TMUA MOCK TEST 5
Solution Book

Paper 1 Styled

- All Topics

ThrivingScholars

1. How many positive roots does the function $f(x) = x^4 - 8x^3 + 22x^2 - 24x$ have?

A. 0

B. 1

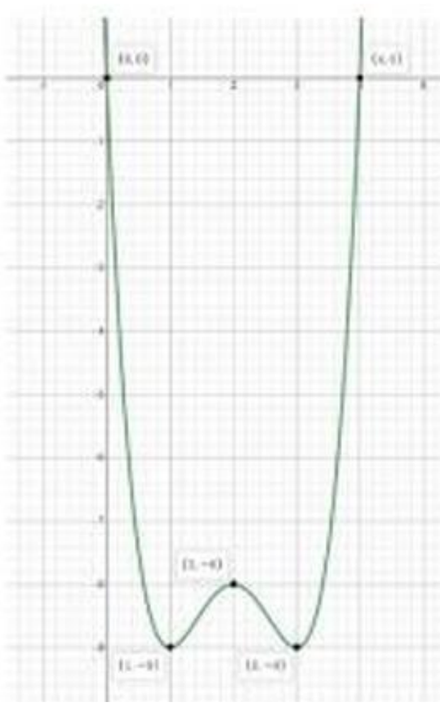
C. 2

D. 3

E. None of the above

B

Factoring the given expression will prove fruitless. Instead, identify its stationary points. Differentiating, we see that $f'(x) = 4(x^3 - 6x^2 + 11x - 6) = 4(x-1)(x-2)(x-3)$, which we factorise by inspection (or by using the factor theorem). Now we can establish $f(1), f(2), f(3)$, and using the asymptotic limits of $+\infty$ for both large positive and negative x, and the obvious root at 0, we have the following sketch., from which we see one positive root.



2. Given that, in the expansion of $(3x+b)^7$, the coefficient of x^4 is the same as the coefficient of x^2 in $(3b+x)^4$, find the positive constant b.

A. $\frac{2}{105}$

B. $\frac{105}{2}$

C. $\frac{107}{3}$

D. $\frac{3}{107}$

E. $\frac{109}{4}$

A

In the first expansion, the coefficient of x^4 is given by $\binom{7}{4}3^4b^3 = 2835b^3$ and in the second, the coefficient of x^2 is $\binom{4}{2}(3b)^2 = 54b^2$, and equating, we see that $b = \frac{54}{2835} = \frac{2}{105}$.

3. Consider the tangent to the curve $y = x^2 + bx$ at $x = 2$. For what values of b is the x intercept greater than 4?

A. $-3 < b < 3$
B. $3 < b < 4$
C. $b > 3$
D. $-4 < b < -3$
E. $b > -3$

D

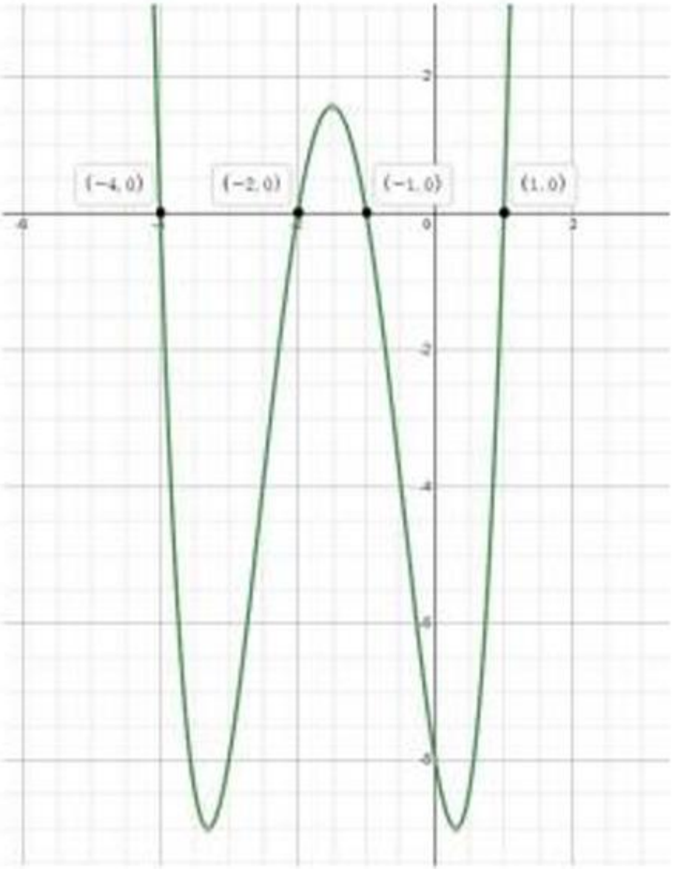
First calculate the equation of the tangent. By differentiating, we find $\frac{dy}{dx} = 2x + b$, and so at $x = 2, \frac{dy}{dx} = 4 + b$. The equation is this $y - (4 + 2b) = (4 + b)(x - 2)$ using the equation for a straight line though the point $(2, 4 + 2b)$. Now the x intercept is found by setting $y = 0$, and we find this to be at $\frac{4}{4+b}$. Now letting this be greater than 4, we find that $-4 < b < -3$.

4. In which of the following ranges is $(x^2 - 1)(x + 2)(x + 4) > 0$?

A. $-2 < x < 1$
B. $-1 < x < 2$
C. $-2 < x < -1$
D. $x \geq 1$
E. $x < -3$

C

We may factorise the function as $(x - 1)(x + 1)(x + 2)(x + 4)$, which allows us to graph it.



5. Suppose $5^{4+6+\cdots+2x} = 0.04^{-14}$. Given x is a positive integer, what is x?

A. 5
B. 6
C. 7
D. 8
E. 5.5

A

Note $0.04 = \frac{1}{25}$. Consider the arithmetic series: $2 + 4 + 6 + \cdots + 2x = 2(1 + \cdots + x) = x(x + 1)$, so the exponent on the left is x(x+1)-2 So $5^{x(x+1)-2} = (5^{-2})^{-14} = 5^{28}$. Taking logarithms (to base 5): $x(x + 1) - 2 = 28$, giving $x^2 + x - 30 = 0$, which has positive root $x = 5$.

6. An arithmetic geometric series is defined by

$$x_1 = 2$$

$$x_{n+1} = x_n + q$$

Given x_{100} is 13, find the sum to infinity of a series with common ratio q, and first term 5

A. $\frac{78}{5}$

B. $\frac{102}{3}$

C. $\frac{36}{7}$

D. $\frac{52}{9}$

E. $\frac{45}{8}$

E

We have an arithmetic progression, and $x_{100} = x_1 + 99q$, giving $q = \frac{1}{9}$. The sum to infinity is thus $\frac{a}{1-r} = \frac{5}{1-\frac{1}{9}} = \frac{45}{8}$.

7. Which of the following is a line of symmetry of the graph $y = \dfrac{1}{\sin(4x + \frac{\pi}{3})}$?

A. $x = \dfrac{13\pi}{2}$

B. $x = \dfrac{\pi}{2}$

C. $x = \pi$

D. $x = \dfrac{13\pi}{24}$

E. $y = \dfrac{5\pi}{24}$

D

The lines of symmetry of $\sin(x)$ lie at $\frac{\pi}{2} + n\pi$, and so of $\sin(4x + \frac{\pi}{3})$ at $\frac{1}{4}\left(\frac{\pi}{2} + n\pi - \frac{\pi}{3}\right) = \frac{1}{4}(\frac{\pi}{6} + n\pi)$. Taking $n = 2$ gives $\frac{13\pi}{24}$.

8. Define a recurrent sequence by $x_{n+1} = \begin{cases} \frac{x_n}{2} & for\ x_n = even \\ 3x_n + 1 & for\ x_n = odd \end{cases}$.

Given $x_1 = 12$, what is x_{100}?

A. 1
B. 4
C. 2
D. 12
E. 0

A

This is an example of the Collatz Conjecture. Applying the rule a few times we get the sequence 12, 6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1… and we see it repeats. We see $x_8 = 4, x_9 = 2, x_{10} = 1$, and from then on x_n depends only on the remainder when *n* is divided by 3. Since the remainder is 1 when dividing 100 by 3, we see the answer is 1.

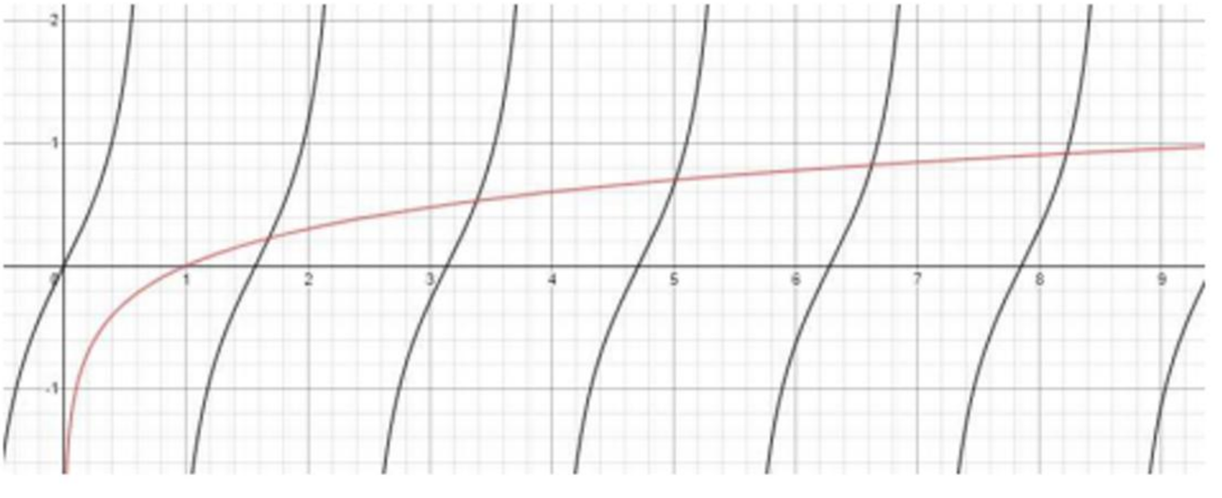
9

How many solutions does the equation $\cos 2x \times \log x = \sin 2x$ have in the range $0 < x < 3\pi$?

A. 0
B. 3
C. 5
D. 6
E. 7

C

Note that we may divide through by $\cos 2x$ to obtain the equation $\log x = \tan 2x$. We have to be careful though that we have no solutions when $\cos 2x = 0$, but this is clear since if $\cos 2x = 0$, then $\sin 2x = \pm 1$. Now draw a graph of $\log x$, and of $\tan 2x$ and count the number of intersections in the range. There are 5.



10. Let f be an integrable function on $\mathbb{R}$ with

$$\int_0^4 f(x)\,dx = 3,$$

and suppose f is **even**, i.e. $f(-x) = f(x)$.

Evaluate

$$I = \int_{-2}^{6} \left[f(x-2) - f(2-x) + 1\right] dx$$

A. 0
B. 3
C. 6
D. 8
E. 12

D

- $u = x - 2$:
$$\int_{-2}^{6} f(x-2)\,dx = \int_{-4}^{4} f(u)\,du = 2\int_0^4 f = 2 \cdot 3 = 6 \text{ (evenness).}$$
- $v = 2 - x$:
$$\int_{-2}^{6} f(2-x)\,dx = \int_{4}^{-4} f(v)(-dv) = \int_{-4}^{4} f(v)\,dv = 6.$$
- Constant term: $\displaystyle\int_{-2}^{6} 1\,dx = 8.$

So $I = 6 - 6 + 8 = 8.$

Sanity check with a concrete even function: take $f(x) = \frac{3}{4}$ (then $\int_0^4 f = 3$). Then $I = \int_{-2}^{6} (0.75 - 0.75 + 1)\,dx = \int_{-2}^{6} 1\,dx = 8.$

11. Find the sum of the x-coordinates of the six points of intersection of

$$y = \pi x\,(x-1)\,(x-2)\,(x-3)\,(x-4)\,(x-5)$$

and

$$y = \frac{5}{17}x + \frac{\pi}{3}$$

(A) -15

(B) $-\frac{44\pi}{51} + 1$

(C) $-\frac{7}{15}$

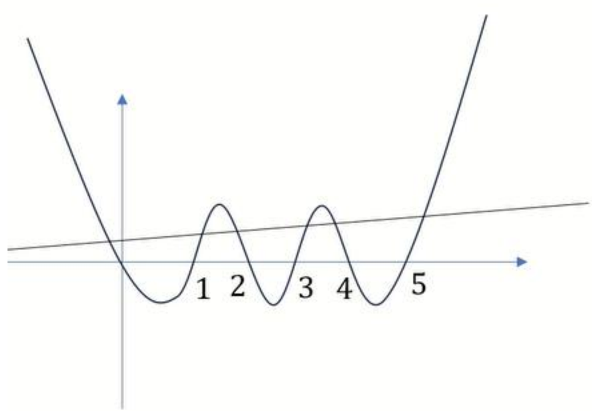
(D) 0

(E) $\frac{7}{15}$

(F) $\frac{44\pi}{51} - 1$

(G) 12

(H) 15



H

Since the polynomial

$$\mathbf{f(x) = \pi x(x-1)(x-2)(x-3)(x-4)(x-5)}$$

has roots at $\mathbf{x} = \mathbf{0, 1, 2, 3, 4, 5}$, it is a degree 6 polynomial that is symmetric about the vertical line $\mathbf{x} = \frac{\mathbf{5}}{\mathbf{2}}$. The line

$$\mathbf{y = \frac{5}{17}x + \frac{\pi}{3}}$$

is linear and intersects the curve exactly six times, as confirmed by the graph.

By symmetry, the six $\mathbf{x}$-coordinates of intersection occur in three symmetric pairs, each summing to $\mathbf{5}$. So the total sum of the six $\mathbf{x}$-coordinates is:

$$\mathbf{3 \times 5 = 15}$$

12. A cubic function $f(x)$ is such that

$f(1) = 1, \quad f(2) = 1, \quad f(-1) = -5,$

and the coefficient of x^3 is 1.

Find $f(3)$.

(A) -7 (B) 0 (C) 4 (D) 6 (E) 7 (F) 36

E

Since the cubic function is given to be

$$\mathbf{f(x) = x^3 + bx^2 + cx + d}$$

and we are told

$$\mathbf{f(1) = 1, \quad f(2) = 1, \quad f(-1) = -5}$$

we substitute into the general form to form a system of equations.

From $\mathbf{f(1) = 1}$:

$$\mathbf{1 + b + c + d = 1 \Rightarrow b + c + d = 0}$$

From $\mathbf{f(2) = 1}$:

$$\mathbf{8 + 4b + 2c + d = 1 \Rightarrow 4b + 2c + d = -7}$$

From $\mathbf{f(-1) = -5}$:

$$\mathbf{-1 + b - c + d = -5 \Rightarrow b - c + d = -4}$$

Now we solve the system: ...

Substitute $\mathbf{c = 2}$ into $\mathbf{3b + c = -7}$:

$$\mathbf{3b + 2 = -7 \Rightarrow b = -3}$$

Then:

$$\mathbf{d = -(-3) - 2 = 3 - 2 = 1}$$

So the cubic function is:

$$\mathbf{f(x) = x^3 - 3x^2 + 2x + 1}$$

Now calculate $\mathbf{f(3)}$:

$$\mathbf{f(3) = 27 - 27 + 6 + 1 = 7}$$

13. Non-zero integers a, b, c, each greater than 1, satisfy

$$abc + ab + bc + ac + a + b + c = 104$$

What is the value of $a + b + c$?

(A) 10

(B) 11

(C) 12

(D) 13

(E) 14

(F) 15

C

Since the expression

$$\mathbf{abc + ab + bc + ac + a + b + c + 1}$$

expands to

$$\mathbf{(a + 1)(b + 1)(c + 1),}$$

we add 1 to both sides of the given equation:

$$\mathbf{abc + ab + bc + ac + a + b + c = 104 \Rightarrow (a + 1)(b + 1)(c + 1) = 105}$$

Now factor $\mathbf{105 = 3 \times 5 \times 7}$.

So the only reasonable solution (with all variables > 1) is:

$$\mathbf{a + 1 = 3, \quad b + 1 = 5, \quad c + 1 = 7 \Rightarrow a = 2, \quad b = 4, \quad c = 6}$$

Therefore, the value of $\mathbf{a + b + c}$ is:

$$\mathbf{2 + 4 + 6 =} \boxed{12}$$

14. Let $u_n = 3u_{n-1} + 2u_{n-2}$, where $u_1 = 1$ and $u_2 = 5$.

What does the value of $\frac{u_k}{u_{k-1}}$ tend toward as $k \to \infty$?

(A) 2

(B) $\sqrt{5}$

(C) $\frac{3+\sqrt{17}}{2}$

(D) $\frac{3\pm\sqrt{17}}{2}$

(E) $\frac{3+\sqrt{5}}{2}$

(F) $1 \pm \sqrt{2}$

(G) $3 + \sqrt{2}$

C

Assume the ratio tends to a limit

$$\mathbf{L} = \lim_{k\to\infty} \frac{u_k}{u_{k-1}}$$

Given the recurrence relation

$$u_k = 3u_{k-1} + 2u_{k-2},$$

we divide both sides by u_{k-1}:

$$\frac{u_k}{u_{k-1}} = 3 + 2 \cdot \frac{u_{k-2}}{u_{k-1}}$$

As $k \to \infty$, both ratios approach L, so we get:

$$L = 3 + \frac{2}{L}$$

Multiply both sides by L:

$$L^2 = 3L + 2 \Rightarrow L^2 - 3L - 2 = 0$$

Solve the quadratic equation:

$$L = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}$$

Since the sequence grows positively, take the positive root:

$$L = \frac{3 + \sqrt{17}}{2}$$

15. For any integer $n \geq 0$,

$$\int_n^{n+1} f(x)\,dx = 2n+1$$

Evaluate:

$$\int_0^2 f(x)\,dx + \int_1^3 f(x)\,dx + \int_4^3 f(x)\,dx$$

(A) 1

(B) 3

(C) 5

(D) 7

(E) 9

(F) 11

C

Since the expression

$$\int_0^2 f(x)\,dx + \int_1^3 f(x)\,dx + \int_4^3 f(x)\,dx$$

involves definite integrals, we use the given identity:

$$\int_n^{n+1} f(x)\,dx = 2n+1$$

Break into unit intervals:

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = 1+3=4$$

$$\int_1^3 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx = 3+5=8$$

$$\int_4^3 f(x)\,dx = -\int_3^4 f(x)\,dx = -7$$

$$4+8-7=\boxed{5}$$

16. In the simplified expansion of $(2+3x)^{12}$, how many of the terms have a coefficient that is divisible by 48?

(A) 0

(B) 2

(C) 5

(D) 9

(E) 10

(F) 12

(G) 13

D

We want terms in $(2+3x)^{12}$ whose coefficients are divisible by $48 = 2^4 \cdot 3$.

The general term:

$$\binom{12}{k} \cdot 2^{12-k} \cdot 3^k$$

- Divisible by 3 when $k \geq 1$
- Divisible by 2^4 when $12 - k \geq 4 \Rightarrow k < 8$
- So $k = 1$ to 8 work (8 terms)
- $k = 9$ also works since $\binom{12}{9}$ adds an extra factor of 2

  → Total = **9 valid terms**

Answer: (D) 9

17. The graph of $y=\sqrt{5x-2}$ undergoes the below transformations in the given order.

I Translated horizontally left by 4
II Translated vertically down by 6
III Vertical reflection in the axis $y = 0$
IV Stretch factor $\frac{1}{3}$ in the y-axis
V Stretch factor 2 in the x-axis

Which of the following equations describes the transformed graph?

A. $y=\sqrt{2-\frac{5x}{18}}-2$

B. $y=\frac{\sqrt{-10x-6}-4}{3}$

C. $y=-\frac{\sqrt{\frac{5}{2}x-22}-6}{3}$

D. $y=-\frac{\sqrt{\frac{5}{2}x+18}-6}{3}$

E. $y=\frac{\sqrt{\frac{5}{2}x-22}-6}{3}$

F. $y=-\frac{\sqrt{\frac{5}{2}x+22}-6}{3}$

G. $y=\frac{\sqrt{\frac{5}{2}x+14}}{3}$

D

- Translated horizontally left by 4
  - Replace x with $(x+4)$
  - $y=\sqrt{5(x+4)-2}$
  - $y=\sqrt{5x+18}$
- Translated vertically down by 6
  - Replace y with $y+6$
  - $y+6=\sqrt{5x+18}$
- Vertical reflection in the axis $y=0$
  - Replace y with $-y$
  - $-y=\sqrt{5x+18}-6$
- Stretch factor $\frac{1}{3}$ in the y-axis
  - Replace y with $3y$
  - $-3y=\sqrt{5x+18}-6$
- Stretch factor 2 in the x-axis
  - $-3y=\sqrt{5(\frac{1}{2}x)+18}-6$

Thus, $y=-\frac{\sqrt{\frac{5}{2}x+18}-6}{3}$

18. How many values satisfy the following equation for $-2\pi \leq x \leq 2\pi$?

$$2 \sin x \cos x = \cos x$$

A. 4
B. 2
C. 1
D. 8
E. 16
F. 3
G. 6

D

$2 \sin x \cos x = \cos x$

$2 \sin x \cos x - \cos x = 0$

$\cos x\,(2 \sin x - 1) = 0$

$\cos x = 0$ and $\sin x = \frac{1}{2}$

Between -2π and 2π, there are four solutions for $\sin x = \frac{1}{2}$ and 4 solutions for $\cos x = 0$

19. State the minimum and maximum points of the graph of:

$y = -2\sin(-3x + \frac{3\pi}{2}) + 7$ $\quad\quad$ $0 \le x < \pi.$

A. $(-\pi/3, 8), (0, 6), (\pi/3, 8)$
B. $(0, 9), (\pi/3, 5), (2\pi/3, 9)$
C. $(-\pi/3, 9), (0, 5), (\pi/3, 9)$
D. $(0, 8), (\pi/3, 6), (2\pi/3, 8)$
E. $(0, 8), (\pi/6, 6), (2\pi/6, 8)$
F. $(0, 9), (\pi/6, 6), (2\pi/6, 9)$

B

Looking at the equation, $y = -2sin(-3x + \frac{3\pi}{2}) + 7$
Recognise $y = -sin\left(x + \frac{3\pi}{2}\right) = cos(x)$
Therefore $y = -2sin\left(-3x + \frac{3\pi}{2}\right) + 7 = 2cos(-3x) + 7$
The range needs to be changed from $0 \le x < \pi$ to $-3\pi < x \le 0$
For y = cos(x): in the new range, the solutions are $(-2\pi, 1)$, $(-\pi, -1)$, $(0,1)$

- For y = cos(-3x):
- $(2\pi/3, 1), (\pi/3, -1), (0, 1)$
- For y = 2cos(-3x) + 7:
- $(2\pi/3, 9), (\pi/3, 5), (0, 9)$

20. Let $f_0(x) = x$, and $f_{n+1}(x) = |f_n(x) - k|$ for non-negative integers n, and real number k. Let α and β respectively equal the least and greatest values of x for which $f_n(x) = 0$. Find the value of:

$$\int_\alpha^\beta f_n(x)\,\mathrm{d}x$$

for $n > 0$, in terms of n and k.

(A) nk^2

(B) $nk^2 - k^2$

(C) kn^2

(D) nk

(E) $k(n-1)^2$

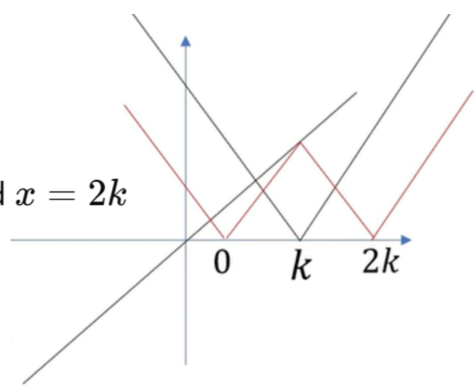
(F) $nk^2 - 1$

B

We are given $f_0(x) = x$, and $f_{n+1}(x) = |f_n(x) - k|$.
The function reflects around $y = k$ at each step, creating piecewise linear "V" shape

Using the plot, we observe:

- For $f_1(x) = |x - k|$, zero at $x = k$
- For $f_2(x) = ||x - k| - k|$, zeros at $x = 0$ and $x = 2k$



So $f_n(x) = 0$ at $x = 0$ and $x = 2k$ for all $n \geq 2$

We compute:

$$\int_0^{2k} f_n(x)\,dx$$

From the plot for $n = 2$, area is a triangle of base $2k$, height k, so:

$$\frac{1}{2}(2k)(k) = k^2$$

Each further iteration adds a layer of V-shaped segments, increasing area by k^2 per step.

So total area:

$$k^2 + k^2 + \cdots + k^2 \text{ (n times)} - k^2 = nk^2 - k^2$$

Answer: (B) $nk^2 - k^2$