $f''(x)$ changes **sign** through a **point** of **inflection**



Copyright © Save My Exams. All Rights Reserved

- It is important to understand that the first condition is **not** sufficient on its own to locate a point of inflection
  - points where $f''(x) = 0$ could be **local minimum** or **maximum** points
    - the **first derivative** test would be needed
  - However, if it is already known $f(x)$ has a point of inflection at $x = a$, say, then $f''(a) = 0$

**What about the first derivative, like with turning points?**

- A **point** of **inflection**, unlike a turning point, does not necessarily have to have a first derivative value of 0 ( $f'(x) = 0$ )
  - If it does, it is also a **stationary point** and is often called a **horizontal point** of **inflection**
    - the tangent to the curve at this point would be horizontal

www.thrivingscholars.com



- The **normal distribution** is an example of a commonly used function that has a graph with two non-stationary points of inflection

**How do I find the coordinates of a point of inflection?**

- For the function $f(x)$

**STEP 1**

Differentiate $f(x)$ **twice** to find $f''(x)$ and solve $f''(x)=0$ to find the $x$-coordinates of possible points of inflection

**STEP 2**

Use the **second derivative** to **test** the **concavity** of $f(x)$ either side of $x=a$

- If $f''(x)<0$ then $f(x)$ is concave down
- If $f''(x)>0$ then $f(x)$ is concave up

If concavity changes, $x=a$ is a **point of inflection**

**STEP 3**

If required, the $y$-coordinate of a point of inflection can be found by substituting the $x$-coordinate into $f(x)$



Exam Tip

- You can find the *x*-coordinates of the point of inflections of $y=f(x)$ by drawing the graph $y=f'(x)$ and finding the *x*-coordinates of any local maximum or local minimum points
- Another way is to draw the graph $y=f''(x)$ and find the *x*-coordinates of the points where the graph crosses (not just touches) the *x*-axis

www.thrivingscholars.com



## Worked Example

Find the coordinates of the point of inflection on the graph of $y = 2x^3 - 18x^2 + 24x + 5$.

Fully justify that your answer is a point of inflection.

**STEP 1:** Differentiate twice, solve $f''(x) = 0$

$f(x) = 2x^3 - 18x^2 + 24x + 5$

$f'(x) = 6x^2 - 36x + 24$

$f''(x) = 12x - 36$

$12x - 36 = 0$ when $x = 3$

**STEP 2:** Use the second derivative to test concavity

$f''(3) = 0$

$f''(2.9) < 0$ (concave down)

$f''(3.1) > 0$ (concave up)

$\therefore$ concavity changes through $x = 3$

**STEP 3:** The y-coordinate is required

$f(3) = 2(3)^3 - 18(3)^2 + 24(3) + 5 = -31$

Since $f''(3) = 0$ AND the graph of $y = f(x)$ changes concavity through $x = 3$, the point $(3, -31)$ is a point of inflection.

Use your GDC to plot the graph of $y = f(x)$ and to help see if your answer is sensible

www.thrivingscholars.com



# Derivatives & Graphs

## How are derivatives and graphs connected?

- If the graph of a function $y = f(x)$ is known, or can be sketched, then it is also possible to sketch the graphs of the derivatives $y = f'(x)$ and $y = f''(x)$
- The key properties of a graph include
  - the **$y$-axis intercept**
  - the **$x$-axis intercepts** – the **roots** of the function; where $f(x) = 0$
  - **stationary points**; where $f'(x) = 0$
    - turning points – (local) **minimum** and **maximum** points
    - (horizontal) **points of inflection**
  - (non-stationary, $f'(x) \neq 0$) points of inflection
  - **asymptotes** – **vertical** and **horizontal**
  - intervals where the graph is **increasing** and **decreasing**
  - intervals where the graph is **concave down** and **concave up**
- **Not** all graphs have all of these properties and **not** all can be determined without knowing the expression of the function
- However questions will provide enough information to sketch
  - the **shape** of the graph
  - **some** of the **key properties** such as **roots** or **turning points**

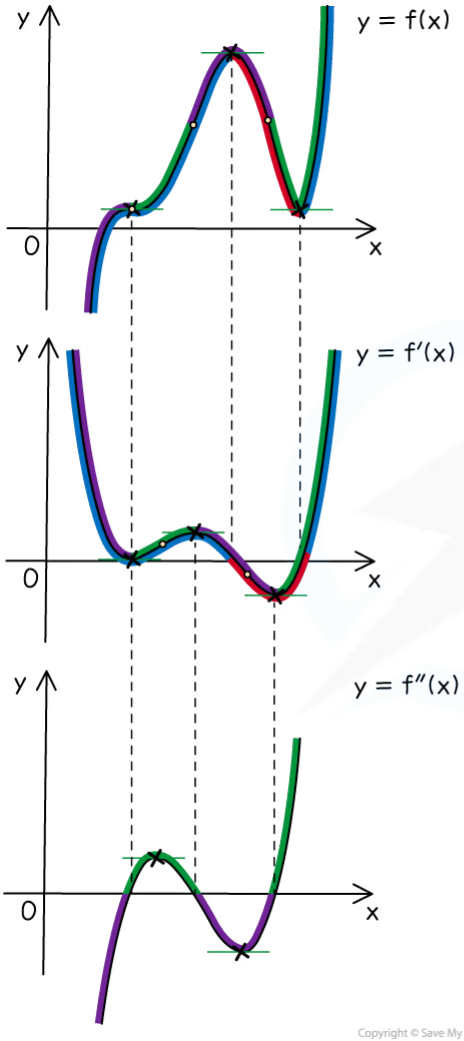
## How do I sketch the graph of $y = f'(x)$ from the graph of $y = f(x)$?

- The graph of $y = f'(x)$ will have its
  - $x$-axis **intercepts** at the $x$-coordinates of the **stationary points** of $y = f(x)$
  - **turning points** at the $x$-coordinates of the **points of inflection** of $y = f(x)$
- For intervals where $y = f(x)$ is concave up, $y = f'(x)$ will be increasing
- For intervals where $y = f(x)$ is concave down , $y = f'(x)$ will be decreasing
- For intervals where $y = f(x)$ is increasing, $y = f'(x)$ will be positive
- For intervals where $y = f(x)$ is decreasing, $y = f'(x)$ will be negative

## How do I sketch the graph of $y = f''(x)$ from the graph of $y = f(x)$?

- First sketch the graph of $y = f'(x)$ from $y = f(x)$, as per the above process
- Then, using the same process, sketch the graph of $y = f''(x)$ from the graph of $y = f'(x)$
- There are a couple of things you can deduce about the graph of $y = f''(x)$ directly from the graph of $y = f(x)$
  - The graph of $y = f''(x)$ will have its $x$-axis **intercepts** at the $x$-coordinates of the **points of inflection** of $y = f(x)$
  - For intervals where $y = f(x)$ is concave up, $y = f''(x)$ will be positive
  - For intervals where $y = f(x)$ is concave down, $y = f''(x)$ will be negative

www.thrivingscholars.com




| | |
|---|---|
| y = f(x) | CONCAVE UP |
| y = f′(x) | INCREASING |
| y = f″(x) | POSITIVE |
| y = f(x) | CONCAVE DOWN |
| y = f′(x) | DECREASING |
| y = f″(x) | NEGATIVE |
| y = f(x) | INCREASING |
| y = f′(x) | POSITIVE |
| y = f(x) | DECREASING |
| y = f′(x) | NEGATIVE |

NOTE:
THE GRAPHS OF y = f(x) AND y = f′(x) HAVE POINTS OF INFLECTION AT CHANGE OF CONCAVITY (∘)
THESE BECOME TURNING POINTS ON THE GRAPHS OF y = f′(x) AND y = f″(x) RESPECTIVELY
BUT WITHOUT FURTHER INFORMATION THE x–COORDINATES OF THESE POINTS CANNOT BE DEDUCED.

Copyright © Save My Exams. All Rights Reserved

## Is it possible to sketch the graph **of y = f(x) from the graph of a derivative?**

- It is possible to **sketch** a graph of $y = f(x)$ by considering the reverse of the above
  - For intervals where $y = f'(x)$ is **positive**, $y = f(x)$ will be **increasing** but is **not** necessarily positive
  - For intervals where $y = f'(x)$ is **negative**, $y = f(x)$ will be **decreasing** but is **not** necessarily negative
  - **Roots** of $y = f'(x)$ give the $x$-coordinates of the **stationary points** of $y = f(x)$
- There are some properties of the graph of $y = f(x)$ that **cannot** be determined from the graph of $y = f'(x)$
  - the ***y*-axis intercept**
  - the **intervals** for which $y = f(x)$ is **positive** and **negative**
  - the roots of $y = f(x)$
- Unless a **specific** point the **curve passes through** is **known**, the **constant** of **integration** **cannot** be determined
  - the *exact* location of the curve will remain unknown
  - but it will still be possible to **sketch** its **shape**

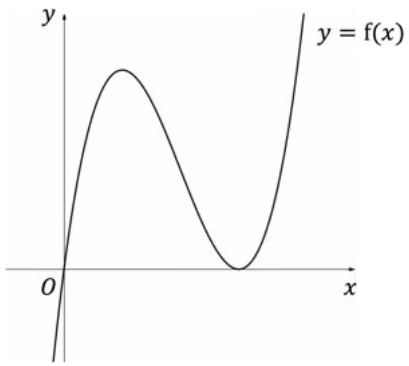
www.thrivingscholars.com

- If starting from the graph of the **second derivative,** $y = f''(x)$, it is easier to sketch the graph of $y = f'(x)$ first, then sketch $y = f(x)$

www.thrivingscholars.com

## Worked Example

The graph of $y = f(x)$ is shown in the diagram below.



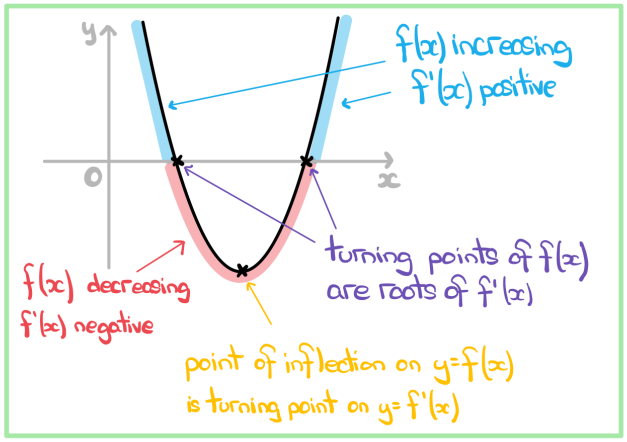

On separate diagrams sketch the graphs of $y = f'(x)$ and $y = f''(x)$, labelling any roots and turning points.

www.thrivingscholars.com

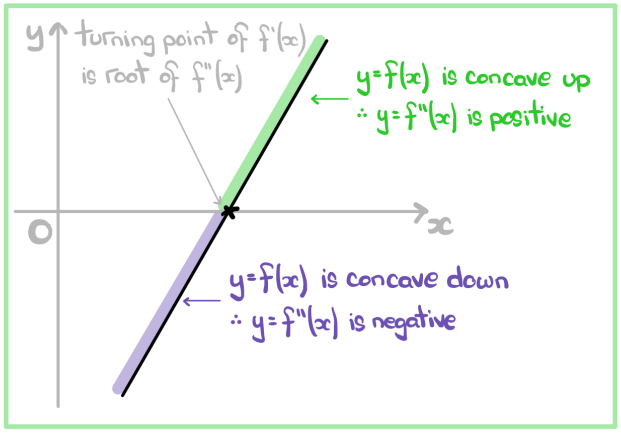
Key features from graph of $y=f(x)$ are:

- local maximum and local minimum
- $f(x)$ is increasing 'at either end'
- $f(x)$ is decreasing between max. and min.
- point of inflexion (non-stationary), graph changes from concave down to concave up

Graph of $y=f'(x)$:




Graph of $y=f''(x)$:



www.thrivingscholars.com

# 5.3 Integration

## 5.3.1 Introduction to Integration

### Introduction to Integration

### What is integration?

- **Integration** is the opposite to **differentiation**
  - Integration is referred to as **antidifferentiation**
  - The result of integration is referred to as the **antiderivative**
- **Integration** is the process of finding the expression of a function (**antiderivative**) from an expression of the **derivative** (**gradient function**)

### What is the notation for integration?

- An **integral** is normally written in the form

$$\int f(x)\,\mathrm{d}x$$

  - the large operator $\int$ means "integrate"
  - "$\mathrm{d}x$" indicates which variable to integrate with respect to
  - $f(x)$ is the function to be integrated (sometimes called the integrand)
- The **antiderivative** is sometimes denoted by $\mathrm{F}(x)$
  - there's then no need to keep writing the whole integral; refer to it as $\mathrm{F}(x)$
- $\mathrm{F}(x)$ may also be called the **indefinite integral** of $f(x)$

### What is the constant of integration?

- Recall one of the special cases from **Differentiating Powers of x**
  - If $f(x) = a$ then $f'(x) = 0$
- This means that integrating 0 will produce a **constant** term in the antiderivative
  - a zero term wouldn't be written as part of a function
  - **every** function, when integrated, potentially has a **constant** term
- This is called the **constant** of **integration** and is usually denoted by the letter $c$
  - it is often referred to as "plus $c$"
- Without more information it is impossible to deduce the value of this constant
  - there are endless antiderivatives, $\mathrm{F}(x)$, for a function $f(x)$

www.thrivingscholars.com

# Integrating Powers of x

## How do I integrate powers of x?

- Powers of $x$ are integrated according to the following formulae:
  - If $f(x) = x^n$ then $\int f(x)\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$ where $n \in \mathbb{Q}$, $n \neq -1$ and $c$ is the **constant of integration**
  - This is given in the **formula booklet**
- If the power of $x$ is multiplied by a constant then the integral is also multiplied by that constant
  - If $f(x) = ax^n$ then $\int f(x)\,\mathrm{d}x = \frac{ax^{n+1}}{n+1} + c$ where $n \in \mathbb{Q}$, $n \neq -1$ and $a$ is a constant and $c$ is the **constant of integration**
- $\frac{\mathrm{d}y}{\mathrm{d}x}$ notation can still be used with integration
- Note that the formulae above do not apply when $x = -1$ as this would lead to division by zero
- Remember the special case:
  - $\int a\,\mathrm{d}x = ax + c$
    - e.g. $\int 4\,\mathrm{d}x = 4x + c$
  - This allows **constant** terms to be integrated
- Functions involving **roots** will need to be rewritten as **fractional powers** of $x$ first
  - eg. If $f(x) = 5\sqrt[3]{x}$ then rewrite as $f(x) = 5x^{\frac{1}{3}}$ and integrate
- Functions involving **fractions** with **denominators** in **terms** of $\boldsymbol{x}$ will need to be rewritten as **negative powers** of $x$ first
  - e.g. If $f(x) = \frac{4}{x^2} + x^2$ then rewrite as $f(x) = 4x^{-2} + x^2$ and integrate
- The formulae for integrating powers of $x$ apply to **all rational numbers** so it is possible to integrate any expression that is a sum or difference of powers of $x$
  - e.g. If $f(x) = 8x^3 - 2x + 4$ then $\int f(x)\,\mathrm{d}x = \frac{8x^{3+1}}{3+1} - \frac{2x^{1+1}}{1+1} + 4x + c = 2x^4 - x^2 + 4x + c$
- **Products** and **quotients** cannot be integrated this way so would need **expanding/simplifying** first
  - e.g. If $f(x) = 8x^2(2x - 3)$ then
    $\int f(x)\,\mathrm{d}x = \int (16x^3 - 24x^2)\,\mathrm{d}x = \frac{16x^4}{4} - \frac{24x^3}{3} + c = 4x^4 - 8x^3 + c$

## What might I be asked to do once I've found the anti-derivative (integrated)?

- With more information the **constant of integration**, $c$, can be found
- The **area under** a **curve** can be found using integration

www.thrivingscholars.com

## Exam Tip

- You can speed up the process of integration in the exam by committing the pattern of basic integration to memory
  - In general you can think of it as 'raising the power by one and dividing by the new power'
  - Practice this lots before your exam so that it comes quickly and naturally when doing more complicated integration questions

## Worked Example

Given that

$$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - \frac{1}{\sqrt{x}}$$

find an expression for $y$ in terms of $x$.

Firstly rewrite all terms as powers of $x$

$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - x^{-\frac{1}{2}}$ ← fractional AND negative!

$y = \int \left(3x^4 - 2x^2 + 3 - x^{-\frac{1}{2}}\right) dx$

$\therefore y = \frac{3x^5}{5} - \frac{2x^3}{3} + 3x - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

Special case ↑ take care with negatives, $-\frac{1}{2}+1=\frac{1}{2}$ ↑ constant of integration

$$\therefore y = \frac{3}{5}x^5 - \frac{2}{3}x^3 + 3x - 2\sqrt{x} + c$$

www.thrivingscholars.com

# 5.6 Kinematics

## 5.6.1 Kinematics Toolkit

### Displacement, Velocity & Acceleration

## What is kinematics?

- **Kinematics** is the branch of mathematics that models and analyses the **motion** of objects
- Common words such as **distance, speed** and **acceleration** are used in kinematics but are used according to their technical definition

## What definitions do I need to be aware of?

- Firstly, only motion of an object in a **straight line** is considered
  - this could be a **horizontal** straight line
    - the **positive** direction would be to the **right**
  - or this could be a **vertical** straight line
    - the **positive** direction would be **upwards**

**Particle**

- A **particle** is the general term for an **object**
  - some questions may use a **specific** object such as a **car** or a **ball**

**Time** $t$ seconds

- **Displacement**, **velocity** and **acceleration** are all **functions** of **time** $t$
- **Initially** time is zero $t = 0$

**Displacement** $s$ m

- The **displacement** of a particle is its **distance relative** to a **fixed point**
  - the fixed point is often (but not always) the particle's **initial position**
- **Displacement** will be **zero** $s = 0$ if the object is at or has returned to its initial position
- **Displacement** will be negative if its **position relative** to the **fixed point** is in the **negative direction** (left or down)

**Distance** $d$ m

- Use of the word **distance** needs to be considered carefully and could refer to
  - the distance **travelled** by a particle
  - the **(straight line)** distance the particle is from a **particular point**
- Be careful not to confuse **displacement** with **distance**
  - if a bus route starts and ends at a bus depot, when the bus has returned to the depot, its **displacement** will be **zero** but the distance the bus has travelled will be the length of the route
- **Distance** is always **positive**

**Velocity** $v$ m s$^{-1}$

- The **velocity** of a particle is the **rate of change** of its **displacement** at time $t$

www.thrivingscholars.com

- **Velocity** will be **negative** if the **particle** is moving in the **negative direction**
- A **velocity** of **zero** means the particle is **stationary** $v = 0$

**Speed** $|v|$ m s$^{-1}$

- **Speed** is the **magnitude** (a.k.a. absolute value or modulus) of **velocity**
  - as the particle is **moving** in a **straight line, speed** is the **velocity ignoring** the **direction**
    - if $v = 4$, $|v| = 4$
    - if $v = -6$, $|v| = 6$

**Acceleration** $a$ m s$^{-2}$

- The **acceleration** of a particle is the **rate of change** of its **velocity** at time $t$
- Acceleration can be **negative** but this alone cannot fully describe the particle's motion
  - if **velocity** and **acceleration** have the **same** sign the particle is **accelerating** (speeding up)
  - if **velocity** and **acceleration** have **different** signs then the particle is **decelerating** (slowing down)
  - if **acceleration** is **zero** $a = 0$ the particle is moving with **constant** velocity
  - in all cases the **direction** of **motion** is determined by the **sign** of **velocity**

## Are there any other words or phrases in kinematics I should know?

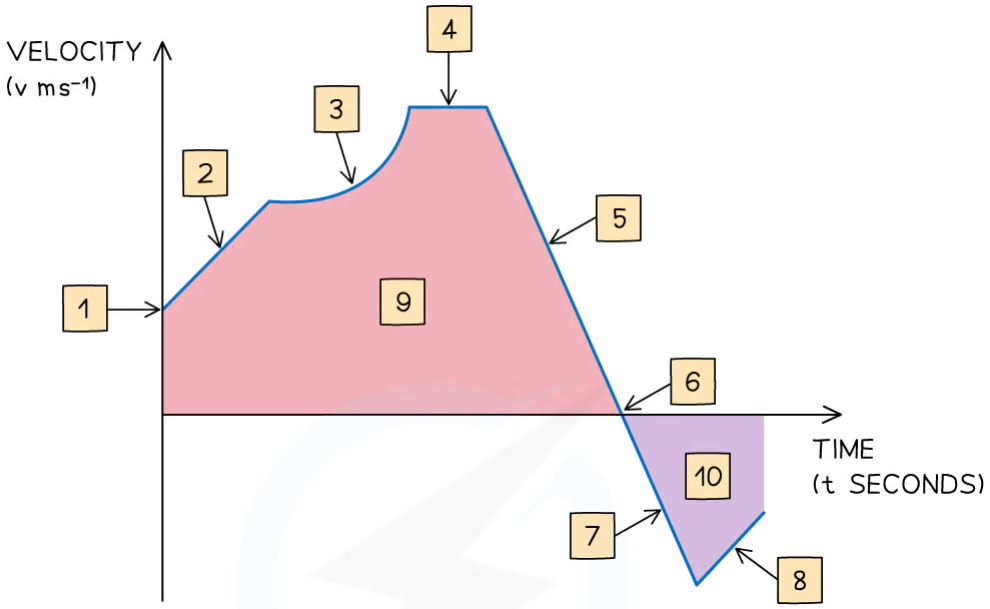
- Certain words and phrases can imply values or directions in kinematics
  - a particle described as "at **rest**" means that its velocity is zero, $\boldsymbol{v = 0}$
  - a particle described as moving "**due east**" or "**right**" or would be moving in the **positive horizontal** direction
    - this also means that $\boldsymbol{v > 0}$
  - a particle "**dropped from the top of a cliff**" or "**down**" would be moving in the **negative vertical** direction
    - this also means that $\boldsymbol{v < 0}$

## What are the key features of a velocity-time graph?

- The **gradient** of the graph equals the **acceleration** of an object
- A **straight line** shows that the object is **accelerating** at a **constant rate**
- A **horizontal** line shows that the object is moving at a **constant velocity**
- The **area** between graph and the x-axis tells us the **change in displacement** of the object
  - Graph **above** the x-axis means the object is moving **forwards**
  - Graph **below** the x-axis means the object is moving **backwards**

- The **total displacement** of the object from its starting point is the sum of the **areas above** the x-axis **minus** the sum of the **areas below** the x-axis
- The **total distance travelled** by the object is the sum of **all** the **areas**
- If the graph **touches** the **x-axis** then the object is **stationary** at that time
- If the graph is **above** the **x-axis** then the object has positive velocity and is **travelling forwards**
- If the graph is **below** the **x-axis** then the object has negative velocity and is **travelling backwards**

www.thrivingscholars.com




| | |
|---|---|
| 1 | INITIAL VELOCITY |
| 2 | CONSTANT ACCELERATION |
| 3 | VARIABLE ACCELERATION |
| 4 | CONSTANT VELOCITY |
| 5 | DECELERATING (SLOWING DOWN BUT STILL MOVING FORWARDS) |
| 6 | INSTANTANEOUSLY AT REST (STATIONARY FOR AN INSTANT) |
| 7 | SPEEDING UP BUT MOVING BACKWARDS |
| 8 | SLOWING DOWN BUT STILL MOVING BACKWARDS |
| 9 | DISTANCE TRAVELLED FORWARDS |
| 10 | DISTANCE TRAVELLED BACKWARDS |

Copyright © Save My Exams. All Rights Reserved

## Exam Tip

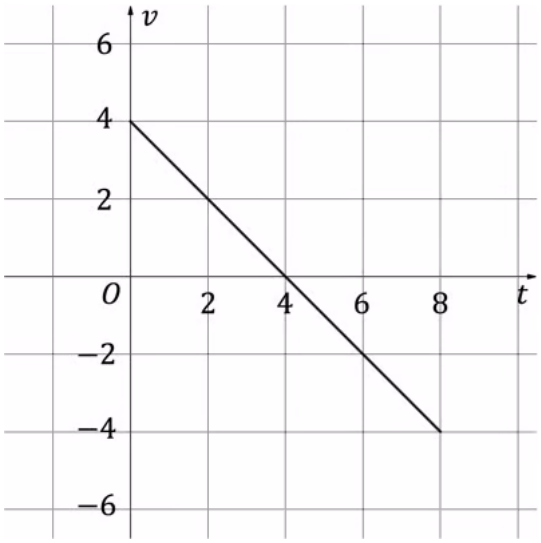
- In an exam if you are given an expression for the velocity then sketching a velocity-time graph can help visualise the problem

www.thrivingscholars.com

## Worked Example

A particle is projected vertically upwards from ground level, taking 8 seconds to return to the ground.

The velocity-time graph below illustrates the motion of the particle for these 8 seconds.




i)
How many seconds does the particle take to reach its maximum height?
Give a reason for your answer.

ii)
State, with a reason, whether the particle is accelerating or decelerating at time $t = 3$.

i. At maximum height, velocity is zero

v = 0 at t = 4

∴ The particle takes 4 seconds to reach its maximum height. This is because its velocity is 0 m s$^{-1}$ at 4 seconds.

ii. At t = 3, velocity is POSITIVE

Acceleration is the gradient of velocity

At t = 3, acceleration is NEGATIVE

∴ At 3 seconds the particle is decelerating as its velocity and acceleration have different signs.

www.thrivingscholars.com

www.thrivingscholars.com

# 5.7 Basic Limits & Continuity

## 5.7.1 Basic Limits & Continuity

### Limits

### What are limits in mathematics?

- When we consider a **limit** in mathematics we look at the tendency of a mathematical process as it approaches, but never quite reaches, an 'end point' of some sort
- We use a special limit notation to indicate this
  - For example $\lim_{x \to 3} f(x)$ denotes 'the limit of the function *f*(*x*) as *x* goes to (or approaches) 3'
    - I.e., what value (if any) *f*(*x*) gets closer and closer to as *x* takes on values closer and closer to 3
    - We are not concerned here with what value (if any) *f*(*x*) takes when *x* is equal to 3 – only with the behaviour of *f*(*x*) as *x* gets close to 3
- The sum of an infinite geometric sequence is a type of limit
  - When you calculate $S_\infty$ for an infinite geometric sequence, what you are actually finding is $\lim_{n \to \infty} S_n$
    - I.e., what value (if any) the sum of the first *n* terms of the sequence gets closer and closer to as the number of terms (*n*) goes to infinity
    - The sum never actually reaches $S_\infty$, but as more and more terms are included in the sum it gets closer and closer to that value
- In this section of the IB course we will be considering the limits of functions
  - This may include finding the limit at a point where the function is undefined
    - For example, $f(x) = \frac{\sin x}{x}$ is undefined when *x* = 0, but we might want to know how the function behaves as *x* gets closer and closer to zero
  - Or it may include finding the limit of a function *f*(*x*) as *x* gets infinitely big in the positive or negative direction
    - For this type of limit we write $\lim_{x \to \infty} f(x)$ or $\lim_{x \to -\infty} f(x)$ (the first one can also be written as $\lim_{x \to +\infty} f(x)$ to distinguish it from the second one)
  - These sorts of limits are often used to find the **asymptotes** of the graph of a function

### How do I find a simple limit?

- STEP 1: To find $\lim_{x \to a} f(x)$ begin by substituting *a* into the function *f*(*x*)
  - If *f*(*a*) exists with a well-defined value, then that is also the value of the limit
  - For example, for $f(x) = \frac{x-1}{x}$ we may find the limit as *x* approaches 3 like this:

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x-1}{x} = \frac{3-1}{3} = \frac{2}{3}$$

www.thrivingscholars.com



    - In this case, is simply equal to $f(3)$
- STEP 2: If $f(a)$ does not exist, it may be possible to use algebra to simplify $f(x)$ so that substituting $a$ into the simplified function gives a well-defined value
    - In that case, the well-defined value at $x = a$ of the simplified version of the function is also the value of the limit of the function as $x$ goes to $a$
    - For example, $f(x) = \frac{x^2}{x}$ is not defined at $x = 0$, but we may use algebra to find the limit as $x$ approaches zero:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} \frac{x}{1} \text{ (cancelling the } x\text{'s)} = \frac{0}{1} = 0$$

        - Note that $f(x) = \frac{x^2}{x}$ and $g(x) = x$ are **not** the same function!
        - They are equal for all values of $x$ except zero
        - But for $x = 0$, $g(0) = 0$ while $f(0)$ is undefined
        - However $f(x)$ gets closer and closer to zero as $x$ gets closer and closer to zero
- If neither of these steps gives a well-defined value for the limit you may need to consider more advanced techniques to evaluate the limit
    - For example **l'Hôpital's Rule** or using **Maclaurin series**

## How do I find a limit to infinity?

- As $x$ goes to $+\infty$ or $-\infty$, a typical function $f(x)$ may **converge** to a well-defined value, or it may **diverge** to $+\infty$ or $-\infty$
    - Other behaviours are possible – for example $\lim_{x \to \infty} \sin x$ is simply undefined, because $\sin x$ continues to oscillate between 1 and -1 as $x$ gets larger and larger
- There are two key results to be used here:
    - $\lim_{x \to \pm\infty} \frac{k}{x^n}$ **converges** to 0 for all $n > 0$ and all $k \in \mathbb{R}$
    - $\lim_{x \to +\infty} x^n$ **diverges** to $+\infty$ for all $n > 0$
        - $\lim_{x \to -\infty} x^n$ for $n > 0$ will need to be considered on a case-by-case basis, because of the differing behaviour of $x^n$ for different values of $n$ when $x$ is negative
- STEP 1: If necessary, use algebra to rearrange the function into a form where one or the other of the key results above may be applied
- STEP 2: Use the key results above to evaluate your limit
- For example:

$$\lim_{x \to \infty} \frac{3x^2 - 2x + 1}{4x^2 - x + 2} = \lim_{x \to \infty} \frac{3 - \frac{2}{x} + \frac{1}{x^2}}{4 - \frac{1}{x} + \frac{2}{x^2}} = \frac{3 - 0 + 0}{4 - 0 + 0} = \frac{3}{4}$$

- Or:

www.thrivingscholars.com

$$\lim_{x \to +\infty} \frac{x^2 + 5x - 2}{32x + 3} = \lim_{x \to +\infty} \frac{x + 5 - \frac{2}{x}}{32 + \frac{3}{x}} = \frac{(+\infty) + 5 - 0}{32 + 0} = +\infty$$

  - I.e., the limit diverges to $+\infty$ (because $\frac{x^2 + 5x - 2}{32x + 3}$ it gets bigger and bigger without limit as $x$ gets bigger and bigger)
- Remember that neither $\frac{0}{0}$ nor $\frac{\pm\infty}{\pm\infty}$ has a well-defined value!
  - If you attempt to evaluate a limit and get one of these two forms, you will need to try another strategy
  - This may just mean different or additional algebraic rearrangement
  - But it may also mean that you need to consider using **l'Hôpital's Rule** or **Maclaurin series** to evaluate the limit
- It is also worth remembering that if $\lim_{x \to \infty} f(x) = \infty$, then $\lim_{x \to \infty} \frac{k}{f(x)} = 0$ for any non-zero $k \in \mathbb{R}$
  - This can be useful for example when evaluating the limits of functions containing exponentials
    - $\lim_{x \to \infty} e^{px} = \infty$ for any $p > 0$, so we immediately have $\lim_{x \to \infty} e^{-px} = \lim_{x \to \infty} \frac{1}{e^{px}} = 0$ for $p > 0$
    - See the worked example below for a more involved version of this

## Do limits ever have 'directions'?

- Yes they do!
- The notation $\lim_{x \to a^+} f(x)$ means 'the limit of $f(x)$ as $x$ **approaches *a* from above**'
  - I.e., this is the limit as $x$ comes 'down' towards $a$, only considering the function's behaviour for values of $x$ that are greater than $a$
- The notation $\lim_{x \to a^-} f(x)$ means 'the limit of $f(x)$ as $x$ **approaches *a* from below**'
  - I.e., this is the limit as $x$ comes 'up' towards $a$, only considering the function's behaviour for values of $x$ that are less than $a$
- One place these sorts of limits appear is for functions defined piecewise
  - In this case the limits 'from above' and 'from below' may well be different for values of $x$ at which the different 'pieces' of the function are joined
- But also be aware of a situation like the following, where the limits from above and below may also be different:
  - $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ (because $\frac{1}{x} > 0$ for $x > 0$, with $\frac{1}{x}$ becoming bigger and bigger in the positive direction as $x$ gets closer and closer to zero 'from above')

www.thrivingscholars.com

- $\lim_{x \to 0-} \frac{1}{x} = +\infty$ (because $\frac{1}{x} < 0$ for $x < 0$, with $\frac{1}{x}$ becoming bigger and bigger in the negative direction as $x$ gets closer and closer to zero 'from below')
- The graph of $y = \frac{1}{x}$ shows this limiting behaviour as $x$ approaches zero from the two different directions

www.thrivingscholars.com

## Worked Example

a)

Consider the function

$$f(x)=\frac{3-4x-5x^4}{2x^4+x^3+7},$$

find $\lim_{x\to\infty} f(x)$.

$$\frac{3-4x-5x^4}{2x^4+x^3+7}\cdot\frac{1/x^4}{1/x^4}=\frac{\frac{3}{x^4}-\frac{4}{x^3}-5}{2+\frac{1}{x}+\frac{7}{x^4}}$$

$$\lim_{x\to\infty}\frac{3-4x-5x^4}{2x^4+x^3+7}=\lim_{x\to\infty}\frac{\frac{3}{x^4}-\frac{4}{x^3}-5}{2+\frac{1}{x}+\frac{7}{x^4}}$$

$$=\frac{0-0-5}{2+0+0}=-\frac{5}{2}$$

b)

Consider the function

$$g(x)=\begin{cases}\dfrac{1-5x}{x^2}, & x<5\\ x^2-4x-6, & x\geq 5\end{cases}$$

find (i) $\lim_{x\to 5^-} g(x)$, and (ii) $\lim_{x\to 5^+} g(x)$.

www.thrivingscholars.com

(i) For $\lim_{x \to 5^-}$, we only consider $x < 5$

$$\lim_{x \to 5^-} g(x) = \lim_{x \to 5} \frac{1 - 5x}{x^2}$$

$$= \frac{1 - 5(5)}{(5)^2} = \boxed{-\frac{24}{25}}$$

(ii) For $\lim_{x \to 5^+}$, we only consider $x > 5$

$$\lim_{x \to 5^+} g(x) = \lim_{x \to 5} (x^2 - 4x - 6)$$

$$= (5)^2 - 4(5) - 6 = \boxed{-1}$$

c)
Consider the function

$$h(x) = \frac{2e^{3x} - 3}{4 - 5e^{3x}}$$

find $\lim_{x \to \infty} h(x)$.

www.thrivingscholars.com

$$\frac{2e^{3x}-3}{4-5e^{3x}} \cdot \frac{1/e^{3x}}{1/e^{3x}} = \frac{2-\frac{3}{e^{3x}}}{\frac{4}{e^{3x}}-5}$$

$$\lim_{x\to\infty} \frac{2e^{3x}-3}{4-5e^{3x}} = \lim_{x\to\infty} \frac{2-\frac{3}{e^{3x}}}{\frac{4}{e^{3x}}-5}$$

$$= \frac{2-0}{0-5} = \boxed{-\frac{2}{5}}$$

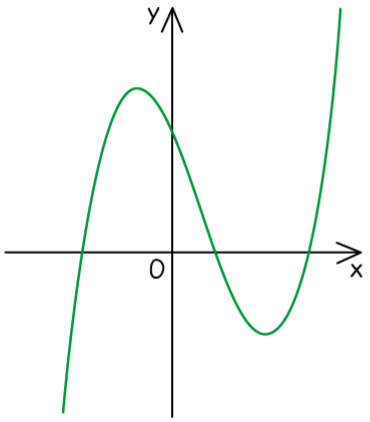
www.thrivingscholars.com

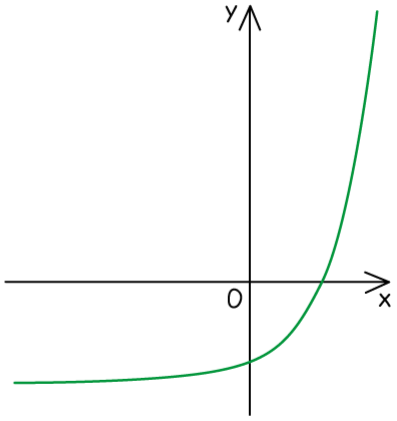
# Continuity & Differentiability

## What does it mean for a function to be continuous at a point?

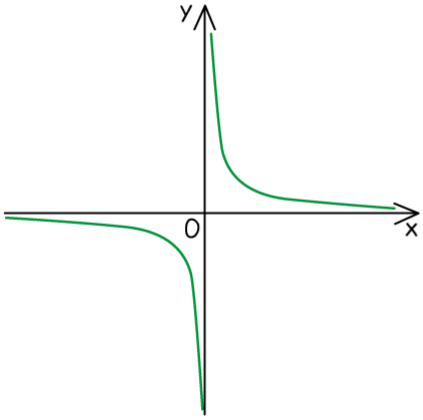
- If a function is **continuous** at a point then the graph of the function does not have any 'holes' or any sudden 'leaps' or 'jumps' at that point
  - One way to think about this is to imagine sketching the graph
    - So long as you can sketch the graph without lifting your pencil from the paper, then the function is continuous at all the points that your sketch goes through
    - But if you would have to lift your pencil off the paper at some point and continue drawing the graph from another point, then the function is not continuous at any such points where the function 'jumps'

CONTINUOUS AT ALL POINTS

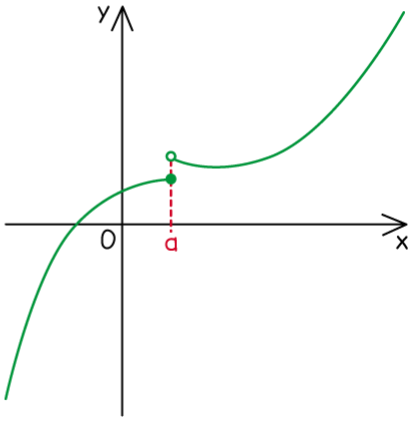



NOT CONTINUOUS AT x = 0




NOT CONTINUOUS AT x = a




- There are two main ways a function can fail to be continuous at a point:
  - If the function is not defined for a particular value of $x$ then it is not continuous at that value of $x$

www.thrivingscholars.com

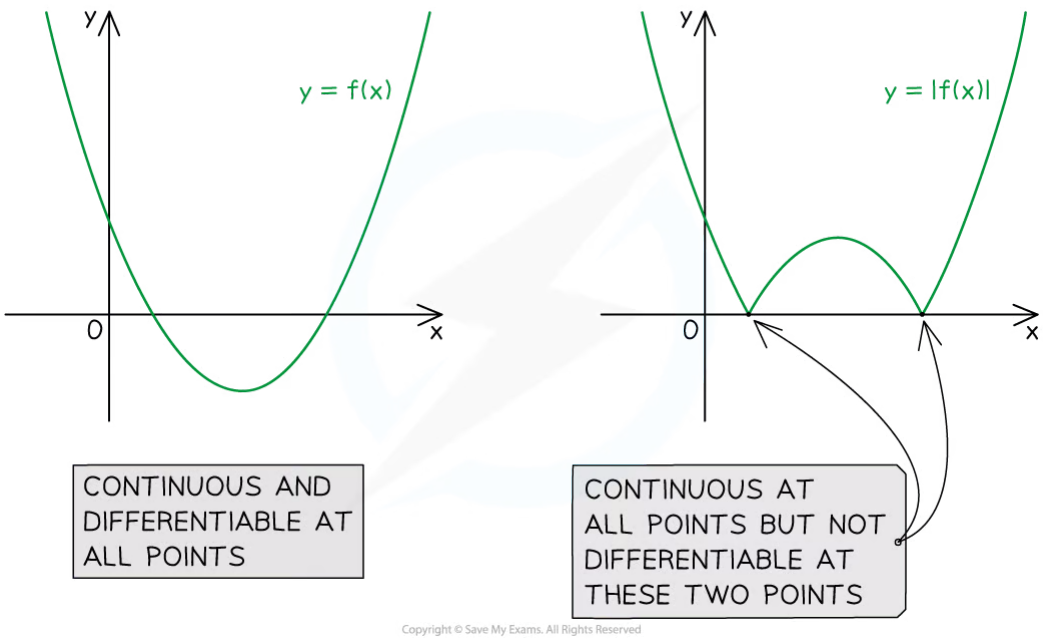


    - For example, $f(x) = \frac{1}{x}$ is not continuous at $x = 0$
  - If the function is defined for a particular value of $x$, but then the value of the function 'jumps' as $x$ moves away from that x value in the positive or negative direction, then the function is not continuous at that value of $x$
    - This type of discontinuity can occur in a piecewise function, for example, where the different pieces of the function's graph don't 'join up'
- You can use limits to show that a function is continuous at a point
  - Let $f(x)$ be a function defined at $x = a$, such that $f(a) = b$
    - If $\lim_{x \to a^-} f(x) = b$ and $\lim_{x \to a+} f(x) = b$, then $f(x)$ is continuous at $x = a$
    - If either of those limits is not equal to $b$, then $f(x)$ is not continuous at $x = a$
  - This is a slightly more formal way of expressing the 'you don't have to lift your pencil from the paper' idea!

## What does it mean for a function to be differentiable at a point?

- We say that a function $f(x)$ is **differentiable** at a point with $x$-coordinate $x_0$, if the derivative $f'(x)$ exists and has a well-defined value $f'(x_0)$ at that point
- To be differentiable at a point a function has to be continuous at that point
  - So if a function is not continuous at a point, then it is also not differentiable at that point
- But continuity by itself does not guarantee differentiability
  - This means that differentiability is a stronger condition than continuity
  - If a function is differentiable at a point, then the function is also continuous at that point
  - But a function may be continuous at a point without being differentiable at that point
  - This means there are functions that are continuous everywhere but are not differentiable everywhere
- In addition to being continuous a point, differentiability also requires that the function be **smooth** at that point
  - 'Smooth' means that the graph of the function does not have any 'corners' or sudden changes of direction at the point
  - An obvious example of a function that is not smooth at certain points is a modulus function $|f(x)|$ at any values of $x$ where $f(x)$ changes sign from positive to negative
    - At any such point a modulus function will not be differentiable

www.thrivingscholars.com




Copyright © Save My Exams. All Rights Reserved

## Exam Tip

- On the exam you will not usually be asked to test a function for **continuity** at a point
  - You should however be familiar with the basic ideas about continuity outlined above
- On the exam you will not be asked to test a function for **differentiability** at a point
  - You should however be familiar with the basic ideas about differentiability and its relationship with continuity as outlined above

www.thrivingscholars.com

## Worked Example

Consider the function $f$ defined by

$$f(x)=\begin{cases} x^2-2x-1, & x<3 \\ 2 & x=3 \\ \dfrac{x+2}{2}, & x>3 \end{cases}$$

a)

use limits to show that $f$ is not continuous at $x=3$.

$f(3) = 2$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} (x^2-2x-1) = (3)^2-2(3)-1 = 2$

↑ The limit 'from below'

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3} \left(\frac{x+2}{2}\right) = \frac{3+2}{2} = \frac{5}{2}$

↑ The limit 'from above'

$\lim_{x\to 3^+} f(x) \neq f(3)$, therefore f is not continuous at $x = 3$.

b)

Hence explain why $f$ cannot be differentiable at $x=3$.

In order to be differentiable at a point, a function must be continuous at that point.

f is not continuous at $x = 3$, therefore it cannot be differentiable at $x = 3$.

www.thrivingscholars.com

# 5.8 Advanced Differentiation

## 5.8.1 First Principles Differentiation

### First Principles Differentiation

### What is differentiation from first principles?

- Differentiation from **first principles** uses the **definition** of the **derivative** of a function ***f(x)***
- The definition is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- $\lim_{h \to 0}$ means the '*limit as* ***h*** *tends to zero*'
- When $h = 0$, $\frac{f(x+h) - f(x)}{h} = \frac{f(x) - f(x)}{0} = \frac{0}{0}$ which is **undefined**
  - Instead we consider what happens as ***h*** gets closer and closer to zero
- Differentiation **from first principles** means using that definition to show what the **derivative** of a function is
- The first principles definition (formula) is in the **formula booklet**

### How do I differentiate from first principles?

**STEP 1:** Identify the function ***f(x)*** and substitute this into the first principles formula

e.g. Show, from first principles, that the derivative of $3x^2$ is $6x$

$$f(x) = 3x^2 \text{ so } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$$

**STEP 2:** Expand ***f(x+h)*** in the numerator

$$f'(x) = \lim_{h \to 0} \frac{3(x^2 + 2hx + h^2) - 3x^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3x^2 + 6hx + 3h^2 - 3x^2}{h}$$

**STEP 3:** Simplify the numerator, factorise and cancel h with the denominator

$$f'(x) = \lim_{h \to 0} \frac{h(6x + 3h)}{h}$$

**STEP 4:** Evaluate the remaining expression as ***h*** *tends to zero*

$$f'(x) = \lim_{h \to 0} (6x + 3h) = 6x \quad \text{As } h \to 0, (6x + 3h) \to (6x + 0) \to 6x$$

www.thrivingscholars.com

$\therefore$ The derivative of $3x^2$ is $6x$

## Exam Tip

- Most of the time you will not use first principles to find the derivative of a function (there are much quicker ways!)
  However, you can be asked to demonstrate differentiation from first principles
- To get full marks make sure you are are writing lim h -> 0 right up until the concluding sentence!

## Worked Example

Prove, from first principles, that the derivative of $5x^3$ is $15x^2$.

STEP 1: For $f(x) = 5x^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is given in the formula booklet

$$= \lim_{h \to 0} \frac{5(x+h)^3 - 5x^3}{h}$$

STEP 2:

$$= \lim_{h \to 0} \frac{5(x^3 + 3x^2h + 3xh^2 + h^3) - 5x^3}{h}$$

Expand $(x+h)^3$ using binomial theorem

$$= \lim_{h \to 0} \frac{5x^3 + 15x^2h + 15xh^2 + 5h^3 - 5x^3}{h}$$

STEP 3:

$$= \lim_{h \to 0} \frac{15x^2h + 15xh^2 + 5h^3}{h}$$

$$= \lim_{h \to 0} (15x^2 + 15xh + 5h^3)$$

STEP 4: As $h \to 0$

$$(15x^2 + 15xh + 5h^2) \to (15x^2 + 15x(0) + 5(0)^2) = 15x^2$$

$\therefore$ The derivative of $5x^3$ is $15x^2$

www.thrivingscholars.com

# 5.9 Advanced Integration

## 5.9.1 Integrating Further Functions

As with other problems in integration the results in this revision note may have further uses such as

- evaluating a definite integral
- finding the constant of integration
- finding areas under a curve, between a line and a curve or between two curves

www.thrivingscholars.com



# Integrating with Reciprocal Trigonometric Functions

**cosec** *(cosecant, csc)*, **sec** *(secant)* and **cot** *(cotangent)* are the reciprocal functions of sine, cosine and tangent respectively.

## What are the antiderivatives involving reciprocal trigonometric functions?

- $\int \sec^2 x \, \mathrm{d}x = \tan x + c$
- $\int \sec x \tan x \, \mathrm{d}x = \sec x + c$
- $\int \operatorname{cosec} x \cot x \, \mathrm{d}x = -\operatorname{cosec} x + c$
- $\int \operatorname{cosec}^2 x \, \mathrm{d}x = -\cot x + c$
- These are **not** given in the **formula booklet** directly
  - they are listed the other way round as 'standard derivatives'
  - be careful with the negatives in the last two results
  - and remember "+c" !

## How do I integrate these if a *linear* function of x is involved?

- All integration rules could apply alongside the results above
- The use of reverse chain rule is particularly common
  - For linear functions the following results can be useful
    - $\int \sec(ax+b)\tan(ax+b) \, \mathrm{d}x = \frac{1}{a}\sec(ax+b) + \mathrm{c}$
    - $\int \operatorname{cosec}(ax+b)\cot(ax+b) \, \mathrm{d}x = -\frac{1}{a}\operatorname{cosec}(ax+b) + \mathrm{c}$
    - $\int \cot(ax+b) \, \mathrm{d}x = -\frac{1}{a}\operatorname{cosec}^2(ax+b) + \mathrm{c}$
  - These are **not** in the formula booklet
    - they can be deduced by spotting reverse chain rule
    - they are not essential to remember but can make problems easier

### Exam Tip

- Even if you think you have remembered these antiderivatives, always use the formula booklet to double check
  - those squares, negatives and "1 over"'s are easy to get muddled up!
- Remember to use 'adjust' and 'compensate' for reverse chain rule when coefficients are involved

www.thrivingscholars.com

## Worked Example

The graph of $y = f(x)$ where $f(x) = \int 2\sec^2 5x \, dx$ passes through the point $\left(\frac{\pi}{3}, 0\right)$.

Show that $5y = 2\left(\sqrt{3} + \tan 5x\right)$.

Reverse chain rule is needed

$\int 2\sec^2 5x \, dx = 2 \times \frac{1}{5} \int 5\sec^2 5x \, dx$

'compensate' '1/5' '5' 'adjust'

$\therefore y = \frac{2}{5}\tan 5x + c$ "$\int \sec^2 x \, dx = \tan x + c$"

At $x = \frac{\pi}{3}$, $y = 0$, $0 = \frac{2}{5}\tan\frac{5\pi}{3} + c$ ($\tan\frac{5\pi}{3} = -\sqrt{3}$)

$c = \frac{2\sqrt{3}}{5}$

$\therefore y = \frac{2}{5}\tan 5x + \frac{2}{5}\sqrt{3}$

$y = \frac{2}{5}\left(\tan 5x + \sqrt{3}\right)$

$\therefore 5y = 2\left(\sqrt{3} + \tan 5x\right)$

www.thrivingscholars.com

# Integrating with Inverse Trigonometric Functions

**arcsin**, **arccos** and **arctan** are (one-to-one) functions defined as the inverse functions of sine, cosine and tangent respectively.

## What are the antiderivatives involving the inverse trigonometric functions?

- $\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + c$
- $\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + c$
- Note that the antiderivative involving $\arccos x$ would arise from

$$\int -\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arccos x + c$$

  - However, the negative can be treated as a coefficient of -1 and so

$$\int -\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\arcsin x + c$$

  - Similarly,

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\int -\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\arccos x + c$$

- Unless a question requires otherwise, stick to the first two results
- These are listed in the **formula booklet** the other way round as 'standard derivatives'
- For the antiderivative involving $\arctan x$, note that $(1+x^2)$ is the same as $(x^2+1)$

## How do I integrate these expressions if the denominator is not in the correct form?

- Some problems involve integrands that look very similar to the above but the denominators start with a number other than one
  - denominators of the form $a^2 \pm (bx)^2$ (with or without the square root!)
  - the integrand can be rewritten by taking a factor of $a^2$
    - this means the denominator will start with 1
    - e.g. $9+4x^2 = 9\left(1+\frac{4}{9}x^2\right) = 9\left(1+\left(\frac{2}{3}x\right)^2\right)$
    - $\frac{2}{3}x$ is a linear function of $x$, so use 'adjust and compensate'
  - Another type of problem has a quadratic denominator
    - denominators of the form $ax^2+bx+c$
      - a rearrangement of this is more likely but it is still quadratic
    - the integrand can be rewritten by completing the square
    - e.g. $5-x^2+4x = 5-(x^2+4x) = 5-[(x+2)^2-4] = 9-(x+2)^2$
    - This can then be dealt with like the first type of problem above

www.thrivingscholars.com

### Exam Tip

- Always start integrals involving the inverse trig functions by rewrtiting the denominator into the correct form
  - With or without the square root, the denominator should be of the form $1 \pm [g(x)]^2$
  - The numerator can be dealt with afterwards using 'adjust' and 'compensate' if necessary

www.thrivingscholars.com

## Worked Example

a) Find $\int \frac{1}{9+4x^2}\,\mathrm{d}x$.

Rewrite the integral so the denominator is in the form $1+(bx)^2$

$$I = \int \frac{1}{9+4x^2}\,dx = \int \frac{1}{9\left(1+\frac{4}{9}x^2\right)}\,dx$$

$$= \frac{1}{9}\int \frac{1}{1+\left(\frac{2}{3}x\right)^2}\,dx$$

'Adjust and compensate' for $\frac{2}{3}x$

$$I = \frac{1}{9}\times\frac{3}{2}\int \frac{\frac{2}{3}}{1+\left(\frac{2}{3}x\right)^2}\,dx$$

('adjust' $\frac{2}{3}$; 'compensate' $\frac{3}{2}$)

Now reverse chain rule applies

$$\therefore I = \frac{1}{6}\arctan\left(\frac{2}{3}x\right)+c$$

b) Find $\int \frac{1}{\sqrt{5-x^2+4x}}\,\mathrm{d}x$.

www.thrivingscholars.com



The denominator is quadratic so complete the square

$$I=\int \frac{1}{\sqrt{5-x^2+4x}}\,dx = \int \frac{1}{\sqrt{5-(x^2-4x)}}\,dx$$

$$= \int \frac{1}{\sqrt{5-[(x-2)^2-4]}}\,dx$$

$$= \int \frac{1}{\sqrt{9-(x-2)^2}}\,dx$$

Now rewrite into an integratable form

$$I=\int \frac{1}{\sqrt{9\left[1-\frac{(x-2)^2}{9}\right]}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{1-\left(\frac{x-2}{3}\right)^2}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-\left(\frac{1}{3}x-\frac{2}{3}\right)^2}}\,dx$$

$\left(\frac{1}{3}x-\frac{2}{3}\right)$ is a linear function of $x$ so 'adjust and compensate'

$$I=\frac{1}{3}\times 3\int \frac{\frac{1}{3}}{\sqrt{1-\left(\frac{1}{3}x-\frac{2}{3}\right)^2}}\,dx$$

'compensate' ↑ '$\frac{1}{3}$' ← 'adjust'

$$\therefore I=\arcsin\left(\tfrac{1}{3}x-\tfrac{2}{3}\right)+c$$

www.thrivingscholars.com

# Integrating Exponential & Logarithmic Functions

Exponential functions have the general form $y = a^x$. Special case: $y = \mathrm{e}^x$.

Logarithmic functions have the general form $y = \log_a x$. Special case: $y = \log_{\mathrm{e}} x = \ln x$.

## What are the antiderivatives of exponential and logarithmic functions?

- Those involving the special cases have been met before
  - $\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + c$
  - $\int \frac{1}{x} \, \mathrm{d}x = \ln |x| + c$
  - These are given in the **formula booklet**
- Also
  - $\int a^x \, \mathrm{d}x = \frac{1}{\ln a} a^x + c$
  - This is also given in the **formula booklet**
- By reverse chain rule
  - $\int \frac{1}{x \ln a} \, \mathrm{d}x = \log_a |x| + c$
  - This is **not** in the formula booklet
- There is also the reverse chain rule to look out for
  - this occurs when the numerator is (almost) the derivative of the denominator
  - $\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \ln | f(x) | + c$

## How do I integrate exponentials and logarithms with a *linear* function of x involved?

- In the case of linear functions of the form $ax + b$
  - $\int \mathrm{e}^{ax+b} \, \mathrm{d}x = \frac{1}{a} \mathrm{e}^{ax+b} + \mathrm{c}$
  - $\int \frac{1}{ax+b} \, \mathrm{d}x = \frac{1}{a} \ln | ax + b| + c$
  - These are **not** in the formula booklet but can be derived using 'adjust and compensate' from **reverse chain rule**

### Exam Tip

- Remember to always use the modulus signs for logarithmic terms in the antiderivative
  - Once it is deduced that $g(x)$ in $\ln |g(x)|$ (say) is guaranteed to be positive, the modulus signs can be replaced with brackets

www.thrivingscholars.com

## Worked Example

a) Find $\int 3^{2x+4}\,dx$.

Using reverse chain rule with 'adjust and compensate'

'adjust'

$$I = \int 3^{2x+4}\,dx = \frac{1}{2}\int 2 \times 3^{2x+4}\,dx$$

'compensate'

$$= \frac{1}{2}\left(\frac{1}{\ln 3} \times 3^{2x+4}\right) + c$$

$$\boxed{\therefore I = \frac{3^{2x+4}}{2\ln 3} + c}$$

b) Given $\frac{dy}{dx} = \frac{5}{1-7x}$ show that $y = -\frac{5}{7}\ln|1-7x| + c$, where $c$ is a constant.

$1-7x \equiv -7x+1$ so is of the form '$ax+b$'

Using reverse chain rule with 'adjust and compensate'

'adjust'

$$y = \int \frac{5}{1-7x}\,dx = 5 \times -\frac{1}{7}\int \frac{-7}{1-7x}\,dx$$

'compensate'

$$\boxed{\therefore y = -\frac{5}{7}\ln|1-7x| + c}$$

www.thrivingscholars.com





# Integration by Substitution

## What is integration by substitution?

- Integration by substitution is used when an integrand where reverse chain rule is either not obvious or is not spotted
  - in the latter case it is like a "back-up" method for reverse chain rule

## How do I use integration by substitution?

- For instances where the substitution is not obvious it will be given in a question
  - e.g. Find $\int \cot x \, \mathrm{d}x$ using the substitution $u = \sin x$
- Substitutions are usually of the form $u = g(x)$
  - in some cases $u^2 = g(x)$ and other variations are more convenient
    - as these would not be obvious, they would be given in a question
  - if need be, this can be rearranged to find $x$ in terms of $u$
- Integration by substitution then involves rewriting the integral, including "$\mathrm{d}x$" in terms of $u$

  STEP 1
  Name the integral to save rewriting it later
  Identify the given substitution $u = g(x)$

  STEP 2
  Find $\frac{\mathrm{d}u}{\mathrm{d}x}$ and rearrange into the form $f(u)\,\mathrm{d}u = g(x)\,\mathrm{d}x$ such that (some of) the integral can be rewritten in terms of $u$

  STEP 3
  If limits are involved, use $u = g(x)$ to change them from $x$ values to $u$ values

  STEP 4
  Rewrite the integral so everything is in terms of $u$ rather than $x$
  This is the step when it may become apparent that $x$ is needed in terms of $u$

  STEP 5
  Integrate with respect to u and either rewrite in terms of $x$ or apply the limits using their $u$ values

- For quotients the substitution usually involves the denominator
- It may be necessary to use 'adjust and compensate' to deal with any coefficients in the integrand
- Although $\frac{\mathrm{d}u}{\mathrm{d}x}$ can be treated like a fraction it should be appreciated that this is a 'shortcut' and the maths behind it is beyond the scope of the IB course

www.thrivingscholars.com

## Exam Tip

- If a substitution is not given in a question, it is usually because it is obvious
  - If you can't see anything obvious, or you find that your choice of substitution doesn't reduce the integrand to something easy to integrate, consider that it may not be a substitution question

## Worked Example

Use the substitution $u = (1 + 2x)$ to evaluate $\int_0^1 x(1+2x)^7 \, dx$.

STEP 1: Name the integral, identify the substitution

$I = \int_0^1 x(1+2x)^7 \, dx$

$u = 1 + 2x$

STEP 2: Find $\frac{du}{dx}$ and rearrange

$\frac{du}{dx} = 2$

$\frac{1}{2}du = dx$

STEP 3: Change limits from $x$ values to $u$ values

$x = 0, \quad u = 1 + 2(0) = 1$

$x = 1, \quad u = 1 + 2(1) = 3$

STEP 4: Rewrite the integral, find $x$ in terms of $u$

$I = \int_1^3 \frac{1}{2}(u-1)u^7 \times \frac{1}{2}du = \frac{1}{4}\int_1^3 (u^8 - u^7)\, du$

$x$ in terms of $u$: $u = 1 + 2x$, $\therefore x = \frac{1}{2}(u-1)$

STEP 5: Integrate and evaluate

$I = \frac{1}{4}\left[\frac{u^9}{9} - \frac{u^8}{8}\right]_1^3$

$I = \frac{1}{4}\left[\left(\frac{3^9}{9} - \frac{3^8}{8}\right) - \left(\frac{1^9}{9} - \frac{1^8}{8}\right)\right]$

$\therefore I = \frac{6151}{18}$

www.thrivingscholars.com

# Integration by Parts

## What is integration by parts?

- Integration by parts is generally used to integrate the product of two functions
  - however reverse chain rule and/or substitution should be considered first
    - e.g. $\int 2x\cos(x^2)\,\mathrm{d}x$ can be solved using reverse chain rule or the substitution $u = x^2$
  - Integration by parts is essentially 'reverse product rule'
    - whilst every product can be differentiated, not every product can be integrated (analytically)

## What is the formula for integration by parts?

- $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$
- This is given in the **formula booklet** alongside its alternative form $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$

## How do I use integration by parts?

- For a given integral $u$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$ (rather than $u$ and $v$) are assigned functions of $x$
- Generally, the function that becomes simpler when differentiated should be assigned to $u$
- There are various stages of integrating in this method
  - only one overall constant of integration ("+c") is required
  - put this in at the last stage of working
  - if it is a definite integral then "+c" is not required at all

STEP 1
Name the integral if it doesn't have one already!
This saves having to rewrite it several times – I is often used for this purpose.
e.g. $I = \int x\sin x\,\mathrm{d}x$

STEP 2
Assign $u$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$.
Differentiate $u$ to find $\frac{\mathrm{d}u}{\mathrm{d}x}$ and integrate $\frac{\mathrm{d}v}{\mathrm{d}x}$ to find $v$

e.g. $u = x$ $\quad v = -\cos x$
$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ $\quad \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$

STEP 3
Apply the integration by parts formula

www.thrivingscholars.com



e.g. $I = -x\cos x - \int -\cos x \, \mathrm{d}x$

STEP 4

Work out the second integral, $\int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

Now include a "+c" (unless definite integration)
e.g. $I = -x\cos x + \sin x + c$

STEP 5
Simplify the answer if possible or apply the limits for definite integration
e.g. $I = \sin x - x\cos x + c$

- In trickier problems other rules of differentiation and integration may be needed
  - chain, product or quotient rule
  - reverse chain rule, substitution

## Can integration by parts be used when there is only a single function?

- Some single functions (non-products) are awkward to integrate directly
  - e.g. $y = \ln x$, $y = \arcsin x$, $y = \arccos x$, $y = \arctan x$
- These can be integrated using parts however
  - rewrite as the product '$1 \times f(x)$' and choose $u = f(x)$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$
  - 1 is easy to integrate and the functions above have standard derivatives listed in the formula booklet

### Exam Tip

- If $\ln x$ or one of the inverse trig functions are one of the functions involved in the product then these should be assigned to "$u$" when applying parts
  - They are (realtively) easy to differentiate (to find $u'$) but are awkward to integrate

www.thrivingscholars.com

## Worked Example

a) Find $\int 5xe^{3x}\,dx$.

STEP 1: Name the integral

$$I = \int 5xe^{3x}\,dx = 5\int xe^{3x}\,dx$$

STEP 2: Assign $u$ and $v'$

Find $u'$ and $v$

$u = x$ $\quad v = \frac{1}{3}e^{3x}$ (reverse chain rule)

$u' = 1$ $\quad v' = e^{3x}$

$x$ becomes simpler when differentiated

STEP 3: Apply the integration by parts formula

$$I = 5\left[\frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}\,dx\right]$$

STEP 4: Work out the second integral

$$I = 5\left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c\right]$$

include "+c" at last working stage

reverse chain rule

STEP 5: Simplify

$$I = \frac{5}{9}e^{3x}(3x-1) + c$$

b) Show that $\int 8x\ln x\,dx = 2x^2(1+\ln x^2) + c$.

www.thrivingscholars.com



STEP 1: Name the integral

$$I = \int 8x \ln x \, dx$$

STEP 2: Assign $u$ and $v'$ – as $\ln$ is involved, $u = \ln x$

Find $u'$ and $v$

$u = \ln x$ $\quad$ $v = 4x^2$

$u' = \frac{1}{x}$ $\quad$ $v' = 8x$

STEP 3: Apply the integration by parts formula

$$I = 4x^2 \ln x - \int 4x^2 \times \frac{1}{x} \, dx = 4x^2 \ln x - \int 4x \, dx$$

STEP 4: Work out the second integral, include "+c" at this stage

$$I = 4x^2 \ln x - 2x^2 + c$$

STEP 5: Simplify

$$I = 2x^2(2 \ln x - 1) + c$$

$$\therefore I = 2x^2(\ln x^2 - 1) + c$$

www.thrivingscholars.com

# Repeated Integration by Parts

## When will I have to repeat integration by parts?

- In some problems, applying integration by parts still leaves the second integral as a product of two functions of $x$
  - integration by parts will need to be applied again to the second integral
- This occurs when one of the functions takes more than one derivative to become simple enough to make the second integral straightforward
  - These functions usually have the form $x^2 g(x)$

## How do I apply integration by parts more than once?

STEP 1
Name the integral if it doesn't have one already!

STEP 2
Assign $u$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$. Find $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $v$

STEP 3
Apply the integration by parts formula

STEP 4
Repeat STEPS 2 and 3 for the second integral

STEP 5
Work out the second integral and include a "+c" if necessary

STEP 6
Simplify the answer or apply limits

## What if neither function never becomes simpler when differentiating?

- It is possible that integration by parts will end up in a seemingly endless loop
  - consider the product $\mathrm{e}^x \sin x$
  - the derivative of $\mathrm{e}^x$ is $\mathrm{e}^x$
    - no matter how many times a function involving $\mathrm{e}^x$ is differentiated, it will still involve $\mathrm{e}^x$
  - the derivative of $\sin x$ is $\cos x$
    - $\cos x$ would then have derivative $-\sin x$, and so on
    - no matter how many times a function involving $\sin x$ or $\cos x$ is differentiated, it will still involve $\sin x$ or $\cos x$
- This loop can be trapped by spotting when the second integral becomes identical to (or a multiple of) the original integral
  - naming the original integral ($I$) at the start helps
  - $I$ then appears twice in integration by parts
    - e.g. $I = g(x) - I$
      where $g(x)$ are parts of the integral not requiring further work
  - It is then straightforward to rearrange and solve the problem

www.thrivingscholars.com

- e.g. $2I = g(x) + c$

  $I = \frac{1}{2} g(x) + c$

www.thrivingscholars.com

## Worked Example

a) Find $\int x^2 \cos x \, dx$.

STEP 1: Name the integral

$$I = \int x^2 \cos x \, dx$$

STEP 2: Assign $u$ and $v'$

Find $u'$ and $v$

$u = x^2$ $\quad v = \sin x$

$u' = 2x$ $\quad v' = \cos x$

$x^2$ becomes 'simpler' when differentiated

STEP 3: Apply the integration by parts formula

$$I = x^2 \sin x - 2\int x \sin x \, dx$$

STEP 4: Repeat STEPS 2 and 3 for the second integral

$u = x$ $\quad v = -\cos x$

$u' = 1$ $\quad v' = \sin x$

$$I = x^2 \sin x - 2\left[-x\cos x - \int -\cos x \, dx\right]$$

STEP 5: Work out the second integral now it is straightforward

$$I = x^2 \sin x + 2x\cos x - 2\sin x + c$$

STEP 6: Simplify

$$\boxed{I = (x^2 - 2)\sin x + 2x\cos x + c}$$

b) Find $\int e^x \sin x \, dx$.

www.thrivingscholars.com



STEP 1: Name the integral

$$I = \int e^x \sin x \, dx$$

STEP 2: Assign $u$ and $v'$. Neither function becomes simpler when differentiated. Find $u'$ and $v$.

$u = e^x$ $\quad$ $v = -\cos x$

$u' = e^x$ $\quad$ $v' = \sin x$

STEP 3: Apply the integration by parts formula

$$I = -e^x \cos x - \int -e^x \cos x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

STEP 4: Repeat STEPS 2 and 3 for the second integral

$u = e^x$ $\quad$ $v = \sin x$

$u' = e^x$ $\quad$ $v' = \cos x$

$$I = -e^x \cos x + \left[ e^x \sin x - \int e^x \sin x \, dx \right]$$

Spot that this is the same as the original question, i.e. $I$

STEP 5: "Work out" the second integral, include "+c" at this stage

$$I = e^x \sin x - e^x \cos x - I + c$$

STEP 6: Simplify

$$2I = e^x (\sin x - \cos x) + c$$

$$\therefore I = \tfrac{1}{2} e^x (\sin x - \cos x) + c_1 \quad (\text{where } c_1 = \tfrac{1}{2} c)$$

www.thrivingscholars.com





# Integrating with Partial Fractions

## What are partial fractions?

- **Partial fractions** arise when a quotient is rewritten as the **sum** of fractions
  - The process is the opposite of adding or subtracting fractions
- Each partial fraction has a denominator which is a **linear factor** of the quotient's denominator
  - e.g. A quotient with a denominator of $x^2 + 4x + 3$
    - factorises to $(x+1)(x+3)$
    - so the quotient will split into two partial fractions
    - one with the (linear) denominator $(x+1)$
    - one with the (linear) denominator $(x+3)$

## How do I know when to use partial fractions in integration?

- For this course, the denominators of the quotient will be of quadratic form
  - i.e. $f(x) = ax^2 + bx + c$
  - check to see if the quotient can be written in the form $\dfrac{f'(x)}{f(x)}$
    - in this case, reverse chain rule applies
- If the denominator does not factorise then the **inverse trigonometric functions** are involved

## How do I integrate using partial fractions?

STEP 1
Write the quotient in the integrand as the sum of partial fractions
This involves factorising the denominator, writing it as an identity of two partial fractions and using values of $x$ to find their numerators

e.g. $I = \displaystyle\int \frac{1}{x^2+4x+3}\,\mathrm{d}x = \int \frac{1}{(x+1)(x+3)}\,\mathrm{d}x = \frac{1}{2}\int \left(\frac{1}{x+1} - \frac{1}{x+3}\right)\mathrm{d}x$

STEP 2
Integrate each partial fraction leading to an expression involving the sum of natural logarithms

e.g. $I = \dfrac{1}{2}\displaystyle\int \left(\frac{1}{x+1} - \frac{1}{x+3}\right)\mathrm{d}x = \frac{1}{2}[\ln|x+1| - \ln|x+3|] + c$

STEP 3
Use the laws of logarithms to simplify the expression and/or apply the limits
(Simplifying first may make applying the limits easier)

e.g. $I = \dfrac{1}{2}\ln\left|\dfrac{x+1}{x+3}\right| + c$

- By rewriting the constant of integration as a logarithm ($c = \ln k$, say) it is possible to write the final answer as a single term

www.thrivingscholars.com



e.g. $I = \frac{1}{2}\ln\left|\frac{x+1}{x+3}\right| + \ln k = \ln\sqrt{\left|\frac{x+1}{x+3}\right|} + \ln k = \ln\left(k\sqrt{\left|\frac{x+1}{x+3}\right|}\right)$

## Exam Tip

- Always check to see if the numerator can be written as the derivative of the denominator
  - If so then it is reverse chain rule, not partial fractions
  - Use the number of marks a question is worth to help judge how much work should be involved

## Worked Example

Find $\int \frac{3x+1}{x^2+3x-10}\,\mathrm{d}x$.

The integrand is NOT of the form $\frac{f'(x)}{f(x)}$ but the denominator does factorise

STEP 1: Write the quotient as partial fractions

$$\frac{3x+1}{x^2+3x-10} \equiv \frac{A}{x+5} + \frac{B}{x-2}$$

$$3x+1 \equiv A(x-2) + B(x+5)$$

Let $x=2$, $7=7B$, $B=1$

Let $x=-5$, $-14=-7A$, $A=2$

$$\therefore I = \int \frac{3x+1}{x^2+3x-10}\,dx = \int \left(\frac{2}{x+5} + \frac{1}{x-2}\right) dx$$

STEP 2: Integrate the partial fractions

$$I = 2\ln|x+5| + \ln|x-2| + c$$

STEP 3: Simplify using laws of logarithms

$$I = \ln(x+5)^2 + \ln|x-2| + c$$

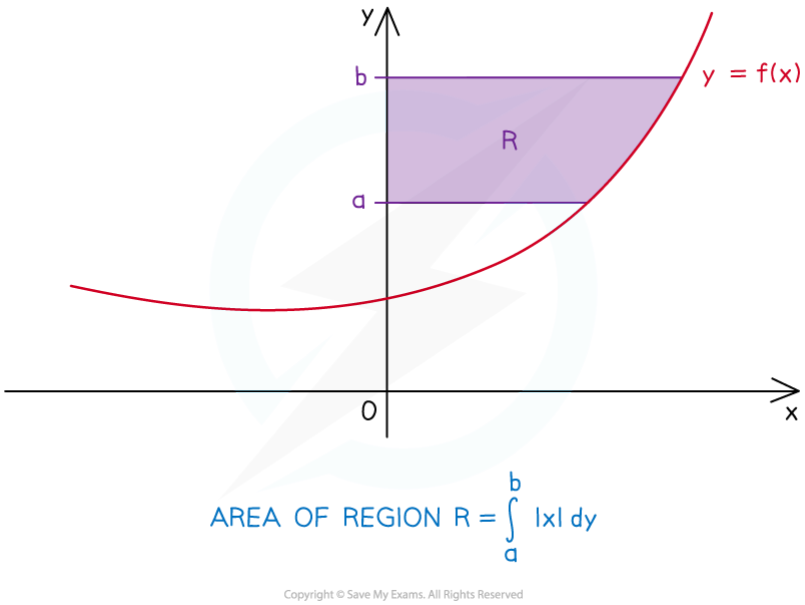
$$\therefore I = \ln\left|(x+5)^2(x-2)\right| + c$$

www.thrivingscholars.com



# Area Between Curve & y-axis

## What is meant by the area between a curve and the y-axis?




Copyright © Save My Exams. All Rights Reserved

- The area referred to is the region bounded by
  - the graph of $y = f(x)$
  - the $y$-axis
  - the horizontal line $y = a$
  - the horizontal line $y = b$
- The exact area can be found by evaluating a definite integral
- The graph of $y = f(x)$ could be a straight line
  - using basic shape area formulae may be easier than integration
    - e.g. area of a trapezium: $A = \frac{1}{2}h(a+b)$

## How do I find the area between a curve and the y-axis?

- Use the formula

$$A = \int_a^b |x| \, \mathrm{d}x$$

  - This is given in the **formula booklet**
  - The function is normally given in the form $y = f(x)$
    - so will need rearranging into the form $x = g(y)$
  - $a$ and $b$ may not be given directly as could involve the $x$-axis ($y = 0$) and/or a root of $x = g(y)$

www.thrivingscholars.com



- use a GDC to plot the curve, sketch it and highlight the area to help

STEP 1
Identify the limits $a$ and $b$
Sketch the graph of $y = f(x)$ or use a GDC to do so, especially if $a$ and $b$ are not given directly in the question

STEP 2
Rearrange $y = f(x)$ into the form $x = g(y)$
This is similar to finding the inverse function $f^{-1}(x)$

STEP 3
Evaluate the formula to evaluate the integral and find the area required
If using a GDC remember to include the modulus ( | … | ) symbols around $x$

- In trickier problems some (or all) of the area may be 'negative'
  - this will be any area that is left of the $y$-axis (negative $x$-values)
  - $|x|$ makes such areas 'positive'
    - a GDC will apply '$|x|$' automatically as long as the | … | are included
    - otherwise, to apply '$|x|$', split the integral into positive and negative parts; write an integral and evaluate each part separately and add the modulus of each part together to give the total area

### Exam Tip

- Longer problems may require you to rotate an area around both the $x$-axis and the $y$-aixs
- Sketch and/or use your GDC to help visualise what the problem looks like

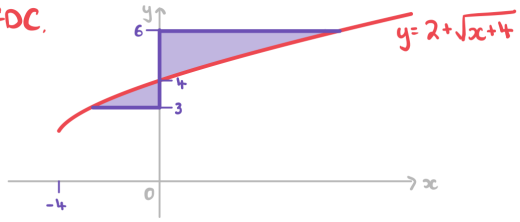
www.thrivingscholars.com

## Worked Example

Find the area enclosed by the curve with equation $y = 2 + \sqrt{x+4}$, the $y$-axis and the horizontal lines with equations $y = 3$ and $y = 6$.

STEP 1: Identify limits, sketch graph/use GDC

From GDC.




STEP 2: Rearrange $y = f(x)$ into $x = g(y)$

$y = 2 + \sqrt{x+4}$

$x = (y-2)^2 - 4 = y^2 - 4y + 4 - 4$

$x = y^2 - 4y$

STEP 3: Evaluate integral to find area

As some area 'is' negative, split the integral

$$A = -\int_3^4 (y^2 - 4y)\,dy + \int_4^6 (y^2 - 4y)\,dy$$

↑ From GDC/sketch this area is 'negative'

If using GDC you can still do this in one go:

$$\int_3^6 |y^2 - 4y|\,dy$$

$$\therefore A = \left[\frac{y^3}{3} - 2y^2\right]_4^6 - \left[\frac{y^3}{3} - 2y^2\right]_3^4$$

$$A = \left[(72-72) - \left(\frac{64}{3} - 32\right)\right] - \left[\left(\frac{64}{3} - 32\right) - (9-18)\right]$$

$$A = \frac{32}{3} - -\frac{5}{3}$$

$$\therefore A = \frac{37}{3} \text{ square units}$$

www.thrivingscholars.com

# Volumes of Revolution Around x-axis

## What is a volume of revolution around the x-axis?

- A **solid** of **revolution** is formed when an **area** bounded by a function $y = f(x)$ (and other boundary equations) is rotated $2\pi$ radians $(360°)$ around the $x$-axis
- The **volume** of **revolution** is the volume of this solid

- Be careful – the 'front' and 'back' of this solid are flat
  - they were created from straight (vertical) lines
  - 3D sketches can be misleading

## How do I solve problems involving the volume of revolution around x-axis?

- Use the formula

$$V = \pi \int_a^b y^2 \, \mathrm{d}x$$

- This is given in the **formula booklet**
- $y$ is a function of $x$
- $x = a$ and $x = b$ are the equations of the (vertical) lines bounding the area
  - If $x = a$ and $x = b$ are not stated in a question, the boundaries could involve the $y$-axis ( $x = 0$) and/or a root of $y = f(x)$
  - Use a GDC to plot the curve, sketch it and highlight the area to help
- Visualising the solid created is helpful
  - Try sketching some functions and their solids of revolution to help

STEP 1
Identify the limits $a$ and $b$
Sketching the graph of $y = f(x)$ or using a GDC to do so is helpful, especially when $a$ and $b$ are not given directly in the question

STEP 2
Square $y$

STEP 3
Use the formula to evaluate the integral and find the volume of revolution
An answer may be required in exact form

### Exam Tip

- If the given function involves a square root(s), problems can seem quite daunting
  - However, this is often deliberate, as the square root will be squared when applying the Volume of Revolution formula, and should leave the integrand as something more manageable
- Whether a diagram is given or not, using your GDC to plot the curve, limits, etc (where possible) can help you to visualise and make progress with problems

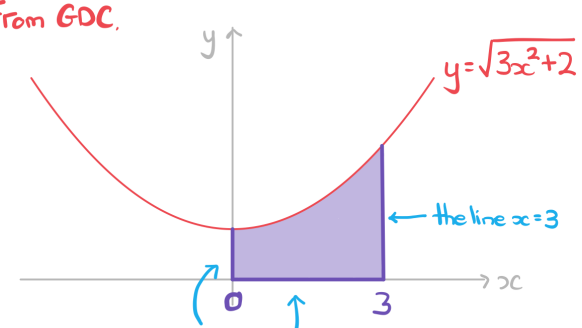
www.thrivingscholars.com

## Worked Example

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of $y = \sqrt{3x^2 + 2}$, the coordinate axes and the line $x = 3$ by $2\pi$ radians around the $x$-axis. Give your answer as an exact multiple of $\pi$.

STEP 1: Identify limits, sketch graph/use GDC

From GDC,




$\therefore a = 0, b = 3$

STEP 2: Square y

$y^2 = \left(\sqrt{3x^2+2}\right)^2 = 3x^2 + 2$

STEP 3: Find the volume

$$V = \pi \int_0^3 (3x^2 + 2)\,dx = \pi \left[x^3 + 2x\right]_0^3$$
$$= \pi (27 + 6)$$

$\therefore V = 33\pi$ cubic units

www.thrivingscholars.com

# Volumes of Revolution Around y-axis

## What is a volume of revolution around the y-axis?

- Very similar to above, this is a **solid** of **revolution** which is formed when an **area** bounded by a function $y = f(x)$ (and other boundary equations) is rotated $2\pi$ radians $(360°)$ around the $y$-axis
- The **volume** of **revolution** is the volume of this solid

## How do I solve problems involving the volume of revolution around y-axis?

- Use the formula

$$V = \pi \int_a^b x^2 \, dy$$

- This is given in the **formula booklet**
- The function is usually given in the form $y = f(x)$
  - so will need rearranging into the form $x = g(y)$
- $a$ and $b$ may not be given directly as could involve the $x$-axis ($y = 0$) and/or a root of $x = g(y)$
  - Use a GDC to plot the curve, sketch it and highlight the area to help
- Visualising the solid created is helpful

STEP 1
Identify the limits $a$ and $b$
Sketching the graph of $y = f(x)$ or using a GDC to do so is helpful, especially if $a$ and $b$ are not given directly in the question

STEP 2
Rearrange $y = f(x)$ into the form $x = g(y)$
This is similar to finding the inverse function $f^{-1}(x)$

STEP 3
Square $x$

STEP 4
Use the formula to evaluate the integral and find the volume of revolution
An answer may be required in exact form

### Exam Tip

- If the given function involves a square root, problems can seem quite daunting
  - This is often deliberate, as the square root will be squared when applying the Volume of Revolution formula and the integrand will then become more manageable
- Whether a diagram is given or not, using your GDC to plot the curve, limits, etc (where possible) can help you to visualise the problem and make progress

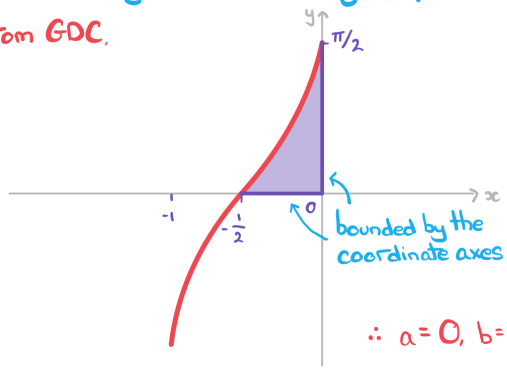
www.thrivingscholars.com

## Worked Example

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of $y = \arcsin(2x+1)$ and the coordinate axes by $2\pi$ radians around the $y$-axis. Give your answer to three significant figures.

STEP 1: Identify limits, sketch graph/use GDC

From GDC.




$\therefore a = 0,\ b = \pi/2$

STEP 2: Rearrange $y = f(x)$ into $x = g(y)$

$y = \arcsin(2x+1)$

$\sin y = 2x+1$

$x = \frac{1}{2}(\sin y - 1)$

STEP 3: Square $x$

$x^2 = \frac{1}{4}(\sin y - 1)^2$

STEP 4: Find the volume

$$V = \pi \int_0^{\pi/2} \frac{1}{4}(\sin y - 1)^2 \, dy$$

As this is awkward, use your GDC but

- your GDC will expect the integrand in terms of $x$
- remember $\pi$ !

$V = 0.279\,754\ldots$

$\therefore V = 0.280$ cubic units (3 s.f.)

www.thrivingscholars.com

## 5.9.5 Modelling with Volumes of Revolution

The volume of the solid of revolution formed by rotating an area through $2\pi$ radians around the $x$-axis is $V = \pi \int_a^b y^2 \, dx$, and for the $y$-axis is $V = \pi \int_a^b x^2 \, dy$. These are both given in the **formula booklet**.

www.thrivingscholars.com

# Adding & Subtracting Volumes

## When would volumes of revolution need to be added or subtracted?

- The 'curve' boundary of an area may consist of **more than one** function of $x$
  - For example
    - the 'curve' boundary from $x = 0$ to $x = 3$ is $y = f(x)$
    - the 'curve' boundary from $x = 3$ to $x = 6$ is $y = g(x)$
  - So the **total volume** would be $V = \pi \int_0^3 [f(x)]^2 \, dx + \pi \int_3^6 [g(x)]^2 \, dx$
- The solid of revolution may have a 'hole' in it
  - e.g. a 'toilet roll' shape would be the **difference** of two cylindrical volumes

## How do I know whether to add or subtract volumes of revolution?

- When the **area** to be **rotated** around the $x$-axis has more than one function defining its boundary it can be trickier to tell whether to **add** or **subtract volumes** of **revolution**
  - It will depend on the **nature** of the **functions** and their **points** of **intersection**
  - With help from a GDC, sketch the graph of the functions and highlight the area required

## How do I solve problems involving adding or subtracting volumes of revolution?

- Visualising the solid created becomes increasingly useful (but also trickier) for shapes generated by separate volumes of revolution
  - Continue trying to sketch the functions and their solids of revolution to help

**STEP 1**

Identify the functions $(y = f(x), \, y = g(x), \, ...)$ involved in generating the volume
Determine whether the separate volumes will need to be added or subtracted
Identify the limits for each volume involved
Sketching the graphs of $y = f(x)$ and $y = g(x)$, or using a GDC to do so, is helpful, especially when the limits are not given directly in the question

**STEP 2**

Square $y$ for all functions $([f(x)]^2, \, [g(x)]^2, \, ...)$
This step is not essential if a GDC can be used to calculate integrals and an exact answer is not required.

**STEP 3**

Use the appropriate volume of revolution formula for each part, evaluate the definite integral and add or subtract as necessary
The answer may be required in exact form

### Exam Tip

- A sketch of the graph, limits, etc is always helpful, whether one has been given in the question or not
  - Use your GDC where possible

www.thrivingscholars.com

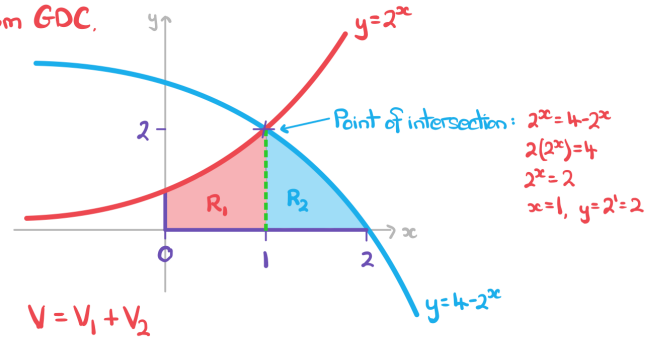
## Worked Example

Find the volume of revolution of the solid formed by rotating the region enclosed by the positive coordinate axes and the graphs of $y = 2^x$ and $y = 4 - 2^x$ by $2\pi$ radians around the $x$-axis. Give your answer to three significant figures.

STEP 1: Identify functions, limits and whether to add or subtract

Use GDC to help sketch the graphs

From GDC,




$V = V_1 + V_2$

For $R_1$, $a=0$, $b=1$

For $R_2$, $a=1$, $b=2$

STEP 2: Square all functions - this step is not required in this question

STEP 3: Use formula for each part, evaluate and add

$$V = \pi\int_0^1 (2^x)^2\,dx + \pi\int_1^2 (4-2^x)^2\,dx$$

Use your GDC to evaluate - to avoid typing errors evaluate each integral separately, store in memory, then add

$V = 6.798\,540\ldots + 4.941\,881\ldots = 11.740\ldots$

$\therefore V = 11.7$ cubic units (3 s.f.)

www.thrivingscholars.com

# Modelling with Volumes of Revolution

## What is meant by modelling volumes of revolution?

- Many everyday objects such as buckets, beakers, vases and lamp shades can be modelled as a **solid of revolution**
- The volume of revolution of the solid can then be calculated
- An object that would usually stand **upright** can be **modelled horizontally** so its **volume of revolution** can be found

## What modelling assumptions are there with volumes of revolution?

- The solids formed are usually the main shape of the body of the object
  - For example, the handle on a bucket would not be included
- The thickness of the solid is negligible relative to the size of the object
  - thickness will depend on the purpose of the object and the material it is made from

## How do I solve modelling problems with volumes of revolution?

- Visualising and sketching the solid formed can help with starting problems
- Familiarity with applying the volume of revolution fomulae
  - around the x-axis: $V = \int_a^b y^2 \, \mathrm{d}x$
  - around the y-axis: $V = \int_a^b x^2 \, \mathrm{d}y$
- The volume of revolution may involve adding or subtracting partial volumes
- Questions may ask related questions in context
  - g. A question about a bucket may ask about its **capacity**
    - this would be measured in litres
    - so a conversion of units may be required
    - ($100\ \mathrm{cm}^3 = 1$ litre)

www.thrivingscholars.com

### Exam Tip

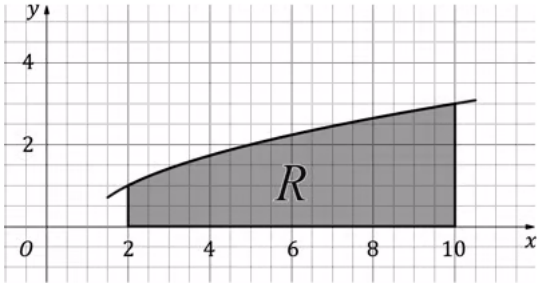
- Remember to answer questions directly
  - In modelling scenarios, interpretation is often needed after finding the 'final answer'
- Modelling questions often ask about assumptions, criticisms and/or improvements
- Examples
  - it is assumed the thickness of the material an object is made from is negligible
  - a 'smooth' curve may not be a good model if the item is being made from a rough material
  - other things may significantly reduce the volume found and impact conclusions
    - e.g. Stones, plants and decorations placed in an aquarium will reduce the volume of water needed to fill it - and hence the number/size/type of fish it can accommodate may be impacted

www.thrivingscholars.com

## Worked Example

The diagram below shows the region *R*, which is bounded by the function $y = \sqrt{x-1}$, the lines $x = 2$ and $x = 10$, and the $x$-axis.

Dimensions are in centimetres.




A mathematical model for a miniature vase is produced by rotating the region *R* through 2π radians around the x-axis.

Find the volume of the miniature vase, giving your answer in litres to three significant figures.

www.thrivingscholars.com



STEP 1 Identify limits

$a = 2$

$b = 10$

STEP 2 Square $y$

$y^2 = \left(\sqrt{x-1}\right)^2 = x - 1$

STEP 3 Evaluate the integral

$$V = \pi \int_2^{10} (x-1)\,dx = \pi \left[0.5x^2 - x\right]_2^{10}$$

$$= \pi \left[(50-10)-(2-2)\right]$$

$$= 40\pi$$

Now we need to interpret this in the context of the miniature vase

$V = 40\pi \text{ cm}^3$

$V = \frac{40\pi}{1000}$ litres $\qquad 1000\text{cm}^3 = 1 \text{ litre}$

$V = 0.125\,663\ldots$

Volume of the miniature vase is 0.126 litres (3 s.f.)

www.thrivingscholars.com

# 5.4 Further Integration

## 5.4.1 Integrating Special Functions

### Integrating Trig Functions

### How do I integrate sin, cos and sec^2?

- The **antiderivatives** for **sine** and **cosine** are

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

where $c$ is the **constant** of **integration**

- Also, from the **derivative** of $\tan x$

$$\int \sec^2 x \, dx = \tan x + c$$

- The **derivatives** of $\sin x$, $\cos x$ and $\tan x$ are in the **formula booklet**
  - so these **antiderivatives** can be easily deduced
- For the **linear** function $ax + b$, where $a$ and $b$ are constants,

$$\int \sin (ax + b) \, dx = -\frac{1}{a} \cos (ax + b) + c$$

$$\int \cos (ax + b) \, dx = \frac{1}{a} \sin (ax + b) + c$$

$$\int \sec^2 (ax + b) \, dx = \frac{1}{a} \tan (ax + b) + c$$

- For **calculus** with **trigonometric** functions **angles must** be **measured** in **radians**
  - Ensure you know how to change the angle mode on your GDC

**Exam Tip**

- The formula booklet can be used to find antiderivatives from the derivatives
  - Make sure you have the page with the section of standard derivatives open
  - Use these backwards to find any antiderivatives you need
  - Remember to add 'c', the constant of integration, for any indefinite integrals

www.thrivingscholars.com

## Worked Example

a)

Find, in the form $F(x) + c$, an expression for each integral

i. $\int \cos x \, dx$

ii. $\int \sec^2\left(3x - \frac{\pi}{3}\right) dx$

i. $\int \cos x \, dx = \sin x + c$

ii. $\int \sec^2\left(3x - \frac{\pi}{3}\right) dx = \frac{1}{3}\tan\left(3x - \frac{\pi}{3}\right) + c$

(Linear function $ax+b$)

b) A curve has equation $y = \int 2\sin\left(2x + \frac{\pi}{6}\right) dx$.

The curve passes through the point with coordinates $\left(\frac{\pi}{3}, \sqrt{3}\right)$.

Find an expression for $y$.

$y = 2\int \sin\left(2x + \frac{\pi}{6}\right) dx$

$y = 2\left[-\frac{1}{2}\cos\left(2x + \frac{\pi}{6}\right)\right] + c$

At $x = \frac{\pi}{3}$, $y = \sqrt{3}$, $\sqrt{3} = -\cos\left(\frac{2\pi}{3} + \frac{\pi}{6}\right) + c$

$c = \cos\left(\frac{5\pi}{6}\right) + \sqrt{3}$

$c = \frac{\sqrt{3}}{2}$

$\therefore y = \frac{\sqrt{3}}{2} - \cos\left(2x + \frac{\pi}{6}\right)$

www.thrivingscholars.com

# Integrating e^x & 1/x

## How do I integrate exponentials and logarithms?

- The **antiderivatives** involving $e^x$ and $\ln x$ are

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

where $c$ is the **constant** of **integration**

  - These are given in the **formula booklet**
- For the **linear** function $(ax + b)$, where $a$ and $b$ are constants,

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c$$

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln|ax+b| + c$$

- It follows from the last result that

$$\int \frac{a}{ax+b} \, dx = \ln|ax+b| + c$$

  - which can be deduced using **Reverse Chain Rule**
- With **ln,** it can be useful to write the constant of integration, $c$, as a logarithm
  - using the laws of logarithms, the answer can be written as a single term
  - $\int \frac{1}{x} \, dx = \ln|x| + \ln k = \ln k|x|$ where $k$ is a constant
  - This is similar to the special case of **differentiating** $\ln(ax + b)$ when $b = 0$

### Exam Tip

- Make sure you have a copy of the formula booklet during revision but don't try to remember everything in the formula booklet
  - However, do be familiar with the **layout** of the formula booklet
    - You'll be able to quickly locate whatever you are after
    - You do not want to be searching every line of every page!
  - For formulae you think you have remembered, use the booklet to double-check

www.thrivingscholars.com

## Worked Example

A curve has the gradient function $f'(x) = \frac{3}{3x+2} + e^{4-x}$.

Given the exact value of $f(1)$ is $\ln 10 - e^3$ find an expression for $f(x)$.

$f(x) = \int \left(\frac{3}{3x+2} + e^{4-x}\right) dx$

$f(x) = 3\int \frac{1}{3x+2}\, dx + \int e^{4-x}\, dx$

$= 3\left[\frac{1}{3}\ln|3x+2|\right] - e^{4-x} + c$

$f(1) = \ln 10 - e^3$, $\ln|3(1)+2| - e^{4-1} + c = \ln 10 - e^3$

$\therefore c = \ln 10 - \ln 5$

$c = \ln\left(\frac{10}{5}\right) = \ln 2$

$\therefore f(x) = \ln|3x+2| - e^{4-x} + \ln 2$

$\left(= \ln 2|3x+2| - e^{4-x}\right)$

www.thrivingscholars.com



# 5.5 Optimisation

## 5.5.1 Modelling with Differentiation

### Modelling with Differentiation

### What can be modelled with differentiation?

- Recall that **differentiation** is about the **rate** of **change** of a function and provides a way of finding **minimum** and **maximum** values of a function
- Anything that involves **maximising** or **minimising** a quantity can be modelled using differentiation; for example
  - minimising the cost of raw materials in manufacturing a product
  - the maximum height a football could reach when kicked
- These are called **optimisation** problems

### What modelling assumptions are used in optimisation problems?

- The quantity being optimised needs to be dependent on a **single** variable
  - If other variables are initially involved, **constraints** or **assumptions** about them will need to be made; for example
    - minimising the cost of the **main** raw material – timber in manufacturing furniture say
    - the cost of screws, glue, varnish, etc can be fixed or considered **negligible**
- Other **modelling assumptions** may have to be made too; for example
  - ignoring air resistance and wind when modelling the path of a kicked football

### How do I solve optimisation problems?

- In **optimisation** problems, letters other than ***x***, ***y*** and ***f*** are often used including capital letters
  - ***V*** is often used for volume, ***S*** for surface area
  - ***r*** for radius if a circle, cylinder or sphere is involved
- **Derivatives** can still be found but be clear about which letter is representing the independent ($x$) variable and which letter is representing the dependent ($y$) variable
  - A GDC may always use $x$ and $y$ but ensure you use the correct letters throughout your working and final answer
- Problems often start by **linking** two connected quantities together – for example **volume** and **surface area**
  - Where more than one variable is involved, **constraints** will be given such that the quantity of interest can be rewritten in terms of **one** variable
- Once the quantity of interest is written as a function of a single variable, **differentiation** can be used to **maximise** or **minimise** the quantity as required

**STEP 1**
Rewrite the quantity to be optimised as a single variable, using any constraints given in the question

**STEP 2**

www.thrivingscholars.com

Differentiate and solve the derivative equal to zero to find the "$x$"-coordinate(s) of any stationary points

**STEP 3**
If there is more than one stationary point, or the requirement to justify the nature of the stationary point, differentiate again

**STEP 4**
Use the second derivative to determine the nature of each stationary point and select the maximum or minimum point as necessary

**STEP 5**
Interpret the answer in the context of the question

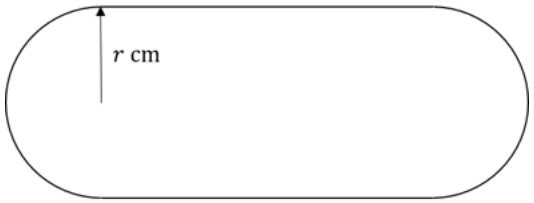
### Exam Tip

- The first part of rewriting a quantity as a single variable is often a "show that" question – this means you may still be able to access later parts of the question even if you can't do this bit
- Even when an algebraic solution is required you can still use your GDC to check answers and help you get an idea of what you are aiming for

www.thrivingscholars.com

## Worked Example

A large allotment bed is being designed as a rectangle with a semicircle on each end, as shown in the diagram below.




The total area of the bed is to be $100\pi \text{ m}^2$.

a)
Show that the perimeter of the bed is given by the formula

$$P = \pi\left(r + \frac{100}{r}\right)$$

www.thrivingscholars.com



**STEP 1:** The width of the rectangle is $2r$ m and its length $L$ m
The AREA of the bed, $100\pi$ m$^2$ is given by

$$\underbrace{\tfrac{1}{2}\pi r^2}_{\text{semi-circle}} + \underbrace{2rL}_{\text{rectangle}} + \underbrace{\tfrac{1}{2}\pi r^2}_{\text{semi-circle}} = \underbrace{100\pi}_{\text{total area (constraint)}}$$

$$\therefore \pi r^2 + 2rL = 100\pi$$

$$2rL = 100\pi - \pi r^2 \qquad \text{Write } L \text{ in terms of } r$$

$$L = \frac{50\pi}{r} - \frac{\pi}{2}r$$

The PERIMETER of the bed is

$$P = \underbrace{\pi r + \pi r}_{\text{semi-circular arcs}} + \underbrace{2L}_{\text{two straight lengths}}$$

Use $L$ from the area constraint to write $P$ in terms of $r$ only

$$P = 2\pi r + 2\left(\frac{50\pi}{r} - \frac{\pi}{2}r\right)$$

$$P = \pi r + \frac{100\pi}{r}$$

$$\therefore P = \pi\left(r + \frac{100}{r}\right)$$

b) Find $\dfrac{dP}{dr}$.

www.thrivingscholars.com



STEP 1: Rewrite P as powers of r

$$P = \pi(r + 100r^{-1})$$

STEP 2: $$\frac{dP}{dr} = \pi\left(1 - 100r^{-2}\right)$$

$$\therefore \frac{dP}{dr} = \pi\left(1 - \frac{100}{r^2}\right)$$

c)

Find the value of $r$ that minimises the perimeter.

STEP 2: $$\pi\left(1 - \frac{100}{r^2}\right) = 0$$

$$r^2 - 100 = 0$$

$$r = 10 \qquad \text{(reject -10 as r is a length)}$$

This is the only stationary point so we can assume it is minimal.

$\therefore r = 10$ m minimises the perimeter

d)

Hence find the minimum perimeter.

STEP 5: Interpret answer in context

Minimum perimeter is when $r = 10$

$$\therefore P = \pi\left(10 + \frac{100}{10}\right) = 20\pi$$

Minimum perimeter is $20\pi$ m

Use your GDC to check

www.thrivingscholars.com



e)
Justify that this is the minimum perimeter.

STEP 4: Use second derivative

$$\frac{d^2P}{dr^2} = \pi\left(200r^{-3}\right)$$

at $r = 10$, $\frac{d^2P}{dr^2} = \frac{\pi}{5} > 0$ $\therefore$ minimum

$\therefore 20\pi$ is the minimum perimeter

www.thrivingscholars.com





# Finding the Constant of Integration

## What is the constant of integration?

- When finding an **anti-derivative** there is a constant term to consider
  - this constant term, usually called $c$, is the **constant** of **integration**
- In terms of **graphing** an **anti-derivative**, there are endless possibilities
  - collectively these may be referred to as the **family** of **antiderivatives** or **family** of **curves**
  - the constant of integration is determined by the **exact** location of the curve
    - if a **point** on the **curve** is **known,** the **constant** of **integration** can be found

## How do I find the constant of integration?

- For $\mathrm{F}(x) + c = \int f(x)\,\mathrm{d}x$, the **constant** of **integration,** $c$ – and so the particular **antiderivative** - can be found if a point the graph of $y = \mathrm{F}(x) + c$ passes through is known

**STEP 1**

If need be, rewrite $f(x)$ into an integrable form

Each term needs to be a power of $x$ (or a constant)

**STEP 2**

Integrate each term of $f'(x)$, remembering the constant of integration, "$+c$"

(Increase power by 1 and divide by new power)

**STEP 3**

Substitute the $x$ and $y$ coordinates of a given point in to $\mathrm{F}(x) + c$ to form an equation in $c$

Solve the equation to find $c$

### Exam Tip

- If a constant of integration can be found then the question will need to give you some extra information
  - If this is given then make sure you use it to find the value of c

www.thrivingscholars.com

## Worked Example

The graph of $y = f(x)$ passes through the point $(3, -4)$. The gradient function of $f(x)$ is given by $f'(x) = 3x^2 - 4x - 4$.

Find $f(x)$.

STEP 1 f'(x) is already in an integrable form

$f'(x) = 3x^2 - 4x - 4$

STEP 2 Integrate, remembering "+c"

$f(x) = \frac{3x^3}{3} - \frac{4x^2}{2} - 4x + c$

$f(x) = x^3 - 2x^2 - 4x + c$

STEP 3 Substitute x and y coordinates to find c

$f(3) = -4$

$\therefore (3)^3 - 2(3)^2 - 4(3) + c = -4$

$27 - 18 - 12 + c = -4$

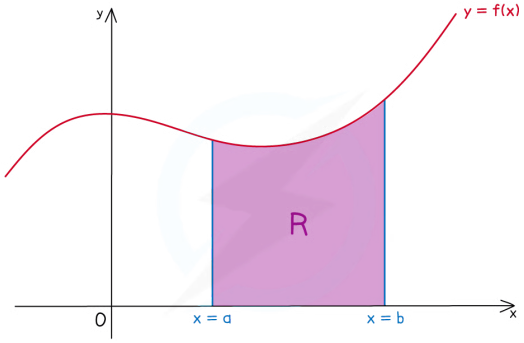
$c = -1$

$\therefore f(x) = x^3 - 2x^2 - 4x - 1$

www.thrivingscholars.com

# Area Under a Curve Basics

## What is meant by the area under a curve?




R IS THE AREA UNDER THE CURVE y = f(x)

Copyright © Save My Exams. All Rights Reserved

- The phrase "**area under a curve**" refers to the area bounded by
  - the graph of $y = f(x)$
  - the $x$-axis
  - the **vertical** line $x = a$
  - the **vertical** line $x = b$
- The **exact area under a curve** is found by evaluating a **definite integral**
- The graph of $y = f(x)$ could be a **straight line**
  - the use of **integration** described below would still apply
    - but the shape created would be a **trapezoid**
    - so it is easier to use "$A = \frac{1}{2}h(a+b)$"

## What is a definite integral?

$$\int_a^b f(x)\,\mathrm{d}x = \mathrm{F}(b) - \mathrm{F}(a)$$

- This is known as the **Fundamental Theorem of Calculus**
- **a** and **b** are called limits
  - **a** is the **lower** limit
  - **b** is the **upper** limit
- $f(x)$ is the **integrand**
- $\mathrm{F}(x)$ is an **antiderivative** of $f(x)$
- The **constant** of **integration** ("$+c$") is not needed in **definite integration**
  - "$+c$" would appear alongside both **F(a)** and **F(b)**
  - subtracting means the "$+c$"'s cancel

## How do I form a definite integral to find the area under a curve?

- The graph of $y = f(x)$ and the $x$-axis should be obvious boundaries for the area so the key here is in finding $a$ and $b$ - the **lower** and **upper** limits of the **integral**

**STEP 1**

www.thrivingscholars.com

Use the given sketch to help locate the limits
You may prefer to plot the graph on your GDC and find the limits from there

**STEP 2**

Look carefully where the 'left' and 'right' boundaries of the area lie
If the boundaries are vertical lines, the limits will come directly from their equations
Look out for the $y$-axis being one of the (vertical) boudnaries – in this case the limit ($x$) will be 0
One, or both, of the limits, could be a root of the equation $f(x) = 0$
i.e. where the graph of $y = f(x)$ crosses the $x$-axis
In this case solve the equation $f(x) = 0$ to find the limit(s)
A GDC will solve this equation, either from the graphing screen or the equation solver

**STEP 3**

The definite integral for finding the area can now be set up in the form

$$A = \int_a^b f(x)\,\mathrm{d}x$$

## Exam Tip

- Look out for questions that ask you to find an **indefinite** integral in one part (so "**+c**" needed), then in a later part use the same integral as a **definite** integral (where "**+c**" is not needed)
- Add information to any diagram provided in the question, as well as axes intercepts and values of limits
  - Mark and shade the area you're trying to find, and if no diagram is provided, **sketch** one!

www.thrivingscholars.com

# Definite Integrals using GDC

## Does my calculator/GDC do definite integrals?

- Modern graphic calculators (and some 'advanced' scientific calculators) have the functionality to evaluate **definite integrals**
  - i.e. they can calculate the **area under a curve** (see above)
- If a calculator has a button for evalutaing definite integrals it will look something like

$$\int_{\square}^{\square} \square$$

- This may be a physical button or accessed via an on-screen menu
- Some GDCs may have the ability to find the area under a curve from the graphing screen
- Be careful with **any** calculator/GDC, they may not produce an **exact** answer

## How do I use my GDC to find definite integrals?

**Without graphing first …**

- Once you know the **definite integral** function your calculator will need three things in order to evaluate it
  - The function to be integrated (**integrand**) ( $f(x)$)
  - The **lower** limit ($a$ from $x = a$)
  - The **upper** limit ($b$ from $x = b$)
- Have a play with the order in which your calculator expects these to be entered – some do not always work left to right as it appears on screen!

**With graphing first …**

- Plot the graph of $y = f(x)$
  - You may also wish to plot the vertical lines $x = a$ and $x = b$
    - make sure your GDC is expecting an "$x =$ " style equation
  - Once you have plotted the graph you need to look for an option regarding "area" or a physical button
    - it may appear as the integral symbol (e.g. $\int \mathrm{d}x$)
    - your GDC may allow you to select the lower and upper limits by moving a cursor along the curve - however this may not be very accurate
    - your GDC may allow you to type the exact limits required from the keypad
      - the lower limit would be typed in first
      - read any information that appears on screen carefully to make sure

### Exam Tip

- When revising for your exams always use your GDC to check any definite integrals you have carried out by hand
  - This will ensure you are confident using the calculator you plan to take into the exam and should also get you into the habit of using you GDC to check your work, something you should do if possible

www.thrivingscholars.com

## Worked Example

a)

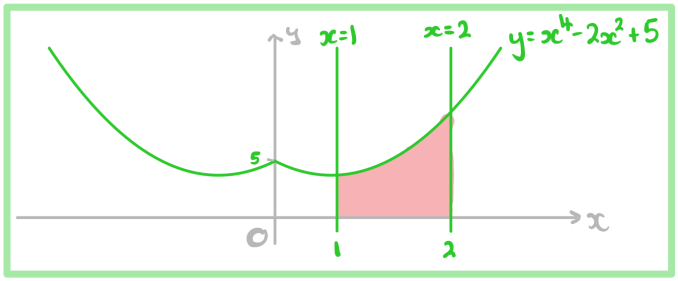
Using your GDC to help, or otherwise, sketch the graphs of

$y = x^4 - 2x^2 + 5$,

$x = 1$ and

$x = 2$ on the same diagram

Use the 'graph' menu on your GDC to plot $y = x^4 - 2x^2 + 5$.
You may then need to change the 'input type' to 'x='
to enter $x = 1$ and $x = 2$.
Plot the graph on your GDC and sketch the result, ensuring
to include all the main properties of each graph.




b)

The area enclosed by the three graphs from part (a) and the $x$-axis is to be found.
Write down an integral that would find this area.

$$\int_1^2 (x^4 - 2x^2 + 5)\, dx$$

c)

Using your GDC, or otherwise, find the exact area described in part (b).

Give your answer in the form $\frac{a}{b}$ where $a$ and $b$ are integers.

$$\text{Area} = \int_1^2 (x^4 - 2x^2 + 5)\, dx = \frac{98}{15} \text{ square units}$$

From the graphing screen on our GDC the integral
value was given as 6·53333333 - not exact!

www.thrivingscholars.com



## 5.4 Further Integration

### 5.4.2 Techniques of Integration

# Integrating Composite Functions (ax+b)

## What is a composite function?

- A **composite function** involves one function being applied after another
- A composite function may be described as a "function of a function"
- This Revision Note focuses on one of the functions being **linear** – i.e. of the form $\boldsymbol{ax+b}$

## How do I integrate linear (*ax+b*) functions?

- A **linear function** (of $x$) is of the form $ax+b$
- The special cases for **trigonometric functions** and **exponential** and **logarithm functions** are
  - $\int \sin(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax+b)+c$
  - $\int \cos(ax+b)\,\mathrm{d}x = \frac{1}{a}\sin(ax+b)+c$
  - $\int \mathrm{e}^{ax+b}\,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax+b}+c$
  - $\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln\lvert ax+b\rvert+c$
- There is one more special case
  - $\int (ax+b)^n\,\mathrm{d}x = \frac{1}{a(n+1)}(ax+b)^{n+1}+c$ where $n\in\mathbb{Q},\ n\neq -1$
- $c$, in all cases, is the **constant** of **integration**
- All the above can be deduced using **reverse chain rule**
  - However, spotting them can make solutions more efficient

### Exam Tip

- Although the specific formulae in this revision note are NOT in the **formula booklet**
  - almost all of the information you will need to apply reverse chain rule is provided
  - make sure you have the formula booklet open at the right page(s) and practice using it

www.thrivingscholars.com

## Worked Example

Find the following integrals

a) $\int 3(7-2x)^{\frac{5}{3}}\,dx$

$I = \int 3(7-2x)^{5/3}\,dx = 3\int (-2x+7)^{5/3}\,dx$

Using $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$,

$I = 3\left[\frac{1}{-2 \times \frac{8}{3}}(-2x+7)^{8/3}\right] + c$ ($\frac{5}{3}+1 = \frac{8}{3}$)

$\therefore I = -\frac{9}{16}(7-2x)^{8/3} + c$

b) $\int \frac{1}{2}\cos(3x-2)\,dx$

$I = \int \frac{1}{2}\cos(3x-2)\,dx = \frac{1}{2}\int \cos(3x-2)\,dx$

Using $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$

$I = \frac{1}{2}\left[\frac{1}{3}\sin(3x-2)\right] + c$

$\therefore I = \frac{1}{6}\sin(3x-2)+c$

www.thrivingscholars.com

# Reverse Chain Rule

## What is reverse chain rule?

- The **Chain Rule** is a way of differentiating two (or more) functions
- **Reverse Chain Rule** (RCR) refers to **integrating** by **inspection**
  - spotting that chain rule would be used in the reverse (differentiating) process

## How do I know when to use reverse chain rule?

- **Reverse chain rule** is used when we have the **product** of a **composite function** and the **derivative** of its **second function**
- Integration is trickier than differentiation; many of the shortcuts do not work
  - For example, in general $\int e^{f(x)}\,\mathrm{d}x \neq \frac{1}{f'(x)}e^{f(x)}$
  - However, this result **is true** if $f(x)$ is linear $(ax+b)$
- Formally, in **function notation**, **reverse chain rule** is used for **integrands** of the form

$$I = \int g'(x) f\big(g(x)\big)\,\mathrm{d}x$$

  - this does not have to be strictly true, but '*algebraically*' it should be
    - if **coefficients** do not match '**adjust** and **compensate'** can be used
    - e.g. $5x^2$ is not quite the derivative of $4x^3$
      - the *algebraic* part $(x^2)$ is '*correct*'
      - but the coefficient 5 is '*wrong*'
      - use '**adjust** and **compensate'** to '*correct*' it
- A particularly useful instance of reverse chain rule to recognise is

$$I = \int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + c$$

  - i.e. the **numerator** is (almost) the **derivative** of the **denominator**
  - '**adjust** and **compensate**' may need to be used to deal with any coefficients
    - e.g.

$$I = \int \frac{x^2+1}{x^3+3x}\,\mathrm{d}x = \frac{1}{3}\int 3\frac{x^2+1}{x^3+3x}\,\mathrm{d}x = \frac{1}{3}\int \frac{3x^2+3}{x^3+3x}\,\mathrm{d}x = \frac{1}{3}\ln|x^3+3x| + c$$

## How do I integrate using reverse chain rule?

- If the product **can** be identified, the **integration** can be done "by **inspection**"
  - there may be some "**adjusting** and **compensating**" to do
- Notice a lot of the "**adjust** and **compensate** method" happens mentally
  - this is indicated in the steps below by quote marks

**STEP 1**

Spot the 'main' function

e.g. $I = \int x(5x^2-2)^6\,\mathrm{d}x$

"the main function is $(\ldots)^6$ which would come from $(\ldots)^7$"

**STEP 2**

www.thrivingscholars.com



'**Adjust'** and '**compensate'** any coefficients required in the integral
e.g. " $(\ldots)^7$ would differentiate to $7(\ldots)^6$"
"chain rule says multiply by the derivative of $5x^2-2$, which is $10x$"
"there is no '7' or '10' in the integrand so adjust and compensate"

$$I=\frac{1}{7}\times\frac{1}{10}\times\int 7\times 10\times x(5x^2-2)^6\,\mathrm{d}x$$

**STEP 3**
**Integrate** and simplify
e.g. $I=\frac{1}{7}\times\frac{1}{10}\times(5x^2-2)^7+c$

$$I=\frac{1}{70}(5x^2-2)^7+c$$

- Differentiation can be used as a means of checking the final answer
- After some practice, you may find Step 2 is not needed
  - Do use it on more awkward questions (negatives and fractions!)
- If the product **cannot** easily be identified, use **substitution**



## Exam Tip

- Before the exam, practice this until you are confident with the pattern and do not need to worry about the formula or steps anymore
  - This will save time in the exam
- You can always check your work by differentiating, if you have time

www.thrivingscholars.com

## Worked Example

A curve has the gradient function $f'(x) = 5x^2\sin(2x^3)$.

Given that the curve passes through the point $(0, 1)$, find an expression for $f(x)$.

$f(x) = \int 5x^2 \sin(2x^3)\, dx$

$f(x) = 5\int x^2 \sin(2x^3)\, dx$ Take 5 out as a factor

This is a product, almost in the form $g'(x)f(g(x))$

STEP 1: Spot the 'main' function

"the main function is sin (...) which would come from cos (...)"

STEP 2: 'Adjust and compensate' coefficients

"cos (...) would differentiate to -sin (...)"

"$2x^3$ would differentiate to $6x^2$"

$f(x) = 5 \times -\times \frac{1}{6} \times \int -\times 6 \times x^2 \sin(2x^3)\, dx$

↑ ↑ Compensate ↑ ↑ adjust

STEP 3: Integrate and simplify

$f(x) = -\frac{5}{6}\cos(2x^3) + c$

www.thrivingscholars.com

# Substitution: Reverse Chain Rule

## What is integration by substitution?

- When reverse chain rule is difficult to spot or awkward to use then **integration** by **substitution** can be used
  - **substitution** simplifies the integral by defining an alternative variable (usually $u$) in terms of the original variable (usually $x$)
  - **everything** (including "$\mathrm{d}x$" and **limits** for **definite integrals**) is then substituted which makes the integration much easier

## How do I integrate using substitution?

**STEP 1**
Identify the substitution to be used – it will be the secondary function in the composite function

So $g(x)$ in $f(g(x))$ and $u = g(x)$

**STEP 2**
Differentiate the substitution and rearrange

$\frac{\mathrm{d}u}{\mathrm{d}x}$ can be treated like a fraction
(i.e. "multiply by $\mathrm{d}x$" to get rid of fractions)

**STEP 3**
Replace all parts of the integral
All $x$ terms should be replaced with equivalent $u$ terms, including $\mathrm{d}x$
If finding a **definite integral** change the limits from $x$-values to $u$-values too

**STEP 4**
Integrate and either
substitute $x$ back in
or
evaluate the definte integral using the $u$ limits (either using a GDC or manually)

**STEP 5**
Find $c$, the constant of integration, if needed

- For **definite integrals**, a GDC should be able to process the integral without the need for a substitution
  - be clear about whether working is required or not in a question

### Exam Tip

- Use your GDC to check the value of a definite integral, even in cases where working needs to be shown

www.thrivingscholars.com

## Worked Example

a)

Find the integral

$$\int \frac{6x+5}{(3x^2+5x-1)^3}\,dx$$

STEP 1: Identify the substitution

The composite function is $(3x^2+5x-1)^3$

The secondary function of this is $3x^2+5x-1$

$\therefore$ Let $u = 3x^2+5x-1$

STEP 2: Differentiate $u$ and rearrange

$$\frac{du}{dx} = 6x+5$$

$$\therefore du = (6x+5)\,dx$$

STEP 3: Replace all parts of the integral

$$I = \int \frac{6x+5}{(3x^2+5x-1)^2}\,dx = \int \frac{du}{u^3}$$

$$= \int u^{-3}\,du$$

STEP 4: Integrate and substitute $x$ back in

(STEP 5 not needed, evaluating $c$ is not required)

$$I = \frac{u^{-2}}{-2} + c$$

$$I = -\frac{1}{2}(3x^2+5x-1)^{-2} + c$$

$$\therefore I = \frac{-1}{2(3x^2+5x-1)^2} + c$$

b)

Evaluate the integral

$$\int_1^2 \frac{6x+5}{(3x^2+5x-1)^3}\,dx$$

giving your answer as an exact fraction in its simplest terms.

www.thrivingscholars.com



Note that you could use your GDC for this part
Certainly use it to check your answer!

From STEP 3 above, $I = \int_{x=1}^{x=2} u^{-3}\, du$

Change limits too, $x=1$, $u = 3(1)^2 + 5(1) - 1 = 7$

$x=2$, $u = 3(2)^2 + 5(2) - 1 = 21$

STEP 4: Integrate and evaluate

$$I = \left[-\tfrac{1}{2}u^{-2}\right]_7^{21} = \left[-\tfrac{1}{2}(21)^{-2}\right] - \left[-\tfrac{1}{2}(7)^{-2}\right]$$

$$\therefore I = \frac{4}{441}$$

www.thrivingscholars.com





# Differentiation for Kinematics

## How is differentiation used in kinematics?

- **Displacement**, **velocity** and **acceleration** are related by calculus
- In terms of differentiation and derivatives
  - **velocity** is the **rate** of **change** of **displacement**
    - $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ or $v(t) = s'(t)$
  - **acceleration** is the **rate** of **change** of **velocity**
    - $a = \frac{\mathrm{d}v}{\mathrm{d}t}$ or $a(t) = v'(t)$
  - so **acceleration** is also the **second derivative** of **displacement**
    - $a = \frac{\mathrm{d}^2s}{\mathrm{d}t^2}$ or $a(t) = s''(t)$
- If a graph is not given you can use your GDC to draw one
  - you can then use your GDC's graphing features to find **gradients**
    - **velocity** is the **gradient** on a **displacement** (-time) graph
    - **acceleration** is the **gradient** on a **velocity** (-time) graph

### Worked Example

The displacement, $s$ m, of a particle at $t$ seconds, is modelled by $s(t) = 2t^3 - 27t^2 + 84t$

i. Find $v(t)$ and $a(t)$.
ii. Find the times at which the particle is at rest.

i. $v(t) = s'(t) = 6t^2 - 54t + 84 = 6(t^2 - 9t + 14)$
$a(t) = v'(t) = 12t - 54 = 6(2t - 9)$

$\therefore v(t) = 6(t-7)(t-2)$
$a(t) = 6(2t-9)$

It's not essential to factorise the final answers

ii. The particle is at rest when $v(t) = 0$
$6(t-7)(t-2) = 0$
$t = 7,\ t = 2$

$\therefore$ The particle is at rest at 2 seconds and 7 seconds

www.thrivingscholars.com

# Integration for Kinematics

## How is integration used in kinematics?

- Since **velocity** is the **derivative** of **displacement** ($v = \frac{\mathrm{d}s}{\mathrm{d}t}$) it follows that

$$s = \int v \, \mathrm{d}t$$

- Similarly, **velocity** will be an **antiderivative** of **acceleration**

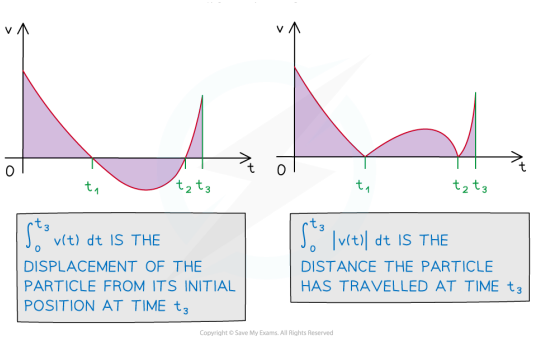
$$v = \int a \, \mathrm{d}t$$

## How would I find the constant of integration in kinematics problems?

- A **boundary** or **initial** condition would need to be known
  - phrases involving the word **"initial"**, or **"initially"** are referring to **time** being **zero,** i.e. $t = 0$
  - you might also be given information about the object at some other time (this is called a **boundary condition**)
  - **substituting** the values in from the **initial or boundary condition** would allow the **constant** of **integration** to be found

## How are definite integrals used in kinematics?

- Definite integrals can be used to find the displacement of a particle between two points in time
  - $\int_{t_1}^{t_2} v(t) \, \mathrm{d}t$ would give the **displacement** of the particle **between** the times $t = t_1$ and $t = t_2$
    - This can be found using a velocity-time graph by **subtracting** the **total area below** the horizontal axis from the **total area above**
  - $\int_{t_1}^{t_2} |v(t)| \, \mathrm{d}t$ gives the **distance** a particle has **travelled** between the times $t = t_1$ and $t = t_2$
    - This can be found using a velocity velocity-time graph by **adding** the **total area below** the horizontal axis to the **total area above**
    - Use a GDC to plot the modulus graph $y = |v(t)|$



Copyright © Save My Exams. All Rights Reserved

www.thrivingscholars.com

### Exam Tip

- Sketching the velocity-time graph can help you visualise the distances travelled using areas between the graph and the horizontal axis

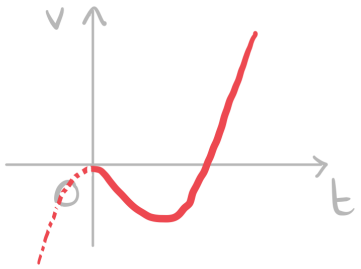
www.thrivingscholars.com

## Worked Example

A particle moving in a straight horizontal line has velocity ($v$ m s$^{-1}$) at time $t$ seconds modelled by $v(t) = 8t^3 - 12t^2 - 2t$.

i. Given that the initial position of the particle is at the origin, find an expression for its displacement from the origin at time $t$ seconds.
ii. Find the displacement of the particle from the origin in the first five seconds of its motion.
iii. Find the distance travelled by the particle in the first five seconds of its motion.

Use your GDC to sketch a velocity(-time) graph and use it to check to see if your answers are sensible.




i. "initial" – $t=0$, "origin" – $s=0$

$s(t) = \int v(t)\,dt = \int (8t^3-12t^2-2t)\,dt$

$s(t) = 2t^4-4t^3-t^2+c$ where $c$ is a constant

at $t=0$, $s=0$, $\therefore c=0$

$\therefore s(t) = 2t^4-4t^3-t^2$

ii. "first five seconds" – $t_1=0$, $t_2=5$

Using a GDC this would be

$s = \int_0^5 (8t^3-12t^2-2t)\,dt$

$s = 725$ m

iii. Using a GDC this would be

$d = \int_0^5 |8t^3-12t^2-2t|\,dt$ d for distance

$d = 736.734\,020\ldots$

$\therefore d = 737$ m (3 s.f.)

www.thrivingscholars.com





# Related Rates of Change

## What is meant by rates of change?

- A rate of change is a measure of how a quantity is changing with respect to another quantity
- Mathematically rates of change are derivatives
  - $\frac{\mathrm{d}V}{\mathrm{d}r}$ could be the rate at which the volume of a sphere changes relative to how its radius is changing
- Context is important when interpreting positive and negative rates of change
  - A positive rate of change would indicate an increase
    - e.g. the change in volume of water as a bathtub fills
  - A negative rate of change would indicate a decrease
    - e.g. the change in volume of water in a leaking bucket

## What is meant by related rates of change?

- Related rates of change are connected by a linking variable or parameter
  - this is often time, represented by $t$
  - seconds is the standard unit for time but this will depend on context
- e.g. Water running into a large hemi-spherical bowl
  - both the height and volume of water in the bowl are changing with time
    - time is the linking parameter between the rate of change of height and the rate of change of volume

## How do I solve problems involving related rates of change?

- Use of chain rule and product rule are common in such problems
- Be clear about which variables are representing which quantities

STEP 1
Write down any variables and derivatives involved in the problem
e.g. $x$, $y$, $t$, $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}x}{\mathrm{d}t}$, $\frac{\mathrm{d}y}{\mathrm{d}t}$

STEP 2
Use an appropriate differentiation rule to set up an equation linking 'rates of change'
e.g. Chain rule: $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$

STEP 3
Substitute in known values
e.g. If, when $t = 3$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 8$, then $8 = \frac{\mathrm{d}y}{\mathrm{d}x} \times 2$

www.thrivingscholars.com



STEP 4

Solve the problem and interpret the answer in context if required

e.g. $\frac{dy}{dx} = \frac{8}{2} = 4$ 'when $t = 3$, $y$ changes at a rate of 4, with respect to $x$'



## Exam Tip

- If you struggle to determine which rate to use then you can look at the units to help
  - e.g. A rate of 5 **cm$^3$** *per* **second** implies **volume** per **time** so the rate would be $\frac{dV}{dt}$

www.thrivingscholars.com

## Worked Example

In a manufacturing process a metal component is heated such that it's cross-sectional area expands but always retains the shape of a right-angled triangle. At time $t$ seconds the triangle has base $b$ cm and height $h$ cm.

At the time when the component's cross-sectional area is changing at 4 cm $s^{-1}$, the base of the triangle is 3 cm and its height is 6 cm. Also at this time, the rate of change of the height is twice the rate of change of the base.

Find the rate of change of the base at this point of time.

STEP 1: List variables and derivatives

$A, b, h, t, \frac{dA}{dt}, \frac{db}{dt}, \frac{dh}{dt}$

$A = \frac{1}{2}bh$

STEP 2: Use a differentiation rule to link 'rates of change'

$A = \frac{1}{2}bh$ is a product - so use product rule

$$\frac{dA}{dt} = \frac{1}{2}\left[b\frac{dh}{dt} + h\frac{db}{dt}\right]$$

STEP 3: Substitute known values

$$4 = \frac{1}{2}\left[3\left(2\frac{db}{dt}\right) + 6\frac{db}{dt}\right]$$

$\frac{dh}{dt} = 2\frac{db}{dt}$ in question

STEP 4: Solve and interpret

$$8 = 12\frac{db}{dt}$$

$\therefore \frac{db}{dt} = \frac{2}{3}$ cm $s^{-1}$

The rate of change of the base is $\frac{2}{3}$ cm per second.

www.thrivingscholars.com

# Differentiating Inverse Functions

## What is meant by an inverse function?

- Some functions are easier to process with $x$ (rather than $y$) as the subject
  - i.e. in the form $x = f(y)$
- This is particularly true when dealing with inverse functions
  - e.g. If $y = f(x)$ the inverse would be written as $y = f^{-1}(x)$
    - finding $f^{-1}(x)$ can be awkward
    - so write $x = f(y)$ instead

## How do I differentiate inverse functions?

- Since $x = f(y)$ it is easier to differentiate "$x$ with respect to $y$" rather than "$y$ with respect to $x$"
  - i.e. find $\frac{dx}{dy}$ rather than $\frac{dy}{dx}$
  - Note that $\frac{dx}{dy}$ will be in terms of $y$ but can be substituted

  STEP 1
  For the function $y = f(x)$, the inverse will be $y = f^{-1}(x)$

  Rewrite this as $x = f(y)$

  STEP 2

  From $x = f(y)$ find $\frac{dx}{dy}$

  STEP 3

  Find $\frac{dy}{dx}$ using $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ - this will usually be in terms of $y$

- If an algebraic solution in terms of $x$ is required substitute $f(x)$ for $y$ in $\frac{dy}{dx}$
- If a numerical derivative (e.g. a gradient) is required then use the $y$-coordinate
  - If the $y$-coordinate is not given, you should be able to work it out from the orginal function and $x$-coordinate

www.thrivingscholars.com



### Exam Tip

- With $x$'s and $y$'s everywhere this can soon get confusing!
  - Be clear of the key information and steps - and set your wokring out accordingly
    - The orginal function, $y = f(x)$
    - Its inverse, $y = f^{-1}(x)$
    - Rewriting the inverse, $x = f(y)$
    - Finding $\frac{dx}{dy}$ first, then finding its reciprocal for $\frac{dy}{dx}$
- Your GDC can help when numerical derivatives (gradients) are required

www.thrivingscholars.com

## Worked Example

a)

Find the gradient of the curve at the point where $y = 3$ on the graph of $y = f^{-1}(x)$ where $f(x) = \sqrt{(5x+1)^3}$.

STEP 1: Rewrite inverse as $x = f(y)$

$f(x) = \sqrt{(5x+1)^3}$

$\therefore$ For $y = f^{-1}(x)$, $x = f(y)$

$x = \sqrt{(5y+1)^3}$

STEP 2: Find $\frac{dx}{dy}$

$x = (5y+1)^{3/2}$ Write as powers

$\frac{dx}{dy} = \frac{3}{2}(5y+1)^{1/2} \times 5$ Using chain rule

$\frac{dx}{dy} = \frac{15}{2}\sqrt{5y+1}$

STEP 3: Find $\frac{dy}{dx}$

A gradient is required - substitute $y=3$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{15}{2}\sqrt{5y+1}} = \frac{2}{15\sqrt{5y+1}}$$

At $y=3$, $\frac{dy}{dx} = \frac{2}{15\sqrt{5(3)+1}}$

$\therefore$ Gradient, at $y=3$, on the graph of $y=f^{-1}(x)$ is $\frac{1}{30}$.

b) Given that $y = e^x$ show that the derivative of $y = \ln x$ is $\frac{1}{x}$.

www.thrivingscholars.com



The key to this question is realising that $e^x$ and $\ln x$ are inverses

$y = e^x$ so $y = \ln x$ will be its inverse

STEP 1: $y = e^x$

$\therefore$ The inverse will be $x = e^y$ "$y = f(x)$"

"$y = f^{-1}(x)$, $x = f(y)$"

STEP 2: $\frac{dx}{dy} = e^y$

STEP 3: $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$

$e^{\ln x} = x$ since $e^x$ and $\ln x$ are inverses

$\therefore$ If $y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$

www.thrivingscholars.com





# Definite Integrals

## What is a definite integral?

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

- This is known as the **Fundamental Theorem of Calculus**
- ***a*** and ***b*** are called limits
  - ***a*** is the lower limit
  - ***b*** is the upper limit
- $f(x)$ is the **integrand**
- $F(x)$ is an **antiderivative** of $f(x)$
- The **constant** of **integration** ("**+c**") is not needed in **definite integration**
  - "+c" would appear alongside both **F(*a*)** and **F(*b*)**
  - subtracting means the "+c"'s cancel

## How do I find definite integrals analytically (manually)?

STEP 1
Give the integral a name to save having to rewrite the whole integral every time
If need be, rewrite the integral into an integrable form

$$I = \int_a^b f(x)\,dx$$

STEP 2
Integrate without applying the limits; you will not need "+c"
Notation: use square brackets [ ] with limits placed at the end bracket

STEP 3
Substitute the limits into the function and evaluate

### Exam Tip

- If a question does not state that you can use your GDC then you must show all of your working clearly, however it is always good practice to check you answer by using your GDC if you have it in the exam

www.thrivingscholars.com

## Worked Example

a)

Show that

$$\int_2^4 3x(x^2-2)\,dx = 144$$

STEP 1: Name the integral and rewrite into an integratable form

$$I = \int_2^4 (3x^3 - 6x)\,dx$$

STEP 2: Integrate

$$I = \left[\frac{3}{4}x^4 - 3x^2\right]_2^4$$

STEP 3: Evaluate

$$I = \left[\frac{3}{4}(4)^4 - 3(4)^2\right] - \left[\frac{3}{4}(2)^4 - 3(2)^2\right]$$

$$I = 144 - 0$$

$$\therefore \int_2^4 3x(x^2-2) = 144$$

b)

Use your GDC to evaluate

$$\int_0^1 3e^{x^2 \sin x}\,dx$$

giving your answer to three significant figures.

Using GDC,

$$\int_0^1 3e^{x^2 \sin x}\,dx = 3.872957\ldots$$

$$\therefore \int_0^1 3e^{x^2 \sin x}\,dx = 3.87 \quad (3\text{ s.f.})$$

www.thrivingscholars.com

# Properties of Definite Integrals

## Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

- Formally,
  - $f(x)$ is **continuous** in the interval $a \le x \le b$
  - $F(x)$ is an **antiderivative** of $f(x)$

## What are the properties of definite integrals?

- Some of these have been encountered already and some may seem obvious …
  - taking **constant** factors outside the integral
    - $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ where $k$ is a constant
    - useful when fractional and/or negative values involved
  - integrating term by term
    - $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
    - the above works for subtraction of terms/functions too
  - equal upper and lower limits
    - $\int_a^a f(x)\,dx = 0$
    - on evaluating, this would be a value, subtract itself !
  - swapping limits gives the same, but negative, result
    - $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
    - compare 8 subtract 5 say, with 5 subtract 8 …
  - splitting the interval
    - $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a \le c \le b$
    - this is particularly useful for areas under multiple curves or areas under the $x$-axis
  - horizontal translations
    - $\int_a^b f(x)\,dx = \int_{a-k}^{b-k} f(x+k)\,dx$ where $k$ is a constant
    - the graph of $y = f(x \pm k)$ is a horizontal translation of the graph of $y = f(x)$ ($f(x+k)$ translates left, $f(x-k)$ translates right)

### Exam Tip

- Learning the properties of definite integrals can help to save time in the exam

www.thrivingscholars.com

## Worked Example

$f(x)$ is a continuous function in the interval $5 \le x \le 15$.

It is known that $\int_5^{10} f(x)\,dx = 12$ and that $\int_{10}^{15} f(x)\,dx = 5$.

a)

Write down the values of

i)

$$\int_7^7 f(x)\,dx$$

ii)

$$\int_{10}^5 f(x)\,dx$$

i. $\int_7^7 f(x)\,dx = 0$ "equal limits" $\int_a^a f(x)\,dx = 0$

ii. $\int_{10}^5 f(x)\,dx = -12$ "swapped limits" $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

b)

Find the values of

i)

$$\int_5^{15} f(x)\,dx$$

ii)

$$\int_5^{10} 6f(x+5)\,dx$$

www.thrivingscholars.com



i. $I = \int_5^{15} f(x)\,dx = \int_5^{10} f(x)\,dx + \int_{10}^{15} f(x)\,dx = 12 + 5 = 17$

"split limits"

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^a f(x)\,dx$$

$$\therefore \int_5^{15} f(x)\,dx = 17$$

ii. $I = \int_5^{10} 6f(x+5) = 6\int_5^{10} f(x+5)\,dx$

"factors"

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

$$I = 6\int_{5+5}^{10+5} f(x)\,dx$$

"horizontal translation"

$$\int_a^b f(x+k)\,dx = \int_{a+k}^{b+k} f(x)\,dx$$

$$I = 6\int_{10}^{15} f(x)\,dx = 6 \times 5 = 30$$

$$\therefore \int_5^{10} 6f(x+5)\,dx = 30$$

www.thrivingscholars.com





# Implicit Differentiation

## What is implicit differentiation?

- An equation connecting x and y is not always easy to write **explicitly** in the form $y = f(x)$ or $x = f(y)$
  - In such cases the equation is written **implicitly**
    - as a function of $x$ and $y$
    - in the form $f(x,y) = 0$
- Such equations can be **differentiated implicitly** using the **chain rule**

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(y)] = f'(y)\frac{\mathrm{d}y}{\mathrm{d}x}$$

- A shortcut way of thinking about this is that '$y$ is a function of a $x$'
  - when differentiating a function of $y$ chain rule says "differentiate with respect to $y$, then multiply by the derivative of $y$" (which is $\frac{\mathrm{d}y}{\mathrm{d}x}$)

www.thrivingscholars.com

# Applications of Implicit Differentiation

## What type of problems could involve implicit differentiation?

- Broadly speaking there are three types of problem that could involve implicit differentiation
  - algebraic problems involving graphs, derivatives, tangents, normals, etc
    - where it is not practical to write $y$ explicitly in terms of $x$
    - usually in such cases, $\frac{\mathrm{d}y}{\mathrm{d}x}$ will be in terms of $x$ and $y$
  - **optimisation** problems that involve **time derivatives**
    - more than one variable may be involved too

      e.g. Volume of a cylinder, $V = \pi r^2 h$

      e.g. The side length and (so) area of a square increase over time
  - any problem that involves differentiating with respect to an extraneous variable
    - e.g. $y = f(x)$ but the derivative $\frac{\mathrm{d}y}{\mathrm{d}\theta}$ is required (rather than $\frac{\mathrm{d}y}{\mathrm{d}x}$)

## How do I apply implicit differentiation to algebraic problems?

- Algebraic problems revolve around values of the derivative (gradient) $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$
  - if not required to find this value it will either be given or implied
- Particular problems focus on special case tangent values
  - horizontal tangents
    - also referred to as tangents parallel to the $x$-axis
    - this is when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
  - vertical tangents
    - also referred to as tangents parallel to the $y$-axis
    - this is when $\frac{\mathrm{d}x}{\mathrm{d}y} = 0$
    - In such cases it may appear that $\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}} = 0$ but this has no solutions; this occurs when for nearby values of $x$, $\frac{\mathrm{d}y}{\mathrm{d}x} \to \pm\infty$

      (i.e. very steep gradients, near vertical)
- Other problems may involve finding equations of (other) tangents and/or normals
- For problems that involve finding the coordinates of points on a curve with a specified gradient the method below can be used

  STEP 1
  Differentiate the equation of the curve implicitly

www.thrivingscholars.com

STEP 2

Substitute the given or implied value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ to create an equation linking $x$ and $y$

STEP 3

There are now two equations

- the original equation
- the linking equation

Solve them simultaneously to find the $x$ and $y$ coordinates as required

## Exam Tip

- After some rearranging, $\frac{dy}{dx}$ will be in terms of both $x$ and $y$
  - There is usually no need (unless asked to by the question) to write $\frac{dy}{dx}$ in terms of $x$ (or $y$) only
- If evalutaing derivatives, you'll need both $x$ and $y$ coordinates, so one may have to be found from the other using the original function

www.thrivingscholars.com

## Worked Example

The curve C has equation $x^2 + 2y^2 = 16$.

a)
Find the exact coordinates of the points where the normal to curve C has gradient 2.

STEP 1: Differentiate implicitly

$2x + 4y\frac{dy}{dx} = 0$

STEP 2: Substitute value of $\frac{dy}{dx}$ in

Gradient of normal is 2
∴ gradient of tangent $\left(\frac{dy}{dx}\right)$ is $-\frac{1}{2}$
(Normal and tangent are perpendicular, so product of their gradients is $-1$)

$\therefore 2x + 4y\left(-\frac{1}{2}\right) = 0$

$2x - 2y = 0$

$x = y$ Equation linking x and y

STEP 3: Solve simultaneously, obtain coordinates

$x^2 + 2y^2 = 16$

$y = x$

$\therefore x^2 + 2x^2 = 16$

$3x^2 = 16$ $x = \pm\frac{4}{\sqrt{3}} = \pm\frac{4}{3}\sqrt{3}$ $\therefore y = \pm\frac{4}{3}\sqrt{3}$

∴ Coordinates are $\left(\frac{4}{3}\sqrt{3}, \frac{4}{3}\sqrt{3}\right)$ and $\left(-\frac{4}{3}\sqrt{3}, -\frac{4}{3}\sqrt{3}\right)$

b)
Find the equations of the tangents to the curve that are
(i) parallel to the x-axis
(ii) parallel to the y-axis.

www.thrivingscholars.com



These are special cases,

in part (i), $\frac{dy}{dx} = 0$ (parallel to x-axis)

in part (ii), $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = 0$ (parallel to y-axis)

$x^2 + 2y^2 = 16$

$2x + 4y\frac{dy}{dx} = 0$

(i) $\frac{dy}{dx} = 0$ $\therefore 2x = 0$

$x = 0$ $\therefore 2y^2 = 16$

$y = \pm\sqrt{8} = \pm 2\sqrt{2}$

(ii) $\frac{dy}{dx} = \frac{-2x}{4y} = \frac{-x}{2y}$

$\therefore \frac{dx}{dy} = \frac{2y}{-x}$

$\frac{dx}{dy} = 0$ $\therefore 2y = 0$

$y = 0$ $\therefore x^2 = 16$

$x = \pm 4$

$\therefore$ Tangents (i) parallel to x-axis are $y = 2\sqrt{2}$ and $y = -2\sqrt{2}$ and (ii) parallel to y-axis are $x = 4$ and $x = -4$

## How do I apply implicit differentiation to optimisation problems?

- For a single variable use chain rule to differentiate implicitly
  - e.g. A square with side length changing over time, $A = x^2$

$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

- For more than one variable use product rule (and chain rule) to differentiate implicitly
  - e.g. A square-based pyramid with base length and height changing over time,

$$V = \frac{1}{3}x^2h$$

$$\frac{dV}{dt} = \frac{1}{3}\left[x^2\frac{dh}{dt} + 2x\frac{dx}{dt}h\right] = \frac{1}{3}x\left(x\frac{dh}{dt} + 2h\frac{dx}{dt}\right)$$

www.thrivingscholars.com



- After differentiating implicitly the rest of the question should be similar to any other optimisation problem
  - be aware of phrasing
    - "the rate of change of the height of the pyramid" (over time) is $\frac{\mathrm{d}h}{\mathrm{d}t}$
  - when finding the location of minimum and maximum problems
    - there is not necessarily a turning point
    - the minimum or maximum could be at the start or end of a given or appropriate interval

### Exam Tip

- If you are struggling to tell which derivative is needed for a question, writing all possibilities down may help you
  - You don't need to work them out at this stage but if you conisder them it may nudge you to the next stage of the solution
  - e.g. For $V = \pi r^2 h$, possible derivatives are $\frac{dV}{dr}$, $\frac{dV}{dh}$ and $\frac{dV}{dt}$

www.thrivingscholars.com

## Worked Example

The radius, $r$ cm, and height, $h$ cm, of a cylinder are increasing with time. The volume, $V$ cm$^3$, of the cylinder at time $t$ seconds is given by $V = \pi r^2 h$.

a) Find an expression for $\frac{dV}{dt}$.

Using implicit differentiation with product rule

$$\frac{dV}{dt} = \pi\left[2r\frac{dr}{dt}h + \frac{dh}{dt}r^2\right]$$

$$\therefore \frac{dV}{dt} = \pi r\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right)$$

b)

At time $T$ seconds, the radius of the cylinder is 4 cm, expanding at a rate of 2 cm s$^{-1}$. At the same time, the height of the cylinder is 10 cm, expanding at a rate of 3 cm s$^{-1}$.

Find the rate at which the volume is expanding at time $T$ seconds.

At time T, $r = 4$, $\frac{dr}{dt} = 2$

$h = 10$, $\frac{dh}{dt} = 3$

$$\therefore \frac{dV}{dt} = \pi(4)(2\times10\times2 + 4\times3)$$

$\therefore$ At time T seconds, the volume is expanding at a rate of $208\pi$ cm$^3$ s$^{-1}$

www.thrivingscholars.com



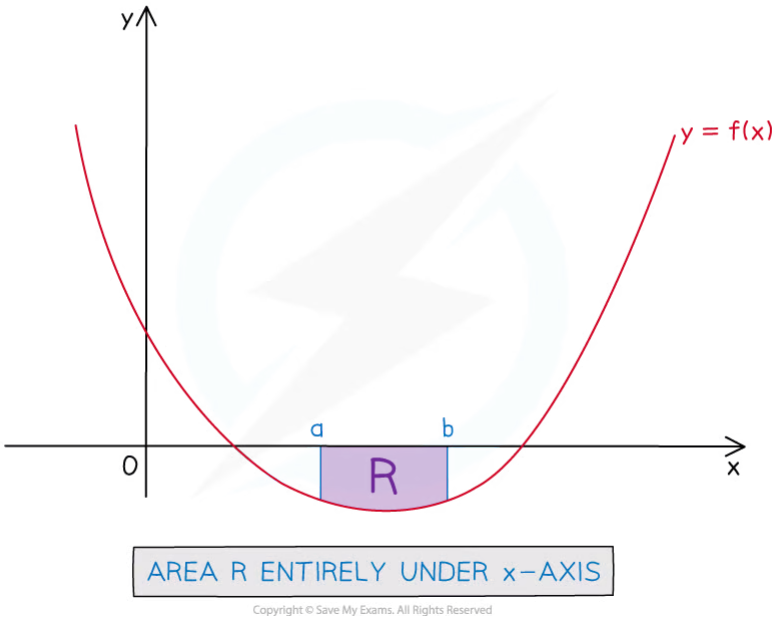
## Negative Integrals

- The area under a curve may appear **fully** or **partially** under the x-axis
  - This occurs when the function $f(x)$ takes **negative** values within the boundaries of the area
- The **definite integrals** used to find such **areas**
  - will be **negative** if the area is **fully** under the $x$-axis
  - possibly **negative** if the area is **partially** under the $y$-axis
    - this occurs if the negative area(s) is/are greater than the positive area(s), their **sum** will be **negative**
- When using a GDC use the modulus (absolute value) function so that all definite integrals have a positive value

$$A = \int_a^b |y| \, \mathrm{d}x$$

  - This is given in the **formula booklet**

### How do I find the area under a curve when the curve is fully under the x-axis?




Copyright © Save My Exams. All Rights Reserved

**STEP 1**

Write the expression for the definite integral to find the area as usual

This may involve finding the lower and upper limits from a graph sketch or GDC and f(x) may need to be rewritten in an integrable form

www.thrivingscholars.com

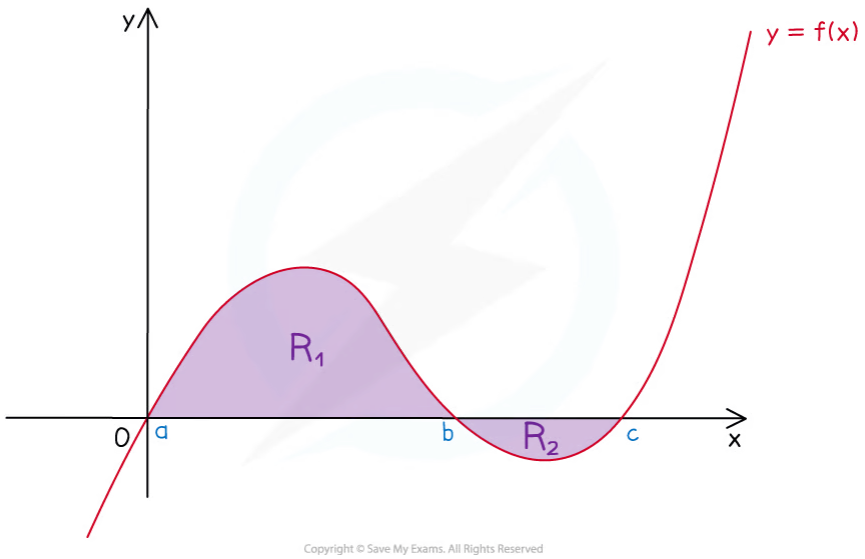
**STEP 2**

The answer to the definite integral will be negative

Area must always be positive so take the modulus (absolute value) of it

e.g. If $I = -36$ then the area would be 36 (square units)

## How do I find the area under a curve when the curve is partially under the x-axis?



Copyright © Save My Exams. All Rights Reserved

- For questions that allow the use of a GDC you can still use

$$A = \int_a^c |f(x)| \, dx$$

- To find the area analytically (manually) use the following method

**STEP 1**

Split the area into parts - the area(s) that are above the x-axis and the area(s) that are below the x-axis

**STEP 2**

Write the expression for the definite integral for each part (give each part a name, $I_1$, $I_2$, etc)

This may involve finding the lower and upper limits of each part from a graph sketch or a GDC, finding the roots of the function (i.e. where $f(x) = 0$) and rewriting $f(x)$ in an integrable form

**STEP 3**

Find the value of each definite integral separately

**STEP 4**

Find the area by summing the modulus (absolute values) of each integral

(Mathematically this would be written $A = |I_1| + |I_2| + |I_3| + ...$)

www.thrivingscholars.com

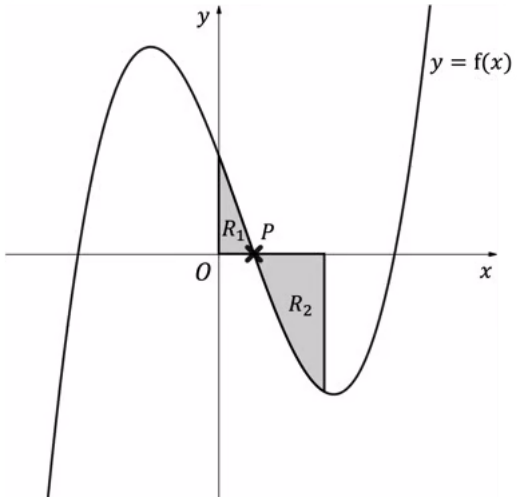
### Exam Tip

- If no diagram is provided, quickly sketch one so that you can see where the curve is above and below the x - axis and split up your integrals accordingly

www.thrivingscholars.com

## Worked Example

The diagram below shows the graph of $y = f(x)$ where $f(x) = (x+4)(x-1)(x-5)$.




The region $R_1$ is bounded by the curve $y = f(x)$, the $x$-axis and the $y$-axis.

The region $R_2$ is bounded by the curve $y = f(x)$, the x-axis and the line $x = 3$.

a)
Determine the coordinates of the point labelled $P$.

a) The x-coordinate of P is a root of f(x)
f(x) = 0
(x+4)(x-1)(x-5) = 0
x = -4, x = 1, x = 5
Clearly from the graph, x = 1 at point P
∴ P(1, 0)

b)
i)
Find a definite integral that would help find the area of the shaded region $R_2$ and briefly explain why this would **not** give the area of the region $R_2$.

ii)
Find the exact area of the shaded region $R_2$.

www.thrivingscholars.com

b) i)

$I_2 = \int_1^3 (x+4)(x-1)(x-5)\ dx$

$R_2$ is underneath the x-axis so the value of the definite integral will be negative. Area cannot be negative.

ii) STEP 1:

$I_2 = \int_1^3 (x+4)(x-1)(x-5)\ dx$

$I_2 = \int_1^3 (x^2+3x-4)(x-5)\ dx$ Rewrite in an integrable form

$I_2 = \int_1^3 (x^3-2x^2-19x+20)\ dx$

$I_2 = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{19x^2}{2} + 20x\right]_1^3$ Integrate (no need for "+c")

$I_2 = \left(\frac{3^4}{4} - \frac{2(3)^3}{3} - \frac{19(3)^2}{2} + 20(3)\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{19}{2} + 20\right)$

$I_2 = -\frac{93}{4} - \frac{121}{12}$

$I_2 = -\frac{100}{3}$

STEP 2: ∴ Area of $R_2$, $A_2 = \frac{100}{3}$ square units

c)

Find the exact total area of the shaded regions, $R_1$ and $R_2$.

c) STEP 1, 2: $A_1 = I_1 = \int_0^1 (x^3-2x^2-19x+20)\ dx$ Use the relevant results from b) ii)

$I_1 = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{19x^2}{2} + 20x\right]_0^1$

STEP 3: $I_1 = \frac{121}{12} - 0$

STEP 4: ∴ $A_1 + A_2 = \frac{121}{12} + \frac{100}{3} = \frac{521}{12}$

∴ Total area shaded = $\frac{521}{12}$ square units

You can check the final answer using your GDC and the formula (in booklet) $A = \int_a^b |y|\ dx$.

Here, $A = \int_0^3 |(x+4)(x-1)(x-5)|\ dx$

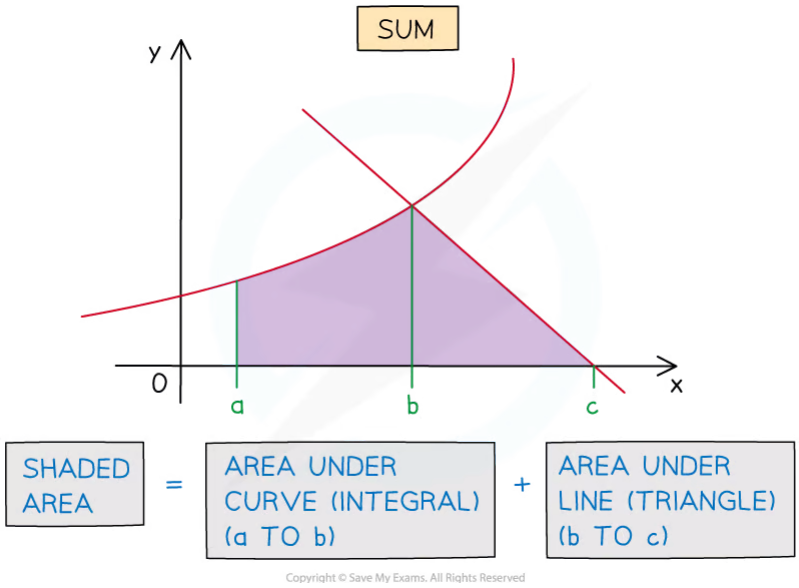
$A = 43.41666...$

(Note that our GDC was not able to produce the exact answer...)

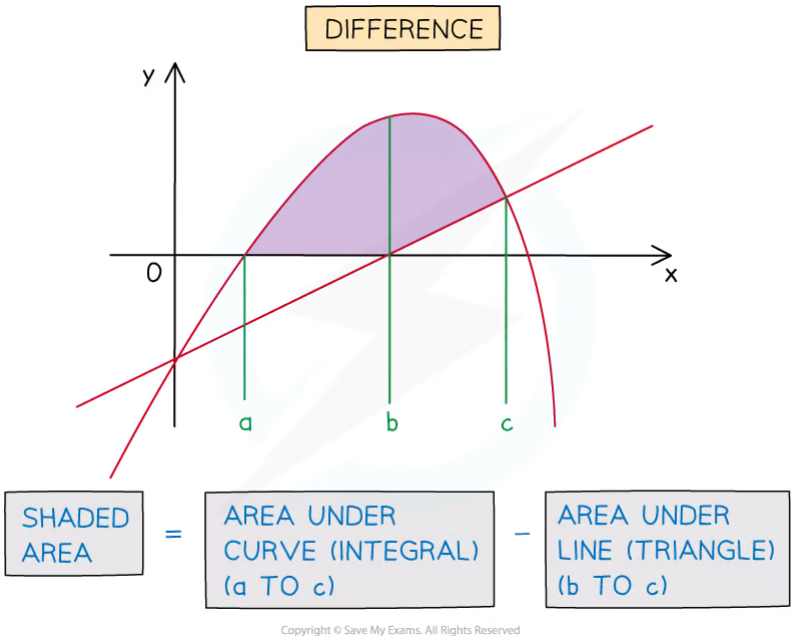
www.thrivingscholars.com

# Area Between a Curve and a Line

- **Areas** whose boundaries include a **curve** and a (non-vertical) **straight line** can be found using integration
  - For an **area** under a **curve** a **definite integral** will be needed
  - For an **area** under a **line** the shape formed will be a **trapezium** or **triangle**
    - **basic area formulae** can be used rather than a definite integral
    - (although a definite integral would still work)
- The **area** required could be the **sum** or **difference** of areas under the curve and line




Copyright © Save My Exams. All Rights Reserved




Copyright © Save My Exams. All Rights Reserved

## How do I find the area between a curve and a line?

**STEP 1**

www.thrivingscholars.com

If not given, sketch the graphs of the curve and line on the same diagram
Use a GDC to help with this step

**STEP 2**
Find the intersections of the curve and the line
If no diagram is given this will help identify the area(s) to be found

**STEP 3**
Determine whether the area required is the sum or difference of the area under the curve and the area under the line
Calculate the area under a curve using a integral of the form

$$\int_a^b y \, \mathrm{d}x$$

Calculate the area under a line using either $A = \frac{1}{2}bh$ for a triangle or $A = \frac{1}{2}h(a+b)$ for a trapezium (y-coordinates will be needed)

**STEP 4**
Evaluate the definite integrals and find their sum or difference as necessary to obtain the area required

### Exam Tip

- Add information to any diagram provided
- Add axes intercepts, as well as intercepts between lines and curves
- Mark and shade the area you're trying to find
- If no diagram is provided, **sketch** one!

www.thrivingscholars.com

## Worked Example

The region $R$ is bounded by the curve with equation $y = 10x - x^2 - 16$ and the line with equation $y = 8 - x$.

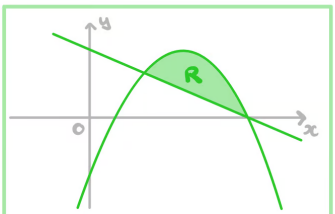
$R$ lies entirely in the first quadrant.

a)

Using your GDC, or otherwise, sketch the graphs of the curve and the line on the same diagram.

Identify and label the region $R$ on your sketch and use your GDC to find the $x$-coordinates of the points of intersection between the curve and the line.

STEP 1:




STEP 2: Points of intersection are (3, 5) and (8, 0)

b)

i)

Write down an integral that would find the area of the region $R$.

ii)

Find the area of the region $R$.

i) STEP 3: Curve is 'upper' boundary of R

$\therefore y_1 = 10x - x^2 - 16$

$y_2 = 8 - x$

$y_1 - y_2 = 10x - x^2 - 16 - (8 - x) = 11x - x^2 - 24$

$\therefore$ Area of R, $A_R = \int_3^8 (11x - x^2 - 24)\,dx$

ii) STEP 4:

$A_R = \int_3^8 (11x - x^2 - 24)\,dx$

$A_R = \left[\frac{11x^2}{2} - \frac{x^3}{3} - 24x\right]_3^8$

$A_R = \left[\frac{11(8)^2}{2} - \frac{(8)^3}{3} - 24(8)\right] - \left[\frac{11(3)^2}{2} - \frac{(3)^3}{3} - 24(3)\right]$

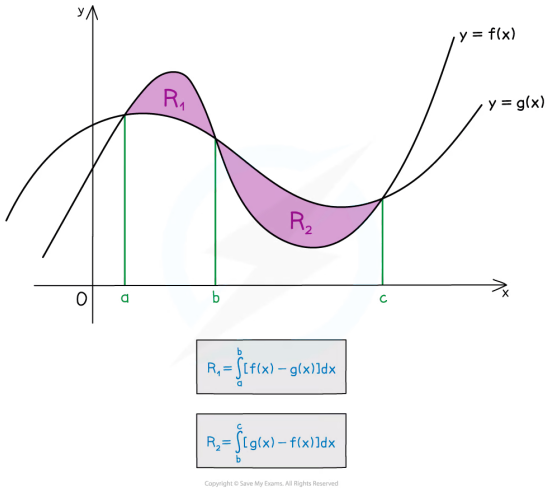
$A_R = -\frac{32}{3} - -\frac{63}{2}$

$\therefore$ Area of region R is $\frac{125}{6}$ square units

www.thrivingscholars.com

# Area Between 2 Curves

- **Areas** whose boundaries include **two curves** can be found by integration
  - The **area between two curves** will be the **difference** of the areas under the two curves
    - both areas will require a **definite integral**
  - Finding points of intersection may involve a more awkward equation than solving for a curve and a line




## How do I find the area between two curves?

**STEP 1**

If not given, sketch the graphs of both curves on the same diagram

Use a GDC to help with this step

**STEP 2**

Find the intersections of the two curves

If no diagram is given this will help identify the area(s) to be found

**STEP 3**

For each area (there may only be one) determine which curve is the 'upper' boundary

For each area, write a definite integral of the form

$$\int_a^b (y_1 - y_2)\,\mathrm{d}x$$

where $y_1$ is the function for the 'upper' boundary and $y_2$ is the function for the 'lower' boundary

Be careful when there is more than one region – the 'upper' and 'lower' boundaries will swap

**STEP 4**

Evaluate the definite integrals and sum them up to find the total area

(Step 3 means no definite integral will have a negative value)

www.thrivingscholars.com

### Exam Tip

- If no diagram is provided sketch one, even if the curves are not accurate
- Add information to any given diagram as you work through a question
- Maximise use of your GDC to save time and maintain accuracy:
  - Use it to sketch the graphs and help you visualise the problem
  - Use it to find definite integrals

www.thrivingscholars.com

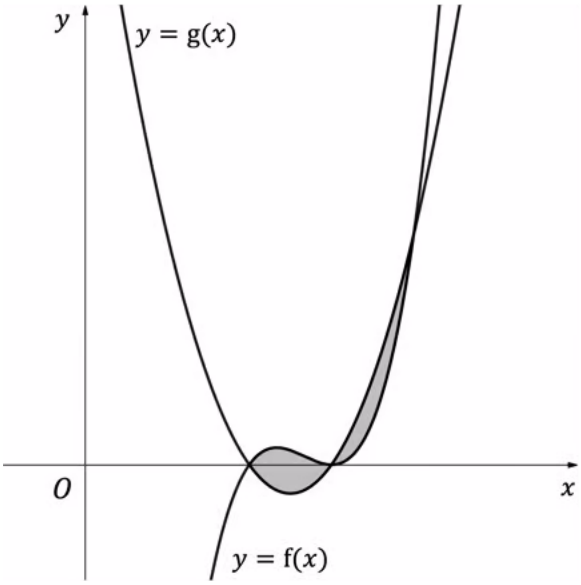
## Worked Example

The diagram below shows the curves with equations $y = \mathrm{f}(x)$ and $y = g(x)$ where

$$\mathrm{f}(x) = (x-2)(x-3)^2$$

$$\mathrm{g}(x) = x^2 - 5x + 6$$

Find the area of the shaded region.



www.thrivingscholars.com

STEP 1: Sketch of graph given

STEP 2: Two intersections are the roots of $f(x)$

$f(x) = (x-2)(x-3)^2 = 0$ at $x = 2$, $(y=0)$

and $x = 3$ $(y=0)$

Solve $f(x) = g(x)$ to find the other intersection

$(x-2)(x-3)^2 = x^2 - 5x + 6$

$(x-2)(x-3)^2 = (x-2)(x-3)$

$x - 3 = 1$

$x = 4$, $y = (4-2)(4-3) = 2$

STEP 3: The area, $A_1$ of the first region is given by

$$A_1 = \int_2^3 \left[(x-2)(x-3)^2 - (x^2-5x+6)\right] dx$$

$$A_1 = \int_2^3 (x-2)(x-3)\left[(x-3) - 1\right] dx \qquad \text{Factorise } (x-2)(x-3)$$

$$A_1 = \int_2^3 (x^2-5x+6)(x-4)\ dx$$

$$A_1 = \int_2^3 (x^3 - 9x^2 + 26x - 24)\ dx$$

$$A_1 = \left[\frac{x^4}{4} - 3x^3 + 13x^2 - 24x\right]_2^3$$

$$A_1 = \left(\frac{(3)^4}{4} - 3(3)^3 + 13(3)^2 - 24(3)\right) - \left(\frac{(2)^4}{4} - 3(2)^3 + 13(2)^2 - 24(2)\right)$$

$$A_1 = -\frac{63}{4} - -16 = \frac{1}{4}$$

For $A_2$, the 'upper' and 'lower' boundaries swap

$$A_2 = \int_3^4 \left[(x^2-5x+6) - (x-2)(x-3)^2\right] dx$$

$$A_2 = \int_3^4 (x-2)(x-3)\left[1 - (x-3)\right] dx$$

$$A_2 = \int_3^4 (x^2-5x+6)(4-x)\ dx$$

$$A_2 = \int_3^4 (-x^3 + 9x^2 - 26x + 24)\ dx$$

$$A_2 = \left[-\frac{x^4}{4} + 3x^3 - 13x^2 + 24x\right]_3^4$$

$$A_2 = \left(-\frac{(4)^4}{4} + 3(4)^3 - 13(4)^2 + 24(4)\right) - \left(-\frac{(3)^4}{4} + 3(3)^3 - 13(3)^2 + 24(3)\right)$$

$$A_2 = 16 - \frac{63}{4} = \frac{1}{4}$$

Total area is $A_1 + A_2$

$\therefore$ Area of shaded region is $\frac{1}{2}$ square unit

www.thrivingscholars.com

# 5.8 Advanced Differentiation

## 5.8.4 Differentiating Further Functions

This Revision Note focuses on the results and derivations of results involving the less common trigonometric, exponential and logarithmic functions. As with any function, questions may go on to ask about gradients, tangents, normals and stationary points.

www.thrivingscholars.com

# Differentiating Reciprocal Trigonometric Functions

## What are the reciprocal trigonometric functions?

- **Secant**, **cosecant** and **cotangent** and abbreviated and defined as

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

- Remember that for calculus, angles need to be measured in **radians**
  - $\theta$ may be used instead of $x$
- $\operatorname{cosec} x$ is sometimes further abbreviated to $\csc x$

## What are the derivatives of the reciprocal trigonometric functions?

- $f(x) = \sec x$
  - $f'(x) = \sec x \tan x$
- $f(x) = \operatorname{cosec} x$
  - $f'(x) = -\operatorname{cosec} x \cot x$
- $f(x) = \cot x$
  - $f'(x) = -\operatorname{cosec}^2 x$
- These are given in the **formula booklet**

## How do I show or prove the derivatives of the reciprocal trigonometric functions?

- For $y = \sec x$
  - Rewrite, $y = \dfrac{1}{\cos x}$
  - Use quotient rule, $\dfrac{dy}{dx} = \dfrac{\cos x(0) - (1)(-\sin x)}{\cos^2 x}$
  - Rearrange, $\dfrac{dy}{dx} = \dfrac{\sin x}{\cos^2 x}$
  - Separate, $\dfrac{dy}{dx} = \dfrac{1}{\cos x} \times \dfrac{\sin x}{\cos x}$
  - Rewrite, $\dfrac{dy}{dx} = \sec x \tan x$
- Similarly, for $y = \operatorname{cosec} x$
  - $y = \dfrac{1}{\sin x}$
  - $\dfrac{dy}{dx} = \dfrac{\sin x(0) - (1)\cos x}{\sin^2 x}$
  - $\dfrac{dy}{dx} = \dfrac{-\cos x}{\sin^2 x}$
  - $\dfrac{dy}{dx} = -\dfrac{1}{\sin x} \times \dfrac{\cos x}{\sin x}$
  - $\dfrac{dy}{dx} = -\operatorname{cosec} x \cot x$

www.thrivingscholars.com



## What do the derivatives of reciprocal trig look like with a linear functions of x?

- For linear functions of the form ax+b
  - $f(x) = \sec(ax + b)$
    - $f'(x) = a \sec(ax + b) \tan(ax + b)$
  - $f(x) = \operatorname{cosec}(ax + b)$
    - $f'(x) = -a \operatorname{cosec}(ax + b) \cot(ax + b)$
  - $f(x) = \cot(ax + b)$
    - $f'(x) = -a \operatorname{cosec}^2(ax + b)$
  - These are not given in the formula booklet
    - they can be derived from chain rule
    - they are not essential to remember

### Exam Tip

- Even if you think you have remembered these derivatives, always use the formula booklet to double check
  - those squares and negatives are easy to get muddled up!
- Where two trig functions are involved in the derivative be careful with the angle multiple; $x$, $2x$, $3x$, etc
  - An example of a common mistake is differentiating $y = cosec\ 3x$
    - $\frac{dy}{dx} = -3\ cosec\ x\ cot\ 3x$ instead of $\frac{dy}{dx} = -3\ cosec\ 3x\ cot\ 3x$

www.thrivingscholars.com

## Worked Example

Curve C has equation $y = 2\cot\left(3x - \frac{\pi}{8}\right)$.

a)
Show that the derivative of $\cot x$ is $-\text{cosec}^2 x$.

$y = \cot x = \frac{\cos x}{\sin x}$

Quotient rule

$u = \cos x \quad v = \sin x$

$u' = -\sin x \quad v' = \cos x$

$\therefore \frac{dy}{dx} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$

$\frac{dy}{dx} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$\sin^2 x + \cos^2 x \equiv 1$

$\therefore \frac{dy}{dx} = -\frac{1}{\sin^2 x}$

$\boxed{\frac{dy}{dx} = -\text{cosec}^2 x}$

b) Find $\frac{dy}{dx}$ for curve C.

Chain rule/Linear function of $x$

$\frac{dy}{dx} = -2\text{cosec}^2\left(3x - \frac{\pi}{8}\right) \times 3$

$\boxed{\therefore \frac{dy}{dx} = -6\text{cosec}^2\left(3x - \frac{\pi}{8}\right)}$

c) Find the gradient of curve C at the point where $x = \frac{7\pi}{24}$.

www.thrivingscholars.com



When $x = \frac{7\pi}{24}$

$$\frac{dy}{dx} = -6\,\text{cosec}^2\left(\frac{\cancel{21}^{7}\pi}{\cancel{24}_{8}} - \frac{\pi}{8}\right)$$

$$\frac{dy}{dx} = \frac{-6}{\sin^2\left(\frac{3\pi}{4}\right)} = \frac{-6}{\left(\frac{\sqrt{2}}{2}\right)^2}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=\frac{7\pi}{24}} = -12$$

Your GDC may be able to do this directly

www.thrivingscholars.com

# Differentiating Inverse Trigonometric Functions

## What are the inverse trigonometric functions?

- **arcsin**, **arccos** and **arctan** are functions defined as the inverse functions of sine, cosine and tangent respectively
  - $\arcsin\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$ which is equivalent to $\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$
  - $\arctan(-1)=\frac{3\pi}{4}$ which is equivalent to $\tan\left(\frac{3\pi}{4}\right)=-1$

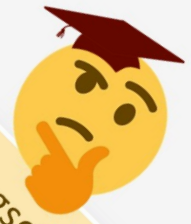
## What are the derivatives of the inverse trigonometric functions?

- $f(x)=\arcsin x$
  - $f'(x)=\frac{1}{\sqrt{1-x^2}}$
- $f(x)=\arccos x$
  - $f'(x)=-\frac{1}{\sqrt{1-x^2}}$
- $f(x)=\arctan x$
  - $f'(x)=\frac{1}{1+x^2}$
- Unlike other derivatives these look completely unrelated at first
  - their derivation involves use of the identity $\cos^2 x+\sin^2 x\equiv 1$
  - hence the squares and square roots!
- All three are given in the **formula booklet**
- Note with the derivative of $\arctan x$ that $(1+x^2)$ is the same as $(x^2+1)$

## How do I show or prove the derivatives of the inverse trigonometric functions?

- For $y=\arcsin x$
  - Rewrite, $\sin y=x$
  - Differentiate implicitly, $\cos y\frac{\mathrm{d}y}{\mathrm{d}x}=1$
  - Rearrange, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\cos y}$
  - Using the identity $\cos^2 y\equiv 1-\sin^2 y$ rewrite, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{1-\sin^2 y}}$
  - Since, $\sin y=x$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{1-x^2}}$
- Similarly, for $y=\arccos x$
  - $\cos y=x$
  - $-\sin y\frac{\mathrm{d}y}{\mathrm{d}x}=1$
  - $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\sin y}$

www.thrivingscholars.com



- $\frac{dy}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}}$
- $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

- Notice how the derivative of $y = \arcsin x$ is positive but is negative for $y = \arccos x$
  - This subtle but crucial difference can be seen in their graphs
    - $y = \arcsin x$ has a positive gradient for all values of $x$ in its domain
    - $y = \arccos x$ has a negative gradient for all values of $x$ in its domain

## What do the derivative of inverse trig look like with a linear function of x?

- For linear functions of the form $ax + b$
- $f(x) = \arcsin(ax + b)$
  - $f'(x) = \frac{a}{\sqrt{1-(ax+b)^2}}$
- $f(x) = \arccos(ax + b)$
  - $f'(x) = \frac{a}{\sqrt{1-(ax+b)^2}}$
- $f(x) = \arctan(ax + b)$
  - $f'(x) = \frac{a}{1+(ax+b)^2}$
- These are **not** in the formula booklet
  - they can be derived from chain rule
  - they are not essential to remember
  - they are not commonly used

### Exam Tip

- For $f(x) = \arctan x$ the terms on the denominator can be reversed (as they are being added rather than subtracted)
  - $f'(x) = \frac{1}{1+x^2} = \frac{1}{x^2+1}$
  - Don't be fooled by this, it sounds obvious but on awkward "show that" questions it can be off-putting!

www.thrivingscholars.com

## Worked Example

a) Show that the derivative of $\arctan x$ is $\dfrac{1}{1+x^2}$

$y = \arctan x$

$\tan y = x$

Differentiate implicitly

$\sec^2 y \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$

Using the identity $\tan^2 y + 1 \equiv \sec^2 y$

$\dfrac{dy}{dx} = \dfrac{1}{\tan^2 y + 1}$

Since $\tan y = x$

$\dfrac{dy}{dx} = \dfrac{1}{x^2+1}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

b)

Find the derivative of $\arctan(5x^3 - 2x)$.

$5x^3 - 2x$ is not a linear function of $x$, use chain rule

$y = \arctan(5x^3 - 2x)$

$\dfrac{dy}{dx} = \dfrac{1}{1+(5x^3-2x)^2} \times (15x^2 - 2)$

$\therefore \dfrac{dy}{dx} = \dfrac{15x^2-2}{1+(5x^3-2x)^2}$

www.thrivingscholars.com



# Differentiating Exponential & Logarithmic Functions

## What are exponential and logarithmic functions?

- Exponential functions have term(s) where the variable ($x$) is the power (exponent)
  - In general, these would be of the form $y = a^x$
    - The special case of this is when $a = e$, i.e. $y = e^x$
- Logarithmic functions have term(s) where the logarithms of the variable ($x$) are involved
  - In general, these would be of the form $y = \log_a x$
    - The special case of this is when $a = e$, i.e. $y = \log_e x = \ln x$

## What are the derivatives of exponential functions?

- The first two results, of the special cases above, have been met before
  - $f(x) = e^x, \ f'(x) = e^x$
  - $f(x) = \ln x, \ f'(x) = \dfrac{1}{x}$
  - These are given in the formula booklet
- For the general forms of exponentials and logarithms
  - $f(x) = a^x$
    - $f'(x) = a^x(\ln a)$
  - $f(x) = \log_a x$
    - $f'(x) = \dfrac{1}{x \ln a}$
  - These are also given in the **formula booklet**

## How do I show or prove the derivatives of exponential and logarithmic functions?

- For $y = a^x$
  - Take natural logarithms of both sides, $\ln y = x \ln a$
  - Use the laws of logarithms, $\ln y = x \ln a$
  - Differentiate, implicitly, $\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = \ln a$
  - Rearrange, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y \ln a$
  - Substitute for $y$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = a^x \ln a$
- For $y = \log_a x$
  - Rewrite, $x = a^y$
  - Differentiate $x$ with respect to $y$, using the above result, $\dfrac{\mathrm{d}x}{\mathrm{d}y} = a^y \ln a$
  - Using $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\dfrac{\mathrm{d}x}{\mathrm{d}y}}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{a^y \ln a}$
  - Substitute for $y$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{a^{\log_a x} \ln a}$

www.thrivingscholars.com



- Simplify, $\frac{dy}{dx} = \frac{1}{x \ln a}$

## What do the derivatives of exponentials and logarithms look like with a linear functions of x?

- For linear functions of the form $px + q$
  - $f(x) = a^{px+q}$
    - $f'(x) = pa^{px+q}(\ln a)$
  - $f(x) = \log_a(px + q)$
    - $f'(x) = \frac{p}{(px+q)\ln a}$
  - These are **not** in the formula booklet
    - they can be derived from chain rule
    - they are not essential to remember

### Exam Tip

- For questions that require the derivative in a particular format, you may need to use the laws of logarithms
  - With ln appearing in denominators be careful with the division law
    - $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
    - but $\frac{\ln a}{\ln b}$ cannot be simplified (unless there is some numerical connection between $a$ and $b$)

www.thrivingscholars.com

## Worked Example

a)

Find the derivative of $a^{3x-2}$.

Chain rule or 'px+q shortcut' is required

$\frac{d}{dx}[a^{3x-2}] = a^{3x-2} \ln a \times 3$

∴ The derivative of $a^{3x-2}$ is $3a^{3x-2} \ln a$

b) Find an expression for $\frac{dy}{dx}$ given that $y = \log_5(2x^3)$

Chain rule is needed.

$\frac{dy}{dx} = \frac{1}{2x^3 \ln 5} \times 6x^2$

$\therefore \frac{dy}{dx} = \frac{3}{x^2 \ln 5}$

www.thrivingscholars.com



# 5.10 Differential Equations

## 5.10.1 Numerical Solutions to Differential Equations

### First Order Differential Equations

### What is a differential equation?

- A **differential equation** is simply an equation that contains derivatives
  - For example $\frac{dy}{dx} = 12xy^2$ is a differential equation
  - And so is $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 7x = 5\sin t$

### What is a first order differential equation?

- A **first order differential equation** is a differential equation that contains first derivatives but no second (or higher) derivatives
  - For example $\frac{dy}{dx} = 12xy^2$ is a first order differential equation
  - But $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 7x = 5\sin t$ is **not** a first order differential equation, because it contains the second derivative $\frac{d^2x}{dt^2}$

### Wait – haven't I seen first order differential equations before?

- Yes you have!
  - For example $\frac{dy}{dx} = 3x^2$ is also a first order differential equation, because it contains a first derivative and no second (or higher) derivatives
  - But for that equation you can just integrate to find the solution $y = x^3 + c$ (where c is a constant of integration)
- In this section of the course you learn how to solve differential equations that can't just be solved right away by integrating

www.thrivingscholars.com

# Euler's Method: First Order

## What is Euler's method?

- **Euler's method** is a numerical method for finding approximate solutions to differential equations
- It treats the derivatives in the equation as being constant over short 'steps'
- The accuracy of the Euler's Method approximation can be improved by making the step sizes smaller

## How do I use Euler's method with a first order differential equation?

- STEP 1: Make sure your differential equation is in $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ form
- STEP 2: Write down the recursion equations using the formulae $y_{n+1} = y_n + h \times f(x_n, y_n)$ and $x_{n+1} = x_n + h$ from the exam formula booklet
  - $h$ in those equations is the **step size**
  - the exam question will usually tell you the correct value of $h$ to use
- STEP 3: Use the recursion feature on your GDC to calculate the Euler's method approximation over the correct number of steps
  - the values for $x_0$ and $y_0$ will come from the boundary conditions given in the question

### Exam Tip

- Be careful with letters – in the equations in the exam, and in your GDC's recursion calculator, the variables may not be $x$ and $y$
- If an exam question asks you how to improve an Euler's method approximation, the answer will almost always have to do with decreasing the step size!

www.thrivingscholars.com

## Worked Example

Consider the differential equation $\frac{dy}{dx} + y = x + 1$ with the boundary condition $y(0) = 0.5$.

a)

Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 1$.

| Euler's method | $y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$ | $h$ is a constant (step length) |
|---|---|---|

} from formula booklet

STEP 1: $\frac{dy}{dx} = \underbrace{x - y + 1}_{f(x,y)}$

STEP 2: $y_{n+1} = y_n + \underbrace{0.2}_{h \text{ (from question)}} \times \underbrace{(x_n - y_n + 1)}_{f(x_n, y_n)}$ $\quad x_{n+1} = x_n + 0.2$

STEP 3: We need to get x from 0 to 1, so we will need $\frac{1-0}{0.2} = 5$ steps.

$y(0) = 0.5$

| n | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 0 | 0.5 |
| 1 | 0.2 | 0.6 |
| 2 | 0.4 | 0.72 |
| 3 | 0.6 | 0.856 |
| 4 | 0.8 | 1.0048 |
| 5 | 1 | 1.16384 |

} from GDC

$y(1) = 1.16$ (3 s.f.)

b)

Explain how the accuracy of the approximation in part (a) could be improved.

Make the step size smaller.

www.thrivingscholars.com



# Separation of Variables

## What is separation of variables?

- **Separation of variables** can be used to solve certain types of first order differential equations
- Look out for equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y)$
  - i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a function of $x$ multiplied by a function of $y$
  - be careful – the 'function of $x$' $g(x)$ may just be a constant!
    - For example in $\frac{\mathrm{d}y}{\mathrm{d}x} = 6y$, $g(x) = 6$ and $h(y) = y$
- If the equation is in that form you can use separation of variables to try to solve it
- If the equation is not in that form you will need to use another solution method

## How do I solve a differential equation using separation of variables?

- STEP 1: Rearrange the equation into the form $\left(\frac{1}{h(y)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$
- STEP 2: Take the integral of both sides to change the equation into the form

$$\int \frac{1}{h(y)}\,\mathrm{d}y = \int g(x)\,\mathrm{d}x$$

  - You can think of this step as 'multiplying the $\mathrm{d}x$ across and integrating both sides'
    - Mathematically that's not *quite* what is actually happening, but it will get you the right answer here!
- STEP 3: Work out the integrals on both sides of the equation to find the **general solution** to the differential equation
  - Don't forget to include a constant of integration
    - Although there are two integrals, you only need to include one constant of integration
  - Look out for integrals that require you to use **partial fractions** to solve them
    - See 'Integrating with Partial Fractions' in 5.9 Advanced Integration
- STEP 4: Use any boundary or initial conditions in the question to work out the value of the integration constant
- STEP 5: If necessary, rearrange the solution into the form required by the question

### Exam Tip

- Be careful with letters – the equation on an exam may not use $x$ and $y$ as the variables
- Unless the question asks for it, you don't have to change your solution into $y = f(x)$ form – sometimes it might be more convenient to leave your solution in another form

www.thrivingscholars.com

## Worked Example

For each of the following differential equations, either (i) solve the equation by using separation of variables giving your answer in the form $y = f(x)$, or (ii) state why the equation may not be solved using separation of variables.

a) $\dfrac{dy}{dx} = \dfrac{e^x + 4x}{3y^2}$.

STEP 1: $3y^2 \frac{dy}{dx} = e^x + 4x$ $\quad g(x) = e^x + 4x$ $\quad h(y) = \frac{1}{3y^2}$

STEP 2: $\int 3y^2 \, dy = \int (e^x + 4x) \, dx$

STEP 3: $y^3 = e^x + 2x^2 + c$ ← Don't forget constant of integration

STEP 4: No boundary conditions given, so skip step

STEP 5: $y = \sqrt[3]{e^x + 2x^2 + c}$ $\quad y = f(x)$

b) $\dfrac{dy}{dx} = 4xy - 2\ln x$.

$4xy - 2\ln x$ is not of the form $g(x)h(y)$, so it may not be solved using separation of variables.

c) $\dfrac{dy}{dx} = 2y^2 + 2y$, given that $y = 2$ when $x = 0$.

www.thrivingscholars.com



STEP 1: $\frac{1}{y^2+y}\frac{dy}{dx} = 2 \qquad g(x) = 2 \quad h(y) = y^2 + y$

STEP 2: $\int \frac{1}{y^2+y}\,dy = \int 2\,dx$

STEP 3: $\int \frac{1}{y^2+y}\,dy = \int \left(\frac{1}{y} - \frac{1}{y+1}\right)dy = \ln\left|\frac{y}{y+1}\right| + c$ (partial fractions)

$\ln\left|\frac{y}{y+1}\right| = 2x + c$ ← Don't forget constant of integration

STEP 4: $\ln\left|\frac{2}{2+1}\right| = 2(0) + c \implies c = \ln\left(\frac{2}{3}\right)$ ← $y = 2$ when $x = 0$

STEP 5: For the boundary condition $y = 2$, $\frac{y}{y+1} > 0$.
Therefore we can drop the modulus sign from $\left|\frac{y}{y+1}\right|$.

$$\frac{y}{y+1} = e^{2x + \ln(2/3)} = \left(e^{2x}\right)\left(e^{\ln(2/3)}\right) = \frac{2}{3}e^{2x}$$

$$\implies \boxed{y = \frac{2e^{2x}}{3 - 2e^{2x}}} \qquad y = f(x)$$

www.thrivingscholars.com



# Homogeneous Differential Equations

## What is a homogeneous first order differential equation?

- If a first order differential equation can be written in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ then it is said to be **homogeneous**

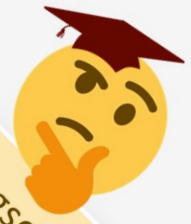
## How do I solve a homogeneous first order differential equation?

- These equations can be solved using the substitution $v = \frac{y}{x} \Leftrightarrow y = vx$
- STEP 1: If necessary, rearrange the equation into the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$
- STEP 2: Replace all instances of $\frac{y}{x}$ in your equation with $v$
- STEP 3: Use the product rule and implicit differentiation to replace $\frac{dy}{dx}$ in your equation with $v + x\frac{dv}{dx}$
  - This is because $y = vx \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx) = v\frac{d}{dx}(x) + x\frac{d}{dx}(v) = v + x\frac{dv}{dx}$
- STEP 4: Solve your new differential equation to find the solution in terms of $v$ and $x$
  - You may need to use other methods for differential equations, such as **separation of variables**, at this stage
- STEP 5: Substitute $v = \frac{y}{x}$ into the solution from Step 4, in order to find the solution in terms of $y$ and $x$

## What else should I know about solving homogeneous first order differential equations?

- After finding the solution in terms of $y$ and $x$ you may be asked to do other things with the solution
  - For example you may be asked to find the solution corresponding to certain initial or boundary conditions
  - Or you may be asked to express your answer in a particular form, such as $y = f(x)$
- It is sometimes possible to solve differential equations that are *not* homogeneous by using the substitution $v = \frac{y}{x}$
  - For such a situation in an exam question, you would be told explicitly to use the substitution
  - You would not be expected to know that you could use the substitution in a case where the differential equation was not homogeneous

www.thrivingscholars.com



### Exam Tip

- Unless the question asks for it, you don't have to change your solution into $y = f(x)$ form – sometimes it might be more convenient to leave your solution in another form

www.thrivingscholars.com

## Worked Example

Consider the differential equation $xy\dfrac{dy}{dx} = y^2 - x^2$ where $y = 3$ when $x = 1$.

a)

Show that the differential equation is homogeneous.

$xy\frac{dy}{dx} = y^2 - x^2$

$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{xy} = \frac{y}{x} - \frac{x}{y} = (y/x) - \frac{1}{(y/x)} = f\left(\frac{y}{x}\right)$

$\Rightarrow$ The equation is homogeneous.

b)

Use the substitution $v = \dfrac{y}{x}$ to solve the differential equation with the given boundary condition.

STEP 1: $\frac{dy}{dx} = (y/x) - \frac{1}{(y/x)}$ } from part (a)

STEP 2: Let $v = \frac{y}{x}$. Then $\frac{dy}{dx} = v - \frac{1}{v}$.

STEP 3: And $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

So $v + x\frac{dv}{dx} = v - \frac{1}{v} \Rightarrow x\frac{dv}{dx} = -\frac{1}{v}$

STEP 4: $\int 2v\,dv = \int \frac{-2}{x}\,dx$ Separation of variables

$v^2 = -2\ln|x| + c = c - \ln(x^2)$ Don't forget constant of integration

STEP 5: $\left(\frac{y}{x}\right)^2 = c - \ln(x^2) \Rightarrow y^2 = x^2(c - \ln(x^2))$

Now use the boundary condition

And $y(1) = 3$, so $(3)^2 = (1)^2(c - \ln(1^2)) \Rightarrow c = 9$

$y^2 = x^2(9 - \ln(x^2))$

Question doesn't ask for solution to be in y=f(x) form, so it's easiest just to keep it like this.

www.thrivingscholars.com



# Integrating Factor

## What is an integrating factor?

- An **integrating factor** can be used to solve a differential equation that can be written in the **standard form** $\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$
  - Be careful – the 'functions of $x$' $p(x)$ and $q(x)$ may just be constants!
    - For example in $\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = \mathrm{e}^{-2x}$, $p(x) = 6$ and $q(x) = \mathrm{e}^{-2x}$
    - While in $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{2x} = 12$, $p(x) = \frac{1}{2x}$ and $q(x) = 12$
- For an equation in standard form, the integrating factor is $\mathrm{e}^{\int p(x)\,\mathrm{d}x}$

## How do I use an integrating factor to solve a differential equation?

- STEP 1: If necessary, rearrange the differential equation into standard form
- STEP 2: Find the integrating factor
  - Note that you don't need to include a constant of integration here when you integrate $\int p(x)\,\mathrm{d}x$
- STEP 3: Multiply both sides of the differential equation by the integrating factor
- This will turn the equation into an **exact differential equation** of the form $\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{\int p(x)\,\mathrm{d}x}\right) = q(x)\mathrm{e}^{\int p(x)\,\mathrm{d}x}$
- STEP 4: Integrate both sides of the equation with respect to $x$
  - The left side will automatically integrate to $y\mathrm{e}^{\int p(x)\,dx}$
  - For the right side, integrate $\int q(x)\mathrm{e}^{\int p(x)\mathrm{d}x}\mathrm{d}x$ using your usual techniques for integration
  - Don't forget to include a constant of integration
    - Although there are two integrals, you only need to include one constant of integration
- STEP 5: Rearrange your solution to get it in the form $y = f(x)$

## What else should I know about using an integrating factor to solve differential equations?

- After finding the general solution using the steps above you may be asked to do other things with the solution
  - For example you may be asked to find the solution corresponding to certain initial or boundary conditions

www.thrivingscholars.com

## Worked Example

Consider the differential equation $\frac{dy}{dx} = 2xy + 5e^{x^2}$ where $y = 7$ when $x = 0$.

Use an integrating factor to find the solution to the differential equation with the given boundary condition.

Integrating factor for $y' + P(x)y = Q(x)$ : $e^{\int P(x)dx}$ } from formula booklet

STEP 1: $\frac{dy}{dx} - 2xy = 5e^{x^2}$ $\quad p(x) = -2x \quad q(x) = 5e^{x^2}$

STEP 2: $e^{\int -2x\,dx} = e^{-x^2}$

STEP 3: $\left(\frac{dy}{dx} - 2xy\right)e^{-x^2} = \left(5e^{x^2}\right)e^{-x^2}$

$$e^{-x^2}\frac{dy}{dx} - 2xye^{-x^2} = 5$$

$\frac{d}{dx}\left(ye^{-x^2}\right) = 5$ Exact differential equation

STEP 4: $ye^{-x^2} = \int 5\,dx = 5x + c$ Don't forget constant of integration

STEP 5: $y = e^{x^2}(5x + c)$

Now use the boundary condition

$7 = e^0(5(0) + c) \implies c = 7$

$$\boxed{y = e^{x^2}(5x + 7)}$$

www.thrivingscholars.com





# Modelling with Differential Equations

## Why are differential equations used to model real-world situations?

- A **differential equation** is an equation that contains one or more derivatives
- Derivatives deal with rates of change, and with the way that variables change with respect to one another
- Therefore differential equations are a natural way to model real-world situations involving change
    - Most frequently in real-world situations we are interested in how things change over time, so the derivatives used will usually be with respect to time $t$

## How do I set up a differential equation to model a situation?

- An exam question may require you to create a differential equation from information provided
- The question will provide a context from which the differential equation is to be created
- Most often this will involve the rate of change of a variable being proportional to some function of the variable
        - For example, the rate of change of a population of bacteria, $P$, at a particular time may be proportional to the size of the population at that time
- The expression 'rate of' ('rate of change of…', 'rate of growth of…', etc.) in a modelling question is a strong hint that a differential equation is needed, involving derivatives with respect to time $t$
        - So with the bacteria example above, the equation will involve the derivative $\frac{\mathrm{d}P}{\mathrm{d}t}$
- Recall the basic equation of proportionality
    - If $y$ is proportional to $x$, then $y = kx$ for some **constant of proportionality** $k$
        - So for the bacteria example above the differential equation needed would be $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$
    - The precise value of $k$ will generally not be known at the start, but will need to be found as part of the process of solving the differential equation
    - It can often be useful to assume that $k > 0$ when setting up your equation
        - In this case, $-k$ will be used in the differential equation in situations where the rate of change is expected to be negative
        - So in the bacteria example, if it were known that the population of bacteria was decreasing, then the equation could instead be written $\frac{\mathrm{d}P}{\mathrm{d}t} = -kP$

www.thrivingscholars.com

## Worked Example

a)

In a particular pond, the rate of change of the area covered by algae, $A$, at any time $t$ is directly proportional to the square root of the area covered by algae at that time. Write down a differential equation to model this situation.

$$\frac{dA}{dt} = k\sqrt{A}$$ (where $k$ is a constant of proportionality)

b)

Newton's Law of Cooling states that the rate of change of the temperature of an object, $T$, at any time $t$ is proportional to the difference between the temperature of the object and the ambient temperature of its surroundings, $T_a$, at that time. Assuming that the object starts off warmer than its surroundings, write down the differential equation implied by Newton's Law of Cooling.

The object is assumed to be warmer than its surroundings, so $T - T_a > 0$

$$\frac{dT}{dt} = -k(T - T_a)$$

(where $k > 0$ is a constant of proportionality)

We expect the temperature to be decreasing, so $-k$ in the equation combined with $k > 0$ assures that $\frac{dT}{dt}$ is negative.

www.thrivingscholars.com

# The Logistic Equation

## What is the logistic equation?

- The differential equation $\frac{\mathrm{d}N}{\mathrm{d}t} = kN$ is a very simple example of a model in which the rate of change of a population at any moment in time is dependent on the size of the population ($N$) at that time
  - The solution is $N = A\mathrm{e}^{kt}$ (where $A > 0$ is a constant)
  - If $k > 0$, this represents unlimited exponential growth of the variable $N$
- In many real-world contexts (for example when considering populations of living organisms), unlimited growth is not a realistic modelling assumption
  - For reproducing populations it is logical to assume that the rate of change of the population will be dependent on the size of the population (more rabbits means more production of baby rabbits!)
  - But there are generally limiting factors on populations that prevent them from growing without limits
    - For example, availability of food or other resources, or the presence of predators or other threats, may limit the population that can exist in a given area
- A **logistic equation** incorporates such limiting factors into the model, and therefore can provide a more realistic model for real-world populations
- The **standard logistic equation** is of the form

$$\frac{\mathrm{d}N}{\mathrm{d}t} = kN(a - N)$$

  - $t$ represents the time (since the moment defined as $t = 0$) that the population has been growing
  - $N$ represents the size of the population at time $t$
  - $k \in \mathbb{R}$ is a constant determining the relative rate of population growth
    - For the models dealt with here it is most common to have $k > 0$, with a larger value of $k$ representing a faster rate of change
  - $a \in \mathbb{R}$ is a constant that places a limit on the maximum size to which the population $N$ can grow
    - For a population model it can be assumed that $a > 0$
    - For $k > 0$ and an initial population $N_0$ such that $0 < N_0 < a$, the population $N$ will grow and will converge to the value $a$ as time $t$ increases
    - For $k > 0$ and an initial population $N_0$ such that $0 < N_0 < a$, the population $N$ will shrink and will converge to the value $a$ as time $t$ increases
- There are other forms of logistic equation
  - The exact form of the logistic equation you are to use will always be given in an exam question

## How do I solve problems that involve a logistic equation model?

- Solving the differential equation will generally involve the technique of **separation of variables**
  - Usually this will also involve rearranging one of the integrals using **partial fractions** (see the worked example below for an example)
- You will usually be given 'boundary conditions' specific to the context of the problem

www.thrivingscholars.com



    - For example, you may be told the initial population at time $t = 0$
    - These conditions will allow you to work out the exact value of any integrating constants that occur while solving the differential equation
- You will need to take account of the context of the question in answering the question or in commenting on the model used

www.thrivingscholars.com

## Worked Example

A group of ecologists are studying a population of rabbits on a particular island. The population of rabbits, $N$, on the island is modelled by the logistic equation

$$\frac{dN}{dt} = 0.0012N(1500 - N)$$

where $t$ represents the time in years since the ecologists began their study. At the time the study begins there are 300 rabbits on the island.

a)

Show that the population of rabbits at time $t$ years is given by $N = \frac{1500e^{1.8t}}{4 + e^{1.8t}}$.

$\int \frac{1}{N(1500-N)}\,dN = 0.0012\int dt$ Separation of variables

$\int \left(\frac{1}{N} - \frac{1}{N-1500}\right)dN = 1.8\int dt$ Partial fractions

$\ln\left|\frac{N}{N-1500}\right| = 1.8t + c$ Don't forget constant of integration

$\frac{N}{1500-N} = Ae^{1.8t}$ $A = e^c$

$0 < N < 1500$, so $\left|\frac{N}{N-1500}\right| = -\frac{N}{N-1500} = \frac{N}{1500-N}$

$N(0) = 300$, so $\frac{300}{1500-300} = Ae^{1.8(0)} \Rightarrow A = \frac{1}{4}$ Use initial condition

$\Rightarrow \frac{N}{1500-N} = \frac{1}{4}e^{1.8t} \Rightarrow \frac{4N}{1500-N} = e^{1.8t}$

$\Rightarrow N = \frac{1500e^{1.8t}}{4 + e^{1.8t}}$ Rearrange

b)

Find the population of rabbits that the model predicts will be on the island two years after the beginning of the study.

www.thrivingscholars.com

$$N(2) = \frac{1500\,e^{1.8(2)}}{4 + e^{1.8(2)}} = \frac{1500\,e^{3.6}}{4 + e^{3.6}} = 1352.210\ldots$$

**1352 rabbits** Round to nearest rabbit

c)
Determine the maximum size that the model predicts the population of rabbits can grow to. Justify your answer by an appropriate analysis of the equation in part (a).

Take limit to determine long-term behaviour :

$$\lim_{t\to\infty} \frac{1500\,e^{1.8t}}{4 + e^{1.8t}} = \lim_{t\to\infty} \frac{1500}{\frac{4}{e^{1.8t}} + 1} = \frac{1500}{0+1} = 1500$$

**1500 rabbits is the maximum population predicted by the model.**

www.thrivingscholars.com

# 5.11 MacLaurin Series

## 5.11.1 Maclaurin Series

### Maclaurin Series of Standard Functions

### What is a Maclaurin Series?

- A Maclaurin series is a way of representing a function as an infinite sum of increasing integer powers of $x$ ($x^1$, $x^2$, $x^3$, etc.)
  - If all of the infinite number of terms are included, then the Maclaurin series is exactly equal to the original function
  - If we **truncate** (i.e., shorten) the Maclaurin series by stopping at some particular power of $x$, then the Maclaurin series is only an approximation of the original function
- A truncated Maclaurin series will always be exactly equal to the original function for $x = 0$
- In general, the approximation from a truncated Maclaurin series becomes less accurate as the value of $x$ moves further away from zero
- The accuracy of a truncated Maclaurin series approximation can be improved by including more terms from the complete infinite series
  - So, for example, a series truncated at the $x^7$ term will give a more accurate approximation than a series truncated at the $x^3$ term

### How do I find the Maclaurin series of a function 'from first principles'?

- Use the **general Maclaurin series formula**

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

- This formula is in your exam formula booklet
- STEP 1: Find the values of $f(0)$, $f'(0)$, $f''(0)$, etc. for the function
  - An exam question will specify how many terms of the series you need to calculate (for example, "up to and including the term in $x^4$")
  - You may be able to use your GDC to find these values directly without actually having to find all the necessary derivatives of the function first
- STEP 2: Put the values from Step 1 into the general Maclaurin series formula
- STEP 3: Simplify the coefficients as far as possible for each of the powers of $x$

### Is there an easier way to find the Maclaurin series for standard functions?

- Yes there is!
- The following Maclaurin series expansions of standard functions are contained in your exam formula booklet:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

www.thrivingscholars.com

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

- Unless a question specifically asks you to derive a Maclaurin series using the general Maclaurin series formula, you can use those standard formulae from the exam formula booklet in your working

## Is there a connection Maclaurin series expansions and binomial theorem series expansions?

- Yes there is!
- For a function like $(1+x)^n$ the binomial theorem series expansion is **exactly the same** as the Maclaurin series expansion for the same function
    - So unless a question specifically tells you to use the general Maclaurin series formula, you can use the binomial theorem to find the Maclaurin series for functions of that type
    - Or if you've forgotten the binomial series expansion formula for $(1+x)^n$ where $n$ is not a positive integer, you can find the binomial theorem expansion by using the general Maclaurin series formula to find the Maclaurin series expansion

www.thrivingscholars.com

## Worked Example

a)

Use the Maclaurin series formula to find the Maclaurin series for $f(x) = \sqrt{1+2x}$ up to and including the term in $x^4$.

$f(x) = \sqrt{1+2x} = (1+2x)^{\frac{1}{2}}$

STEP 1: $f(0) = 1 \quad f'(0) = 1 \quad f''(0) = -1$

$f'''(0) = 3 \quad f^{(4)}(0) = -15$

STEP 2: $f(x) = 1 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(3) + \frac{x^4}{4!}(-15) + \ldots$

STEP 3: Up to the $x^4$ term,

$\sqrt{1+2x} = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4$

Note: This is the same as the binomial theorem expansion of $(1+2x)^{\frac{1}{2}}$

b)

Use your answer from part (a) to find an approximation for the value of $\sqrt{1.02}$, and compare the approximation found to the actual value of the square root.

www.thrivingscholars.com

Up to the $x^4$ term,

$$\sqrt{1+2x} = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4$$

} from part (a)

Let $x = 0.01$. Then $\sqrt{1+2x} = \sqrt{1+2(0.01)} = \sqrt{1.02}$.

So

$$\sqrt{1.02} \approx 1 + (0.01) - \frac{1}{2}(0.01)^2 + \frac{1}{2}(0.01)^3 - \frac{5}{8}(0.01)^4$$

$$\sqrt{1.02} \approx 1.00995049375$$

The exact value of the square root is

$$\sqrt{1.02} = 1.009950493836...$$

The approximation is accurate to 10 d.p. or 11 s.f.

www.thrivingscholars.com



# Maclaurin Series of Composites & Products

## How can I find the Maclaurin series for a composite function?

- A **composite function** is a 'function of a function' or a 'function within a function'
  - For example $\sin(2x)$ is a composite function, with $2x$ as the 'inside function' which has been put into the simpler 'outside function' $\sin x$
  - Similarly $e^{x^2}$ is a composite function, with $x^2$ as the 'inside function' and $e^x$ as the 'outside function'
- To find the Maclaurin series for a composite function:
  - STEP 1: Start with the Maclaurin series for the basic 'outside function'
    - Usually this will be one of the 'standard functions' whose Maclaurin series are given in the exam formula booklet
  - STEP 2: Substitute the 'inside function' every place that $x$ appears in the Maclaurin series for the 'outside function'
    - So for $\sin(2x)$, for example, you would substitute $2x$ everywhere that $x$ appears in the Maclaurin series for $\sin x$
  - STEP 3: Expand the brackets and simplify the coefficients for the powers of $x$ in the resultant Maclaurin series
- This method can theoretically be used for quite complicated 'inside' and 'outside' functions
  - On your exam, however, the 'inside function' will usually not be more complicated than something like $kx$ (for some constant $k$) or $x^n$ (for some constant power $n$)

## How can I find the Maclaurin series for a product of two functions?

- To find the Maclaurin series for a product of two functions:
  - STEP 1: Start with the Maclaurin series of the individual functions
    - For each of these Maclaurin series you should only use terms up to an appropriately chosen power of $x$ (see the worked example below to see how this is done!)
  - STEP 2: Put each of the series into brackets and multiply them together
    - Only keep terms in powers of $x$ up to the power you are interested in
  - STEP 3: Collect terms and simplify coefficients for the powers of $x$ in the resultant Maclaurin series

www.thrivingscholars.com

## Worked Example

a)

Find the Maclaurin series for the function $f(x) = \ln(1+3x)$, up to and including the term in $x^4$.

| Maclaurin series for special functions | | |
|---|---|---|
| | $e^x = 1 + x + \frac{x^2}{2!} + \ldots$ | $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ |
| | $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ | $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ |
| | $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ | |

from exam formula booklet

STEP 1: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

STEP 2: $\ln(1+3x) = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \ldots$

STEP 3: $\ln(1+3x) = 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \ldots$

b)

Find the Maclaurin series for the function $g(x) = e^x \sin x$, up to and including the term in $x^4$.

www.thrivingscholars.com

| Maclaurin series for special functions | | |
|---|---|---|
| | $e^x = 1 + x + \frac{x^2}{2!} + \ldots$ | $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ |
| | $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ | $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ |
| | $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ | |

from exam formula booklet

STEP 1: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$ ← Higher powers of x here will give powers higher than 4 when multiplied by the sin x series.

$\sin x = x - \frac{x^3}{6} + \ldots$ ← Don't need terms in powers of x higher than 4

STEP 2: $e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(x - \frac{x^3}{6} + \ldots\right)$

$= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^3}{6} - \frac{x^4}{6} - \frac{x^5}{12} - \frac{x^6}{36}$

Note that the $x^4$ terms cancel out

Discard terms for powers higher than 4

STEP 3: $e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$

www.thrivingscholars.com

# Differentiating & Integrating Maclaurin Series

## How can I use differentiation to find Maclaurin Series?

- If you differentiate the Maclaurin series for a function $f(x)$ term by term, you get the Maclaurin series for the function's derivative $f'(x)$
- You can use this to find new Maclaurin series from existing ones
  - For example, the derivative of $\sin x$ is $\cos x$
  - So if you differentiate the Maclaurin series for $\sin x$ term by term you will get the Maclaurin series for $\cos x$

## How can I use integration to find Maclaurin series?

- If you integrate the Maclaurin series for a derivative $f'(x)$, you get the Maclaurin series for the function $f(x)$
  - Be careful however, as you will have a constant of integration to deal with
  - The value of the constant of integration will have to be chosen so that the series produces the correct value for $f(0)$
- You can use this to find new Maclaurin series from existing ones
  - For example, the derivative of $\sin x$ is $\cos x$
  - So if you integrate the Maclaurin series for $\cos x$ (and correctly deal with the constant of integration) you will get the Maclaurin series for $\sin x$

www.thrivingscholars.com

## Worked Example

a)

(i)

Write down the derivative of $\arctan x$.

(ii)

Hence use the Maclaurin series for $\arctan x$ to derive the Maclaurin series for $\dfrac{1}{1+x^2}$.

.

| Standard derivatives | |
|---|---|
| $\arctan x$ | $f(x)=\arctan x \Rightarrow f'(x)=\dfrac{1}{1+x^2}$ |

| Maclaurin series for special functions | | |
|---|---|---|
| | $e^x = 1+x+\dfrac{x^2}{2!}+\ldots$ | $\ln(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\ldots$ |
| | $\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots$ | $\cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\ldots$ |
| | $\arctan x = x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots$ | |

from exam formula booklet

(i) $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}$

(ii) $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$

$\Rightarrow \dfrac{1}{1+x^2} = \dfrac{d}{dx}\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots\right)$

$\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$

Note: This is the same as the binomial theorem expansion of $(1+x^2)^{-1}$

b)

(i)

Write down the derivative of $-\sin x$.

(ii)

Hence derive the Maclaurin series for $\cos x$, being sure to justify your method.

www.thrivingscholars.com

| Maclaurin series for special functions | | |
|---|---|---|
| | $e^x = 1 + x + \frac{x^2}{2!} + \ldots$ | $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ |
| | $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ | $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ |
| | $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ | |

from exam formula booklet

(i) $$-\sin x = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots$$

(ii) $-\sin x$ is the derivative of $\cos x$, so we can integrate the Maclaurin series for $-\sin x$ to find the Maclaurin series for $\cos x$.

$$\cos x = \int \left(-x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots\right) dx$$

constant of integration

$$= c - \frac{1}{2}x^2 + \frac{1}{4}\cdot\frac{x^4}{3!} - \frac{1}{6}\cdot\frac{x^6}{5!} + \frac{1}{8}\cdot\frac{x^8}{7!} - \ldots$$

$$= c - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

And $\cos(0) = 1$, so $c - \frac{0^2}{2!} + \frac{0^4}{4!} - \frac{0^6}{6!} + \frac{0^8}{8!} - \ldots = 1 \implies c = 1$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

www.thrivingscholars.com





# Maclaurin Series for Differential Equations

## Can I apply Maclaurin Series to solving differential equations?

- If you have a differential equation of the form $\frac{dy}{dx} = g(x,y)$ along with the value of $y(0)$ it is possible to build up the Maclaurin series of the solution $y = f(x)$ term by term
  - This does not necessarily tell you the explicit function of $x$ that corresponds to the Maclaurin series you are finding
  - But the Maclaurin series you find is the exact Maclaurin series for the solution to the differential equation
- The Maclaurin series can be used to approximate the value of the solution $y = f(x)$ for different values of $x$
  - You can increase the accuracy of this approximation by calculating additional terms of the Maclaurin series for higher powers of $x$

## How can I find the Maclaurin Series for the solution to a differential equation?

- STEP 1: Use **implicit differentiation** to find expressions for $y''$, $y'''$ etc., in terms of $x$, $y$ and lower-order derivatives of $y$
  - The number of derivatives you need to find depends on how many terms of the Maclaurin series you want to find
  - For example, if you want the Maclaurin series up to the term, then you will need to find derivatives up to $y^{(4)}$ (the fourth derivative of $y$)
- STEP 2: Using the given initial value for $y(0)$, find the values of $y'(0)$, $y''(0)$, $y'''(0)$, etc., one by one
  - Each value you find will then allow you to find the value for the next higher derivative
- STEP 3: Put the values found in STEP 2 into the **general Maclaurin series formula**

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

  - This formula is in your exam formula booklet
  - $y = f(x)$ is the solution to the differential equation, so $y(0)$ corresponds to $f(0)$ in the formula, $y'(0)$ corresponds to $f'(0)$, and so on
- STEP 4: Simplify the coefficients for each of the powers of $x$ in the resultant Maclaurin series

www.thrivingscholars.com

## Worked Example

Consider the differential equation $y' = y^2 - x$ with the initial condition $y(0) = 2$.

a)

Use implicit differentiation to find expressions for $y''$, $y'''$ and $y^{(4)}$.

STEP 1:

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(y^2 - x) = 2yy' - 1$$

$$\boxed{y'' = 2yy' - 1}$$

$$y''' = \frac{d}{dx}(y'') = \frac{d}{dx}(2yy' - 1) = 2yy'' + 2(y')^2$$

$$\boxed{y''' = 2yy'' + 2(y')^2}$$

$$y^{(4)} = \frac{d}{dx}(y''') = \frac{d}{dx}(2yy'' + 2(y')^2)$$

$$= 2y'y'' + 2yy''' + 4y'y''$$

$$\boxed{y^{(4)} = 6y'y'' + 2yy'''}$$

b)

Use the given initial condition to find the values of $y'(0)$, $y''(0)$, $y'''(0)$ and $y^{(4)} = 0$.

STEP 2:

$y(0) = 2$, so $y'(0) = 2^2 - 0 = 4$ $\quad y' = y^2 - x$

Then $y''(0) = 2(2)(4) - 1 = 15$ $\quad y'' = 2yy' - 1$

$y'''(0) = 2(2)(15) + 2(4)^2 = 92$ $\quad y''' = 2yy'' + 2(y')^2$

$y^{(4)}(0) = 6(4)(15) + 2(2)(92) = 728$ $\quad y^{(4)} = 6y'y'' + 2yy'''$

$$\boxed{\begin{array}{ll} y'(0) = 4 & y''(0) = 15 \\ y'''(0) = 92 & y^{(4)}(0) = 728 \end{array}}$$

Let $y = f(x)$ be the solution to the differential equation with the given initial condition.

c)

Find the first five terms of the Maclaurin series for $f(x)$.

www.thrivingscholars.com



| Maclaurin series | $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$ |
|---|---|

from exam formula booklet

STEP 3: $f(x) = 2 + x(4) + \frac{x^2}{2!}(15) + \frac{x^3}{3!}(92) + \frac{x^4}{4!}(728) + \ldots$

STEP 4:

$$f(x) = 2 + 4x + \frac{15}{2}x^2 + \frac{46}{3}x^3 + \frac{91}{3}x^4 + \ldots$$

www.thrivingscholars.com

# 5.12 Further Limits (inc l'Hôpital's Rule)

## 5.12.1 Further Limits

### l'Hôpital's Rule

### What is l'Hôpital's Rule?

- **l'Hôpital's rule** is a method involving calculus that allows us to find the value of certain limits
- Specifically, it allows us to attempt to evaluate the limit of a quotient $\frac{f(x)}{g(x)}$ for which our usual limit evaluation techniques would return one of the **indeterminate forms** $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

### How do I evaluate a limit using l'Hôpital's Rule?

- STEP 1: Check that the limit of the quotient results in one of the indeterminate forms given above
  - I.e., check that $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f(a)}{g(a)}=\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$
- STEP 2: Find the derivatives of the numerator and denominator of the quotient
- STEP 3: Check whether the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists
- STEP 4: If that limit does exist, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$
- STEP 5: If $\lim_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}=\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ then you may repeat the process by considering $\lim_{x\to a}\frac{f''(x)}{g''(x)}$ (and possibly higher order derivatives after that)
  - As long as the limits continue giving indeterminate forms you may continue applying l'Hôpital's rule
  - Each time this happens find the next set of derivatives and consider the limit again

**Exam Tip**

- Some limits of an indeterminate form can also be evaluated using the **Maclaurin series** for the numerator and denominator
- If an exam question does not specify a method to use, then you are free to use whichever method you prefer

www.thrivingscholars.com

## Worked Example

Use l'Hôpital's rule to evaluate each of the following limits:

a) $\lim_{x \to 0} \frac{\sin x}{e^x - 1}$.

STEP 1: $\lim_{x \to 0} \frac{\sin x}{e^x - 1} = \frac{\sin(0)}{e^0 - 1} = \frac{0}{0}$ ← indeterminate form

($f(x) = \sin x$, $g(x) = e^x - 1$)

STEP 2: $\frac{d}{dx}(\sin x) = \cos x$ $\quad \frac{d}{dx}(e^x - 1) = e^x$

STEP 3: $\lim_{x \to 0} \frac{\cos x}{e^x} = \frac{\cos(0)}{e^0} = \frac{1}{1} = 1$ ← limit exists

STEP 4: $\lim_{x \to 0} \frac{\sin x}{e^x - 1} = 1$

b) $\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x}$.

STEP 1: $\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x} = \frac{0^3}{-2(0) + \sin 0} = \frac{0}{0}$ ← indeterminate form

STEP 2: $\frac{d}{dx}(x^3) = 3x^2$ $\quad \frac{d}{dx}(-2x + \sin 2x) = -2 + 2\cos 2x$

STEP 3: $\lim_{x \to 0} \frac{3x^2}{-2 + 2\cos 2x} = \frac{3(0)^2}{-2 + 2\cos 0} = \frac{0}{0}$ ← indeterminate form

STEP 4: That limit is still an indeterminate form, so proceed to STEP 5.

STEP 5: $\lim_{x \to 0} \frac{6x}{-4\sin 2x} = \frac{6(0)}{-4\sin 0} = \frac{0}{0}$

$\frac{d}{dx}(3x^2) = 6x$

$\frac{d}{dx}(-2 + 2\cos 2x) = -4\sin 2x$

That's still an indeterminate form, so repeat again:

$\lim_{x \to 0} \frac{6}{-8\cos 2x} = \frac{6}{-8\cos 0} = -\frac{3}{4}$

$\frac{d}{dx}(6x) = 6$

$\frac{d}{dx}(-4\sin 2x) = -8\cos 2x$

And that limit exists, so

$\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x} = -\frac{3}{4}$

www.thrivingscholars.com

# Limits Using a Maclaurin Series

## How do I evaluate a limit using Maclaurin series?

- Limits of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ or $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ may sometimes be evaluated by using Maclaurin series
- Usually this will be in a situation where attempting to evaluate the limit in the usual way returns an **indeterminate form** $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.
- In such a case:
  - STEP 1: Find the Maclaurin series for $f(x)$ and $g(x)$
  - STEP 2: Rewrite $\frac{f(x)}{g(x)}$ using the Maclaurin series in the numerator and denominator
  - STEP 3: Use algebra to simplify your new expression for $\frac{f(x)}{g(x)}$ as far as possible
  - STEP 4: Evaluate the limit using your simplified form of the expression

### Exam Tip

- Some limits of an indeterminate form can also be evaluated using **l'Hôpital's Rule**
- If an exam question does not specify a method to use, then you are free to use whichever method you prefer

www.thrivingscholars.com

## Worked Example

Use Maclaurin series to evaluate the limit

$$\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x}$$

| Maclaurin series for special functions | | |
|---|---|---|
| | $e^x = 1 + x + \frac{x^2}{2!} + \ldots$ | $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ |
| | $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ | $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ |
| | $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ | |

from exam formula booklet

$$\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x} = \frac{0^3}{-2(0) + \sin 0} = \frac{0}{0}$$ ← indeterminate form

STEP 1: $-2x + \sin 2x = -2x + \left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots\right)$

$$= -\frac{4}{3}x^3 + \frac{4}{15}x^5 - \ldots$$

STEP 2: $\frac{x^3}{-2x + \sin 2x} = \frac{x^3}{-\frac{4}{3}x^3 + \frac{4}{15}x^5 - \ldots}$

STEP 3: $\frac{x^3}{-\frac{4}{3}x^3 + \frac{4}{15}x^5 - \ldots} \times \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \frac{1}{-\frac{4}{3} + \frac{4}{15}x^2 - \ldots}$

STEP 4: $\lim_{x \to 0} \frac{1}{-\frac{4}{3} + \frac{4}{15}x^2 - \ldots} = \frac{1}{-\frac{4}{3} + \frac{4}{15}(0)^2 - \ldots} = \frac{1}{-\frac{4}{3}} = -\frac{3}{4}$

higher powers of x will also be zero

$$\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x} = -\frac{3}{4}$$

www.thrivingscholars.com

www.thrivingscholars.com

www.thrivingscholars.com

www.thrivingscholars.com

www.thrivingscholars.com