

TMUA TEST 1

Solution Book

Paper 1 Styled

- Algebra and Functions
- Sequences And Series
- Functions

ThrivingScholars 

Question 1

The expansion of $(a - bx)^c$ is:

$$4 - px + 108x^2 - qx^3 + rx^4$$

where a, b, c, p, q, r are **positive real constants**.

Find the value of $p + q + r$.

Options:

- A. $81 - 84\sqrt{2}$
- B. $81 + 132\sqrt{2}$
- C. $132\sqrt{2} - 81$
- D. $81 + 84\sqrt{2}$

For the expansion to give a quartic polynomial, $c = 4$ is necessary.

Looking at the constant term of $(a - bx)^4$ shows that $a^4 = 4$. Since a is positive and real, $a = \sqrt{2}$ is necessary.

The x^2 term in the expansion of $(\sqrt{2} - bx)^4$ is $6 \times (\sqrt{2})^2 \times (-bx)^2 \equiv 12b^2x^2$, so $12b^2 = 108 \Leftrightarrow b^2 = 9 \Leftrightarrow b = 3$, since b is positive. (Seeing the possible answers and the appearance everywhere of $81 = 3^4$, does suggest the possibility of $b = 3$.)

The expansion is $(\sqrt{2} - 3x)^4 \equiv (\sqrt{2})^4 + 4 \times (\sqrt{2})^3 \times (-3x) + 6 \times (\sqrt{2})^2 \times (-3x)^2 + 4 \times \sqrt{2} \times (-3x)^3 + (-3x)^4 \equiv 4 - 24\sqrt{2}x + 108x^2 - 108\sqrt{2}x^3 + 81x^4$.

From this, $p + q + r = 24\sqrt{2} + 108\sqrt{2} + 81 = 132\sqrt{2} + 81$

The answer is B.

Question 2

The coefficient of x^{11} in the expansion of

$$(2 + x^2 + x^3)^8$$

is equal to **28 times** the coefficient of x^2 in

$$(2 + ax)^6$$

Find all the possible values of the constant a .

Options:

A. $\pm 2\sqrt{7}$

B. $\pm \frac{1}{2}$

C. $\pm \frac{1}{4}$

D. ± 1

Thinking of the ways in which x^{11} terms can be obtained from $(2 + x^2 + x^3)^8$:

$$(x^3)^3(x^2)^1 \times 2^4 \equiv 16x^{11} \text{ happens } \binom{8}{3} \binom{5}{1} = \frac{8 \times 7 \times 6}{3!} \times 5 = 56 \times 5 = 280 \text{ times;}$$

$$(x^3)^1(x^2)^4 \times 2^3 \equiv 8x^{11} \text{ happens } \binom{8}{1} \binom{7}{4} = 8 \times \frac{7 \times 6 \times 5}{3!} = 8 \times 35 = 280 \text{ times.}$$

It follows that the coefficient of x^{11} is $280 \times 16 + 280 \times 8 = 280 \times 24$.

$$\text{The coefficient of } x^2 \text{ in } (2 + ax)^6 \text{ is } \binom{6}{2} \times 2^4 \times a^2 = \frac{6 \times 5}{2!} \times 16 \times a^2 = 240a^2.$$

$$\text{So } 280 \times 24 = 28 \times 240a^2 \Leftrightarrow 10 = 10a^2 \Leftrightarrow a^2 = 1 \Leftrightarrow a = \pm 1.$$

The answer is **D**

Question 3

The coefficient of x^2 in the expansion of

$$(2 + bx)^4$$

is **2 times** the coefficient of x^3 in

$$(1 + bx)^6$$

Given that $b \neq 0$,

What is the value of b ?

Options:

A. $\frac{8}{15}$

B. $\frac{3}{5}$

C. $\frac{3}{20}$

D. $\frac{6}{5}$

The x^2 term in the expansion of $(2 + bx)^4$ is $6 \times 2^2 \times (bx)^2$, so the coefficient is $24b^2$.

The x^3 term in the expansion of $(1 + bx)^6$ is $\binom{6}{3} \times 1^3 \times (bx)^3$, so the coefficient is

$$\frac{6 \times 5 \times 4}{3!} b^3 = 20b^3.$$

Therefore, $24b^2 = 2 \times 20b^3 = 40b^3 \Leftrightarrow 3b^2 = 5b^3 \Leftrightarrow 0 = b^2(5b - 3) \Leftrightarrow b = 0$ or $b = \frac{3}{5}$.

If $b = 0$, then both polynomials become constants (16 and 1, respectively) and there is no solution to the question.

The answer is B.

Question 4

Find the coefficient of:

$$(1 - x)^0 - (1 - x)^1 + (1 - x)^2 - (1 - x)^3 + (1 - x)^4 - \dots + (1 - x)^{50}$$

What is the value of the coefficient?

Options:

A. 51

B. -51

C. -25

D. 25

For any positive integer, n , the expansion of $(1 - x)^n$ begins $1 - nx + \dots$, and $(1 - x)^0 \equiv 1$ (assuming $x \neq 0$).

It follows that the x term in the above expression is $-(-x) + (-2x) - (-3x) + (-4x) - \dots + (-50x)$, and that this has coefficient $1 - 2 + 3 - 4 + \dots - 50$. This sum is 25 groups of $n - (n + 1) = -1$ where $n = 1, 3, \dots, 49$. Therefore the coefficient of x is $25 \times (-1) = -25$. The answer is C.

Question 5

Find the value of the expression:

$$\sqrt{16 + 8\sqrt{3} + 3} - \sqrt{12 - 4\sqrt{3} + 1}$$

What is the value of the expression?

Options:

A. $\sqrt{6} + 12\sqrt{3}$

B. $\sqrt{3} - 5$

C. 3

D. $5 - \sqrt{3}$

Although there are other, much longer, available methods for answering this question, the examiners do drop a hint that there must be an easier approach on this occasion. In particular, for example, why do they write $16 + 8\sqrt{3} + 3$ instead of $19 + 8\sqrt{3}$? This must be done for a reason. If you can quickly spot a value that squares to give $16 + 8\sqrt{3} + 3$, then square rooting simply reverses the process. Considering $(a + b)^2 \equiv a^2 + 2ab + b^2$:

$16 = 4^2$ and $3 = (\sqrt{3})^2$, so it is worth checking the expansion of $(4 \pm \sqrt{3})^2 \equiv 16 \pm 8\sqrt{3} + 3$... you are in luck!

$12 = (2\sqrt{3})^2$ and $1 = 1^2$, so it is worth checking the expansion of $(2\sqrt{3} \pm 1)^2 \equiv 12 \pm 4\sqrt{3} + 1$... you are in luck, again, and grateful not to have 'piled in' to this question!

Therefore, $\sqrt{16 + 8\sqrt{3} + 3} - \sqrt{12 - 4\sqrt{3} + 1} = \sqrt{(4 + \sqrt{3})^2} - \sqrt{(2\sqrt{3} - 1)^2} = 4 + \sqrt{3} - (2\sqrt{3} - 1) = 5 - \sqrt{3}$.

The answer is D.

Question 6

The circles with equations:

$$(x - r)^2 + (y - 2)^2 = r^2 \quad \text{and} \quad (x + r)^2 + (y + r)^2 = 4r^2$$

touch precisely once when:

Options:

A. $r = \sqrt{3} - 1$

B. $r = \frac{8}{5}$

C. $r = \frac{2}{1 + \sqrt{5}}$

D. $r = \frac{1 + \sqrt{5}}{2}$

Neither circle contains the centre of the other circle and so the circles only touch when the centres are distance $3r$ apart (the sum of their radii). It simplifies the algebra to work with the square of the distance between the centres $(r, 2)$ and $(-r, -r)$. Solving $(2r)^2 + (2 + r)^2 = 9r^2$ gives $r = \frac{1 \pm \sqrt{5}}{2}$. Since $r > 0$, the correct answer is D.

Question 7

Find the value(s) of a such that the turning point of the parabola

$$y = x^2 + 2ax + 1$$

is closest to the origin.

What is the value of a ?

Options:

A. $a = \sqrt{2}$

B. $a = \pm\sqrt{2}$

C. $a = \frac{\sqrt{2}}{2}$

D. $a = \pm\frac{1}{\sqrt{2}}$

$x^2 + 2ax + 1 = (x + a)^2 + 1 - a^2$ so the turning point has coordinates $(-a, 1 - a^2)$.

Letting d denote the distance from the turning point to the origin gives $d^2 = a^2 +$

$(1 - a^2)^2 = a^4 - a^2 + 1 = \left(a^2 - \frac{1}{2}\right)^2 + \frac{3}{4}$. So the required values of a are $\pm\frac{1}{\sqrt{2}}$. The correct

answer is D.

Question 8

The line $y = 2x + c$ is such that it intersects the circle

$$x^2 + y^2 = 9$$

at two points A and B . Let M be the **midpoint of the chord AB** of the circle.

Find the equation of the locus of M as c varies between $-3\sqrt{5}$ and $3\sqrt{5}$.

Options:

A. $x^2 + y^2 = 2$

B. $y = -\frac{1}{2}x$

C. $y = \frac{1}{2}x$

D. $y = x^2$

The line $y = 2x + c$ and the circle $x^2 + y^2 = 9$ intersect when $x^2 + (2x + c)^2 = 9$ giving $5x^2 + 4cx + (c^2 - 9) = 0$, so $x = \frac{-4c \pm \sqrt{180 - 4c^2}}{10}$ (note that there are 2 points of intersection provided that $180 - 4c^2 > 0$, and hence $-4\sqrt{5} < c < 4\sqrt{5}$). By considering the means of the coordinates of A and B you obtain $M\left(-\frac{2c}{5}, \frac{c}{5}\right)$ (note that when computing the mean of the x -coordinates of A and B the discriminant cancels, and the mean of the y -coordinates of A and B is given by $2x + c$ with $x = -\frac{2c}{5}$). By inspection, as c varies, the coordinates of M satisfy $y = -\frac{x}{2}$. The correct answer is B.

Question 9

The tangent to the circle

$$x^2 + y^2 + 6x + 2y + 2 = 0$$

at the point $(-5, 1)$ passes through the point $(-3, 3)$.

The **other tangent** to the circle that passes through this point touches the circle at the point:

Options:

- A. $(-3, 2\sqrt{2})$
- B. $(-3, 1)$
- C. $(-1, 2\sqrt{2})$
- D. $(-1, 1)$

The centre-radius form for the circle is $(x + 3)^2 + (y + 1)^2 = 8$, so the circle's centre, $(-3, -1)$, lies vertically below the point $(-3, 3)$. By symmetry, the other tangent meets the circle at the point $(-1, 1)$. The correct answer is D.

Question 10

The circle

$$x^2 + y^2 + 2ax + 2by = c$$

encloses the point (1, 1) precisely when:

Options:

A. $c > 2(1 - a - b)$

B. $a^2 + b^2 < 1$

C. $c > 2(a + b + 1)$

D. $a^2 + 2ab + b^2 < c$

$x^2 + y^2 + 2ax + 2by = c \implies (x + a)^2 + (y + b)^2 = c + a^2 + b^2$. The point (1,1) lies within the circle if and only if $c + a^2 + b^2 > (a - 1)^2 + (b - 1)^2$. Rearranging gives $c > 2(1 - a - b)$. The correct answer is (a).

Question 11

Which of the following is a **tangent** to the circle

$$(x - 2)^2 + (y - 3)^2 = 4 ?$$

Options:

A. $x = 3 + \sqrt{2}$

B. $y = 5 + \sqrt{2} - x$

C. $y = 2(2 - \sqrt{2})$

D. $y = x + 1 - 2\sqrt{2}$

A quick sketch rules out (b) and (d) since the vertical and horizontal tangents are $x = 4$ and $y = 5$ respectively. The point $(2, 3 + \sqrt{2})$ lies inside the circle (since $(x - 2)^2 + (y - 3)^2 = 2$) and hence (c) can be eliminated. There are no obvious interior points for (a) and (e) so use direct substitution and check for a zero discriminant. Considering (a) first, you obtain $(x - 2)^2 + (5 - x)^2 = 4$ which rearranges to $2x^2 - 14x + 25 = 0$ with discriminant -4 , so the line does not intersect the circle. Considering (e) first, you obtain $(x - 2)^2 + (x - 2 - 2\sqrt{2})^2 = 4$ which rearranges to give $2x^2 - (8 + 4\sqrt{2})x + 8\sqrt{2} + 12 = 0$ with discriminant 0. The correct answer is (d)

Question 12

The reflection of the point

$$(a, a)$$

in the line

$$y = \frac{1}{a}x \quad \text{where } a \neq 0$$

is:

Options:

A.

$$\left(\frac{a(a-1)^2}{a^2+1}, \frac{a^2(a^2+3)}{a^2+1} \right)$$

B.

$$\left(\frac{a(a^2+2a-1)}{a^2+1}, \frac{-a(a^2-2a-1)}{a^2+1} \right)$$

C.

$$\left(\frac{a(a-1)^2}{a^2+1}, \frac{-a(a-1)^2}{a^2+1} \right)$$

D.

$$\left(\frac{a(a+1)^2}{a^2+1}, \frac{a^2(a+1)^2}{a^2+1} \right)$$

When $a = 1$ you need the image of the point $(1,1)$ when reflected in the line $y = x$ (which is $(1,1)$). Only option (b) gives this point so the rest can be eliminated. The correct answer is (b).

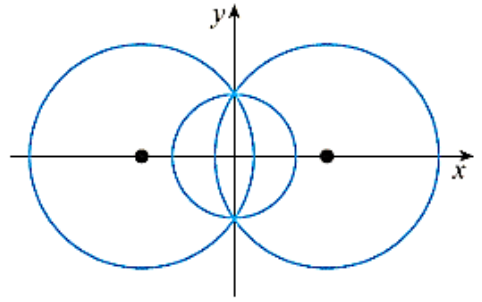
Question 13

The circle C has equation

$$x^2 + y^2 = 1$$

and is intersected at $(0, 1)$ and $(0, -1)$ by two circles of radius r :

- One with center at $(a, 0)$
- The other with center at $(-a, 0)$



(This is shown in the diagram.)

The value of a that results in circle C having **three distinct regions of equal area** satisfies the equation:

A.

$$\frac{\pi(a^2 + 1) \arctan\left(\frac{1}{a}\right)}{90} - 6a = \pi$$

B.

$$\frac{\pi(a^2 + 1) \arctan\left(\frac{1}{a}\right)}{30} - 6a = \pi$$

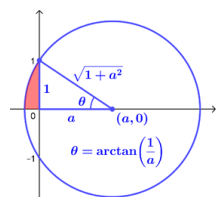
C.

$$\pi(a^2 + 1) \arctan\left(\frac{1}{a}\right) - 6a = \pi$$

D.

$$\frac{(a^2 + 1) \arctan\left(\frac{1}{a}\right)}{30} = 1$$

The area of the shaded region is $\frac{(1+a^2)\pi \arctan(\frac{1}{a})}{360} - \frac{a}{2}$. Setting this equal to $\frac{\pi}{12}$ (so that the area enclosed by the overlapping circles is $\frac{\pi}{3}$) gives $\frac{\pi(a^2+1) \arctan(\frac{1}{a})}{30} - 6a = \pi$. The correct answer is (b).



Question 14

The function f is defined on the **positive integers** as follows:

$$f(1) = 2 \quad \text{and for } n \geq 1 :$$

$$f(n+1) = \begin{cases} 5f(n) + 1 & \text{if } f(n) \text{ is odd} \\ \frac{1}{2}f(n) & \text{if } f(n) \text{ is even} \end{cases}$$

What is the value of

$$\sum_{r=1}^{100} f(r)?$$

Options:

- A. 535
- B. 546
- C. 560
- D. 563

$$f(1) = 2, f(2) = 1, f(3) = 6, f(4) = 3, f(5) = 16, f(6) = 8, f(7) = 4, f(8) = 2$$

Since $f(8) = f(1) = 2$, the values will repeat in sets of the seven values 2, 1, 6, 3, 16, 8, 4

$\frac{100}{7} = 14\frac{2}{7}$ so there will be 14 complete sets followed by 2, 1

$$\sum_{r=1}^{100} f(r) = 14(2 + 1 + 6 + 3 + 16 + 8 + 4) + 1 + 2 = 14 \times 40 + 3$$

The answer is D

Question 15

When

$$2x^2 + 3x - 15$$

is multiplied by $(ax - 4)$, and the resulting product is **divided by** $x + 2$, the **remainder is 182**.

What is the value of a ?

Options:

A. -9

B. -5

C. 5

D. 9

$$\text{Left } f(x) = (ax - 4)(2x^2 + 3x - 15)$$

$$f(-2) = 182$$

$$(-2a - 4)(8 - 6 - 15) = 182$$

$$26(a + 2) = 182$$

$$a + 2 = 7$$

$$a = 5$$

The correct answer is C.

Question 16

Find the **non-zero solution** to the equation:

$$\frac{3^{(16^x)}}{81^{(4^x)}} = \frac{1}{27}$$

Options:

A. $\log_4 3$

B. $2 \log_4 3$

C. 1

D. 2

$$\frac{3^{16^x}}{81^{4^x}} = \frac{1}{27}$$
$$\frac{3^{(4^2)^x}}{(3^4)^{4^x}} = 3^{-3}$$
$$\frac{3^{4^{2x}}}{3^{4^{x+1}}} = 3^{-3}$$

$$3^{(4^{2x} - 4^{x+1})} = 3^{-3}$$

$$4^{2x} - 4^{x+1} = -3$$

$$4^{2x} - 4 \times 4^x + 3 = 0$$

$$\text{Let } a = 4^x$$

$$a^2 - 4a + 3 = 0$$

$$(a - 1)(a - 3) = 0$$

$$a = 1 \Leftrightarrow 4^x = 1 \Leftrightarrow x = 0 \text{ but non-zero solution required}$$

$$a = 3 \Leftrightarrow 4^x = 3 \Leftrightarrow x = \log_4 3$$

The correct answer is A

Question 17

Consider the expression:

$$(x^4 - 5)^n - (x^2 + 1)^{n+1} + (x^2 + 4)^n(3 + x^2)^n$$

For which values of n is the expression divisible by x^2 ?

Options:

- A. Odd n
- B. Even n
- C. No n
- D. All n

$$\text{Let } x^2 = -2$$

By the factor theorem:

$$(4 - 5)^n - (-2 + 1)^{n+1} + (-2 + 4)(3 - 2)^n = 0$$

$$(-1)^n - (-1)^{n+1} + 2 \times 1^n = 0$$

$$2 = (-1)^{n+1} - (-1)^n$$

$$2 = (-1)^n(-1 - 1)$$

$$2 = -2(-1)^n$$

This is true if n is odd.

The correct answer is (a)

Question 18

Let a and b be non-zero integers.

When the polynomial

$$x^2 - 2ax - a^2$$

is divided by $x - b$, the remainder is 1.

Also, the polynomial

$$4bx^2 - 6x + 6$$

has $2x - a$ as a factor.

It follows that $a + b$ equals:

Options:

A. -5

B. -3

C. -1

D. 0

$x^2 - 2ax - a^2$ divided by $x - b$ remainder 1

$4bx^2 - 6x - 10$ divided by $2x - a$ remainder 0

Using the remainder theorem

$$x = b \text{ in } x^2 - 2ax - a^2 \text{ gives } b^2 - 2ab - a^2 = 1 \quad \text{A}$$

Using the factor theorem

$$x = \frac{a}{2} \text{ in } 4bx^2 - 6x - 10 \text{ gives } a^2b - 3a - 10 = 0 \quad \text{B}$$

From A

$$a^2 + 2ab + 1 - b^2 = 0$$

Completing the square for a gives

$$(a + b)^2 + 1 - 2b^2 = 0$$

$$(a + b)^2 = 2b^2 - 1$$

From the options, if $a + b = \pm 5$ (options (a) and (e)) then

$2b^2 - 1 = 25$ but this means that b is not an integer as specified in the question so (a) and (e) can be eliminated.

If $a + b = -3$ then $2b^2 - 1 = 9$ and again, b is not an integer so (b) can be eliminated

If $a + b = -1$ then $2b^2 - 1 = 1$ giving $b = \pm 1$ since $a + b = -1$, b must be $+1$ otherwise $a = 0$. So it is possible that (c) is correct with $b = 1$ and $a = -2$

If $a + b = 0$ then $2b^2 - 1 = 0$ and again, b is not an integer.

(c) is the correct answer.

Question 19

The polynomial $P_n(x)$ is defined by:

$$P_n(x) = (x - 2n + 1) + (2x - 2n + 3) + (3x - 2n + 5) + \cdots + (nx - 1)$$

Given that $n \geq 2$,

What is the **remainder** when $P_n(x)$ is divided by $P_{n-1}(x)$?

Options:

A. $\frac{n^2 - 1}{2}$

B. n

C. -1

D. 1

The answer can be found by using a simple value for n

If $n = 3$ then

$$p_3(x) = (x - 5) + (2x - 3) + (3x - 1) = 6x - 9$$

$$p_2(x) = (x - 3) + (2x - 1) = 3x - 4$$

Using the remainder theorem with $x = \frac{4}{3}$

$$p_3\left(\frac{4}{3}\right) = 6 \times \frac{4}{3} - 9 = 8 - 9 = -1$$

It looks like (c) is the correct answer. To confirm this the other values can be checked

$$n = 3 \text{ (a) } \frac{n^2 - 1}{2} = \frac{9 - 1}{2} = 4, \text{ (b) } n = 3, \text{ (d) } 1 \neq -1, \text{ (e) } \frac{n}{2} = \frac{3}{2}$$

The correct answer is (c)

Question 20

Given that x, y, z are **positive real numbers**, the equations

$$4 \log_z x = y, \quad \log_x z = y, \quad x + \log_y z = 0$$

Which of the following is true?

Options:

- A. Have a unique solution for x , but not for y and z
- B. Have unique solutions for x and y , but infinitely many solutions for z
- C. Have unique solutions for x and z , but infinitely many solutions for y
- D. Have a unique solution for x, y, z

From $4 \log_z x = y$, $x^4 = z^y$ A

From $\log_x z = y$, $z = x^y$ B

From $x + \log_y z = 0$, $z = y^{-x}$ C

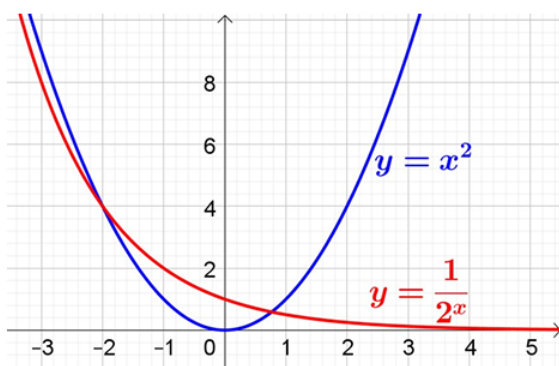
Since x, y and z all feature as the base of a logarithm, all three must be positive.

Substituting B into A gives $x^4 = (x^y)^y$ so $x^{y^2} = x^4$ and $y^2 = 4$ so $y = 2$ since $y > 0$

A becomes $x^4 = z^2$, B becomes $z = x^2$ and C becomes $z = 2^{-x}$ i.e. $z = \frac{1}{2^x}$

Substituting $z = \frac{1}{2^x}$ into $z = x^2$ gives $x^2 = \frac{1}{2^x}$

To see if this gives unique values for x , the graphs of $y = \frac{1}{2^x}$ and $y = x^2$ can be sketched



There is one solution for $x > 0$ (negative values can be ignored since you know that $x > 0$)

As there is a unique value for x , since $z = x^2$, there is also a unique value for z .

x, y and z all have unique solutions.

The correct answer is **D**