AMC 10
MOCK TEST 6
Solution Book

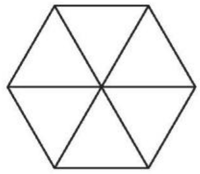
Geometry

ThrivingScholars

1. The figure shows a regular hexagon.

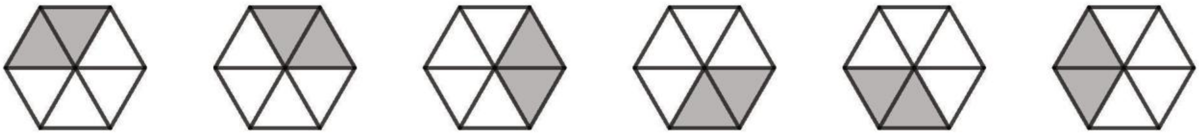
How many parallelograms are there in the figure?

A 2 B 4 C 6 D 8 E more than 8



Solution **C**

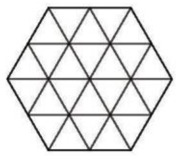
The parallelograms in the figure are made up from pairs of adjacent triangles. Therefore there are six parallelograms in the figure, as shown in the diagrams below.



For investigation

The figure on the right shows a regular hexagon divided into 24 congruent equilateral triangles.

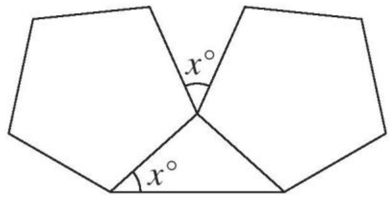
How many parallelograms are there in the figure?



2. The diagram shows two congruent regular pentagons and a triangle. The angles marked $x°$ are equal.

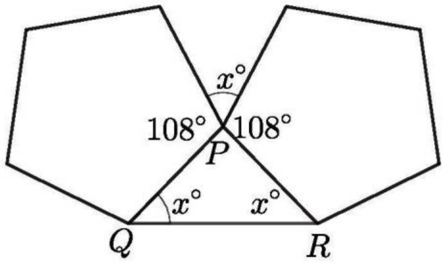
What is the value of x?

A 24   B 30   C 36   D 40   E 45



SOLUTION **C**

Let P, Q and R be the points shown in the diagram.



The interior angles of a regular pentagon are all 108°.

Because the two pentagons are congruent, $PR = PQ$. Therefore $\angle PRQ = \angle PQR = x°$.

Because the sum of the angles in a triangle is 180°, from the triangle PQR, we have $x° + x° + \angle QPR = 180°$. Therefore $\angle QPR = 180° - 2x°$.

The sum of the angles at a point is 360°. Therefore from the angles at the point P, we have

$$(180° - 2x°) + 108° + x° + 108° = 360°.$$

That is

$$396° - x° = 360°.$$

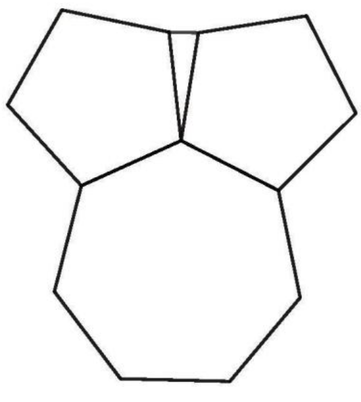
It follows that

$$x° = 396° - 360° = 36°.$$

FOR INVESTIGATION

- Prove that the sum of the angles in a triangle is 180°.
- Prove that each interior angle of a regular pentagon is 108°.
- Find a formula in terms of n for the size of the interior angles of a regular polygon with n sides.
- The diagram shows a regular heptagon, two regular pentagons and a triangle.

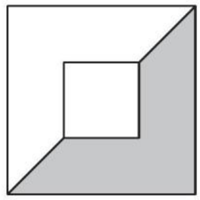
  What are the interior angles of the triangle?



3. The diagram shows two symmetrically placed squares with sides of length 2 and 5.

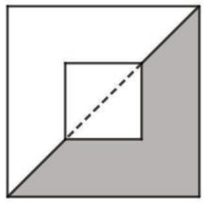
What is the ratio of the area of the small square to that of the shaded region?

A 7 : 24 B 1 : 3 C 8 : 25 D 8 : 21 E 2 : 5



Solution **D**

The small square has area 2^2, that is, 4. The shaded region is made up of half the large square with side length 5, with half the small square with side length 2 removed.



Therefore, the shaded area is $\frac{1}{2}(5^2) - \frac{1}{2}(2^2) = \frac{25}{2} - 2 = \frac{21}{2}$. Hence the ratio of the area of the small square to that of the shaded region is $4 : \frac{21}{2}$, that is, $8 : 21$.

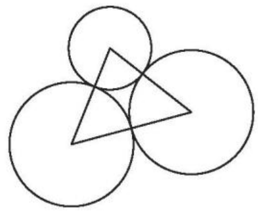
For investigation

- Suppose that the area of the shaded region were one-third of the area of the outer square. What would be the ratio of the area of the inner square to the area of the outer square?
- Suppose that the ratio of the shaded area to the area of the small square is 7 : 18. What is the ratio of the side length of the small square to the side length of the large square?

4. Three circles with radii 2, 3 and 3 touch each other, as shown in the diagram.

What is the area of the triangle formed by joining the centres of these circles?

A 10 B 12 C 14 D 16 E 18

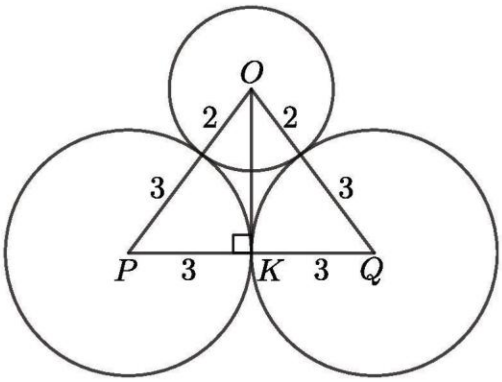


Solution **B**

We let O be the centre of the circle with radius 2, and let P and Q be the centres of the circles with radius 3.

The line joining the centres of touching circles goes through the point where the circles touch. [You are asked to prove this in Problem 5.2.] It follows that both OP and OQ have length $2 + 3 = 5$, and PQ has length $3 + 3 = 6$.

Let K be the midpoint of PQ.



The triangles OPK and OQK are congruent (SSS), and therefore the angles $\angle OKP$ and $\angle OKQ$ are equal, and therefore they are both 90°.

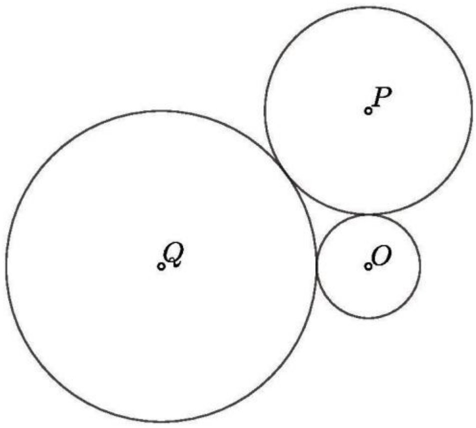
By Pythagoras' Theorem, applied to the triangle OKP, we have $OK^2 = OP^2 - PK^2 = 5^2 - 3^2 = 25 - 9 = 16$. Therefore $OK = 4$.

The triangle OPQ has base PQ of length 6, and height OK of length 4. Therefore the area of this triangle is $\frac{1}{2}(6 \times 4) = 12$.

For investigation

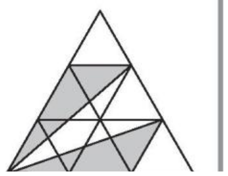
Three circles with centres O, P and Q with radii 1, 2 and 3, respectively, touch each other as shown.

What is the area of the triangle OPQ ?



Prove that the line joining the centres of touching circles goes through the point where the circles touch.

5. The diagram shows an equilateral triangle divided into nine smaller equilateral triangles, with two additional lines.

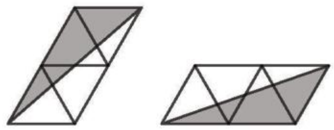


What fraction of the large triangle is shaded?

A $\frac{1}{3}$ B $\frac{1}{4}$ C $\frac{3}{8}$ D $\frac{2}{9}$ E $\frac{4}{9}$

SOLUTION **E**

The shaded region is made up of two triangles. Each of these triangles forms half of a parallelogram which consists of four of the smaller equilateral triangles. Hence each of these triangles has the same area as two of the smaller triangles.



It follows that the shaded region has the same area as four of the smaller triangles. The large equilateral triangle is made up of nine of the smaller triangles.

Therefore the fraction of the large triangle that is shaded is $\frac{4}{9}$.

FOR INVESTIGATION

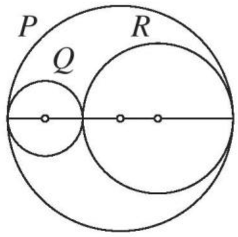
The unshaded region in the large equilateral triangle of this question is made up of three triangles. Find the area of each of these triangles as a fraction of the area of the large triangle. Deduce that $\frac{5}{9}$ths of the area of the large triangle is not shaded.

How many different parallelograms are there in the diagram of this question?

6. The circles P, Q and R are all tangent to each other. Their centres all lie on a diameter of P, as shown in the figure.

What is the value of $\dfrac{\text{circumference of } Q + \text{circumference of } R}{\text{circumference of } P}$?

A 1 B $\frac{1}{2}$ C $\frac{1}{3}$ D $\frac{1}{4}$
E more information needed



Solution **A**

Let the radius of the circle Q be q and the radius of the circle R be r. We see that the diameter of the circle P is $2q + 2r$ It follows that the radius of P is $q + r$.

We now use the formula

$$\text{circumference} = 2\pi \times \text{radius},$$

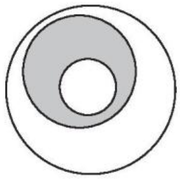
to deduce that

$$\begin{aligned}\frac{\text{circumference of } Q + \text{circumference of } R}{\text{circumference of } P} &= \frac{2\pi q + 2\pi r}{2\pi(q+r)}\\ &= \frac{2\pi(q+r)}{2\pi(q+r)}\\ &= 1\end{aligned}$$

7. The diagram shows three circles with radii 1, 2 and 3.

What is the ratio of the shaded area to the area of the largest circle?

A $1:3$ B $1:2$ C $\sqrt{2}:\sqrt{3}$ D $2:3$ E $4:9$



SOLUTION **A**

The circles have areas $\pi(1^2)$, $\pi(2^2)$ and $\pi(3^2)$, that is, π, 4π and 9π, respectively.

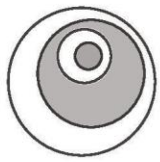
It follows that the shaded area is $4\pi - \pi = 3\pi$.

Therefore the ratio of the shaded area to the area of the largest circle is $3\pi : 9\pi = 1:3$.

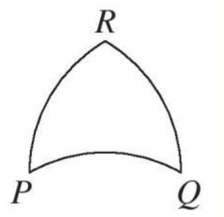
FOR INVESTIGATION

(a) Suppose that the four circles in the diagram have radii 1, 2, 3 and 4. What fraction of the area of the largest circle is shaded?

(b) Find integer values for the radii of the circles shown in the diagram so that half the area of the largest circle is shaded.



8. The diagram shows three arcs of circles of radius 1. P is the centre of the circle of which RQ is an arc and Q is the centre of the circle of which PR is an arc.

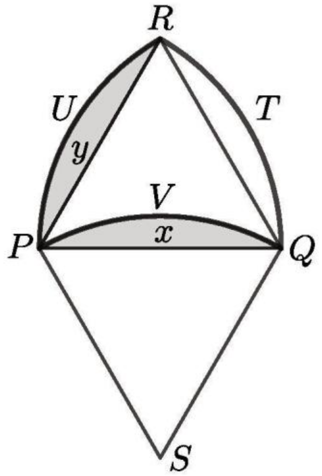


What is the area of this shape?

A $\frac{\sqrt{3}\pi}{10}$ B $\frac{\pi}{6}$ C $\frac{\pi}{8}$ D $\frac{\sqrt{3}}{4}$ E $\frac{\sqrt{2}\pi}{9}$

Solution **B**

In the diagram on the right we have added the labels T, U and V so that we can distinguish between the lines joining P, Q and R and the arcs joining these points. For example we refer to the line joining P and Q as PQ and the arc joining them as PVQ.



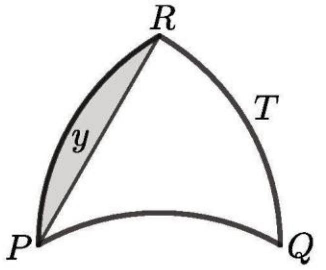
The point S is the centre of the circle that the arc PVQ is part of.

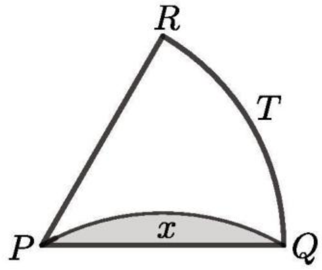
Since PQ, QR and RP all have length 1, the triangle PQR is equilateral. Therefore $\angle RQP = 60°$. Similarly, $\angle QSP = 60°$.

Therefore the regions $RUPQ$ and $QVPS$ are both sectors of circles of radius 1 bounded by arcs that subtend an angle of 60° at the centres of the circles. Hence these two regions are congruent.

The shaded regions, marked x and y in the diagram, are obtained from the regions $RUPQ$ and $QVPS$ by removing the congruent equilateral triangles PQR and PSQ.

It follows that the regions x and y are congruent and hence they have have the same area.





Therefore the shape of the question has the same area as the shape obtained from it by removing the region y and adding the region x.

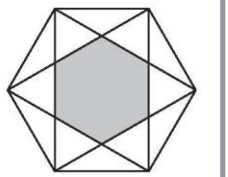
This new shape is the sector $PQTR$. Because $\angle QPR = 60°$ its area is one sixth of the area of the circle centre P and radius 1, namely $\frac{1}{6}\pi(1^2)$, that is $\frac{\pi}{6}$.

Therefore the area of the shape of this question is also $\frac{\pi}{6}$.

For investigation

. What is the area of the regions marked x and y in the diagram above?

9. In the diagram, the outer hexagon is regular and has an area of 216. What is the shaded area?

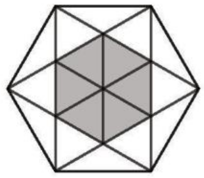


A 108 B 96 C 90 D 84 E 72

Solution **E**

Method 1

In the diagram on the right we have added three lines to the diagram given in the question.



The outer hexagon is now divided into 12 congruent equilateral triangles and 6 congruent triangles with angles 120°, 30° and 30°.

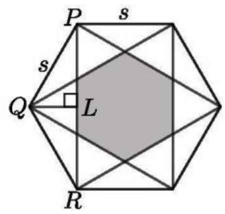
We leave it to the reader to check that each of the triangles with angles 120°, 30°, and 30° has the same area as each of the equilateral triangles (see Problem 8.1).

It follows that the outer hexagon is divided into 18 triangles all with the same area.

The shaded area is made up of 6 of these triangles. Hence its area is $\frac{6}{18}$, that is, $\frac{1}{3}$ of the area of the outer hexagon. Therefore the area of the shaded hexagon is $\frac{1}{3} \times 216 = 72$.

Method 2

We let s be the side length of the outer hexagon, and P, Q and R be three adjacent vertices of this hexagon, as shown.



We let L be the midpoint of PR.

We leave it to the reader to check that PLQ is a right-angled triangle in which $\angle PQL = 60°$ and $PL = \frac{\sqrt{3}}{2}s$ (see Problem 8.2). It follows that $PR = \sqrt{3}s$.

The inner hexagon is regular. The distance between its parallel sides is s. The outer hexagon is regular. The distance between its parallel sides is $\sqrt{3}s$. The ratio of the areas of similar figures is the same as the ratio of the squares of their corresponding lengths. Therefore

$$\text{area of inner hexagon : area of outer hexagon} = s^2 : (\sqrt{3}s)^2 = s^2 : 3s^2 = 1 : 3.$$

It follows that the area of the shaded hexagon is $\frac{1}{3} \times 216 = 72$.

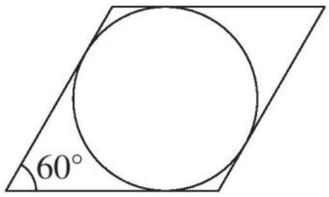
For investigation

- Show that in the diagram of Method 1, all the 18 triangles into which the outer hexagon is divided have the same area.
- Show that in the diagram of Method 2, the triangle PLQ is right-angled, $\angle PQL = 60°$ and $PL = \frac{\sqrt{3}}{2}s$

10. A circle with radius 1 touches the sides of a rhombus, as shown. Each of the smaller angles between the sides of the rhombus is 60°.

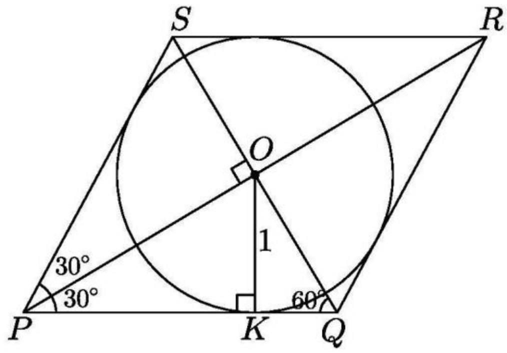
What is the area of the rhombus?

A 6 B 4 C $2\sqrt{3}$ D $3\sqrt{3}$ E $\frac{8\sqrt{3}}{3}$



SOLUTION **E**

Let P, Q, R and S be the vertices of the rhombus.



We leave it to the reader to prove that the diagonals of a rhombus bisect the angles of the rhombus and meet at right angles (see Exercise 12.1).

Let O be the point where the diagonals PR and QS of the rectangle meet.

We also leave it to the reader to prove that the four triangles POQ, QOR, ROS and SOP are congruent (see Exercise 12.1) and that O is the centre of the circle (see Exercise 12.2).

We let K be the point where PQ touches the circle. Then $OK = 1$. Because the radius of a circle is at right angles to the tangent at the point where the radius meets the circle, $\angle PKO = 90°$.

From the right-angled triangle PKO we have

$$\frac{OK}{OP} = \sin 30° = \frac{1}{2},$$

and therefore, since $OK = 1$, it follows that $OP = 2$.

Because $\angle POQ = 90°$, it follows from the triangle POQ that $\angle OQK = 60°$, and hence

$$\frac{OK}{OQ} = \sin 60° = \frac{\sqrt{3}}{2},$$

and therefore $OQ = \frac{2}{\sqrt{3}}$.

It now follows that the area of the triangle POQ is given by

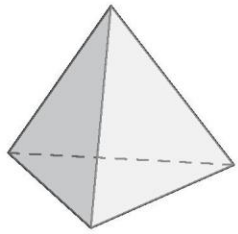
$$\frac{1}{2}(OP \times OQ) = \frac{1}{2}\left(2 \times \frac{2}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}.$$

Therefore, as the rhombus is made up of four triangles each congruent to the triangle POQ, the area of the rhombus is

$$4 \times \frac{2}{\sqrt{3}} = \frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3}.$$

Note: See Exercise 12.4 for a way to remember the values of sin(30°) and sin(60°).

11. A regular tetrahedron is a polyhedron with four faces, each of which is an equilateral triangle, as shown. A solid regular tetrahedron is cut into two pieces by a single plane cut.



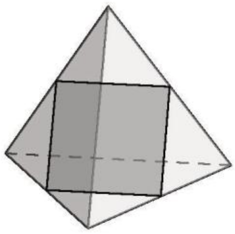
Which of the following could *not* be the shape of the section formed by the cut?

A a pentagon
B a square
C a rectangle that is not a square
D a trapezium
E a triangle that is not equilateral

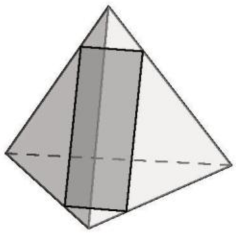
SOLUTION **A**

When the regular tetrahedron is cut by a single plane cut, each of its four faces is cut at most once. The place where each face is cut becomes the edge of the newly formed section. Therefore a pentagon, with five edges, cannot be formed.

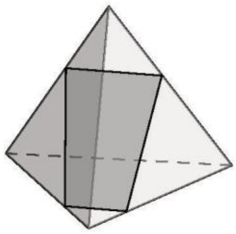
However, each of the other four options is possible, as the following diagrams show.



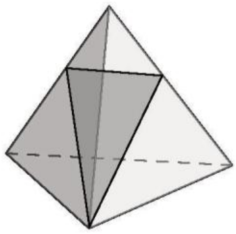
A square



A rectangle that is not a square

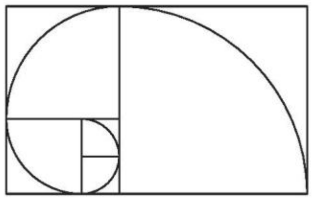


A trapezium



A triangle that is not equilateral

12. Five square tiles are put together side by side. A quarter circle is drawn on each tile to make a continuous curve as shown. Each of the smallest squares has side-length 1.



What is the total length of the curve?

A 6π B 6.5π C 7π D 7.5π E 8π

Solution **A**

The side lengths of the 5 squares are 1, 1, 2, 3 and 5. So the curve is made up of five quarter circles with these radii. The circumference of a circle with radius r is $2\pi r$. Therefore the length of a quarter circle of radius r is $\frac{1}{4}(2\pi r)$, that is, $\frac{1}{2}\pi r$.

Therefore the length of the curve is l, where

$$l = \tfrac{1}{2}\pi 1 + \tfrac{1}{2}\pi 1 + \tfrac{1}{2}\pi 2 + \tfrac{1}{2}\pi 3 + \tfrac{1}{2}\pi 5 = \tfrac{1}{2}\pi(1 + 1 + 2 + 3 + 5) = 6\pi.$$

13. *PQRST* is a regular pentagon. The point U lies on ST such that $\angle QPU$ is a right angle.

What is the ratio of the interior angles in triangle PUT?

A 1 : 3 : 6 B 1 : 2 : 4 C 2 : 3 : 4 D 1 : 4 : 8 E 1 : 3 : 5

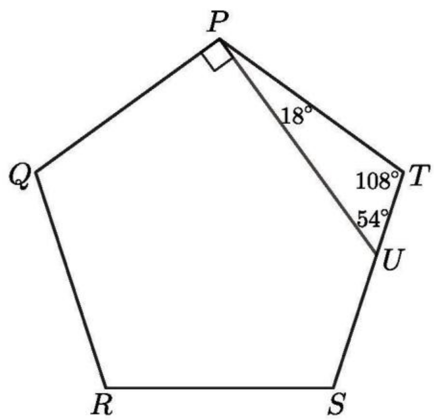
Solution **A**

The interior angles of a regular pentagon are all 108°. [You are asked to prove this in Problem 13.1.] Therefore $\angle PTU = \angle QPT = 108°$.

Hence $\angle UPT = \angle QPT - \angle QPU = 108° - 90° = 18°$.

Because the angles in a triangle have sum 180°, it follows that $\angle TUP = 180° - 108° - 18° = 54°$.

Therefore the ratio of the interior angles in the triangle PUT is 18 : 54 : 108 = 1 : 3 : 6.



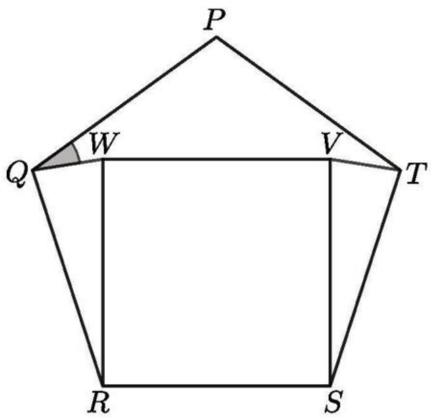
For investigation

(a) Show that the sum of the interior angles of a polygon with n vertices is $(n - 2)180°$.

(b) Deduce that the interior angles of a regular pentagon are equal to 108°.

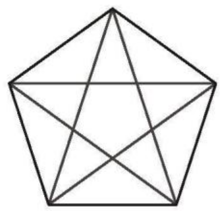
The regular pentagon *PQRST* and the square *RSVW* share the edge *RS*.

What is $\angle WQP$?

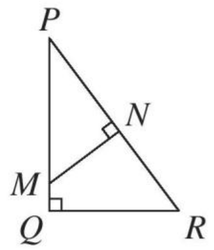


The diagram shows a regular pentagon and all its diagonals.

Find all the angles in the diagram.



14. The diagram shows a triangle PQR with $\angle PQR = 90°$, $PQ = 20$ and $PR = 25$. Point M lies on PQ, point N lies on PR and PNM is a right-angled triangle whose area is half that of triangle PQR.



What is the length of MN?

A $6\sqrt{2}$ B $\frac{15}{2}\sqrt{2}$ C $8\sqrt{2}$ D $\frac{17}{2}\sqrt{2}$ E $10\sqrt{2}$

SOLUTION **B**

We note first that, by Pythagoras' Theorem applied to the right-angled triangle PQR,

$$QR^2 = PR^2 - PQ^2 = 25^2 - 20^2 = 625 - 400 = 225 = 15^2.$$

Therefore $QR = 15$.

Note that PQR is the standard right-angled triangle with sides of lengths 3, 4 and 5 scaled up by the factor 5.

If you notice that $PQ : PR = 4 : 5$, you could deduce that $QR = 15$ without the need to do the above calculation.

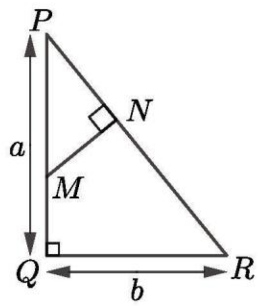
In the triangles PQR and PNM, we have $\angle PQR = \angle PNM = 90°$. These triangles share the angle at P. Because the angles in a triangle have sum 180°, it follows that $\angle PRQ = \angle NMP$. Therefore these triangles are similar.

The areas of similar triangles are proportional to the squares of the lengths of their corresponding sides. [Problem 11.2 asks you to prove this.]

Therefore since area of PNM : area of $PQR = 1 : 2$, it follows that $MN : QR = 1 : \sqrt{2}$. Therefore $MN = \dfrac{QR}{\sqrt{2}} = \dfrac{15}{\sqrt{2}} = \dfrac{15}{2}\sqrt{2}$.

FOR INVESTIGATION

The diagram on the right shows a triangle PQR in which $\angle PQR = 90°$, $PQ = a$ and $QR = b$. M is a point on PQ and N is the point on PR such that MN is perpendicular to PR.



(a) Suppose that area of $PMN = k\times$ area of PQR, with $0 < k < 1$.

Find the ratio $PN : NR$ in terms of a, b and k.

(b) Suppose now that the point M coincides with the point Q.

Find, in terms of a and b, the ratio $PN : NR$.

Prove that the areas of similar triangles is proportional to the squares of their corresponding sides.

15. What is the smallest number of rectangles, each measuring 2 cm by 3 cm, which are needed to fit together without overlap to form a rectangle whose sides are in the ratio 5 : 4 ?

A 10 B 15 C 20 D 30 E 60

Solution **D**

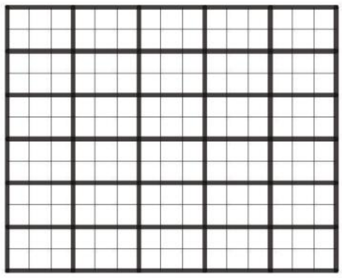
A rectangle whose sides are in the ratio 5 : 4 has dimensions $5k$ cm $\times$ $4k$ cm, for some positive number k. Since we aim to cover this rectangle with 2 cm $\times$ 3 cm rectangles without overlap, k needs to be a positive integer.

The area of the $5k$ cm$\times 4k$ cm rectangle is $(5k \times 4k)\,\text{cm}^2 = 20k^2\,\text{cm}^2$. The 2 cm$\times$3 cm rectangles have area $6\,\text{cm}^2$.

It follows that $20k^2$ needs to be a multiple of 6. The least positive integer k for which this is the case is 3. In this case the larger rectangle has dimensions 15 cm $\times$ 12 cm and area $(15 \times 12)\,\text{cm}^2 = 180\,\text{cm}^2$. Therefore the smallest number of 2 cm $\times$ 3 cm rectangles that are needed would be $\frac{180}{6} = 30$, provided that it is possible to cover a 15 cm $\times$ 12 cm with 30 2 cm $\times$ 3 cm rectangles.

To complete the question we need to show that this is possible.

One way in which this can be done is shown in the diagram on the right.



Therefore the smallest number of 2 cm $\times$ 3 cm rectangles that are needed is 30.

For investigation

Find other ways to fit together 30 rectangles measuring 2 cm$\times$3 cm to make a 15 cm$\times$12 cm rectangle.

16. The points $P\ (d, -d)$ and $Q\ (12 - d, 2d - 6)$ both lie on the circumference of the same circle whose centre is the origin.

What is the sum of the two possible values of d?

A -16 B -4 C 4 D 8 E 16

Solution **E**

Let $O\ (0, 0)$ be the origin.

Because the points P and Q lie on the same circle with centre O, we have $OP^2 = OQ^2$. That is,

$$d^2 + (-d)^2 = (12 - d)^2 + (2d - 6)^2.$$

Expanding both sides of this equation, we obtain

$$d^2 + d^2 = (144 - 24d + d^2) + (4d^2 - 24d + 36).$$

We can rearrange this equation to obtain

$$3d^2 - 48d + 180 = 0.$$

By dividing both sides of this last equation by 3, it follows that

$$d^2 - 16d + 60 = 0.$$

We can now use the fact that the sum of the roots of the quadratic equation $x^2 + px + q = 0$ is $-p$ to deduce that the sum of the two possible values of d is $-(-16)$, that is, 16.

For investigation

(a) Find the two possible values of d by solving the equation $d^2 - 16d + 60 = 0$.

(b) Hence check that the sum of the two possible values of d is 16.

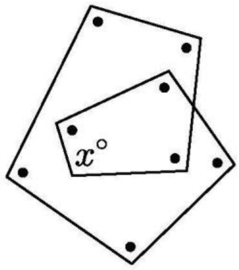
Find the centre of the circle that goes through the points $(4, -14)$, $(-3, -13)$ and $(-7, -11)$.

Prove that the sum of the roots of the quadratic equation $x^2 + px + q = 0$ is $-p$.

17. In the diagram all the angles marked • are equal in size to the angle marked $x°$.

What is the value of x?

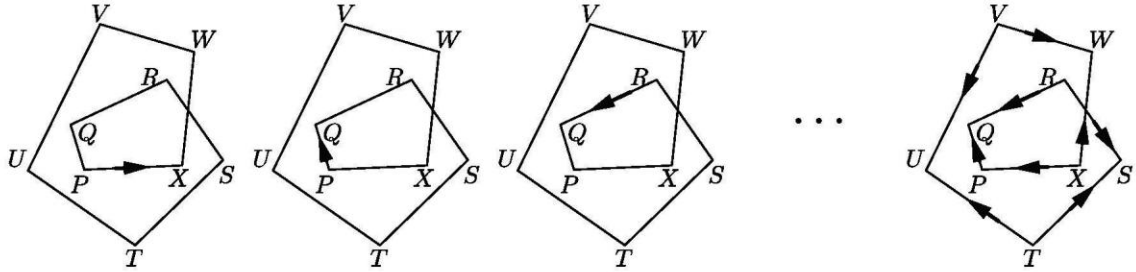
A 100 B 105 C 110 D 115 E 120



Solution **A**

Method 1

We label the vertices of the figure as shown below.



We place an arrow lying along PX in the direction shown in the first figure on the left above. We rotate this arrow anticlockwise about the point P until it lies along PQ in the direction shown in the second figure above. The arrow has been turned anticlockwise through the angle $x°$.

Next we rotate the arrow anticlockwise about the point Q until it lies along RQ in the direction shown in the third figure above. As all the angles marked • are $x°$, the arrow has again been turned anticlockwise through $x°$.

We continue this process, rotating the arrow anticlockwise through $x°$ about the points R, S, T, U, V, W and X in turn. The arrow ends up lying along XP in the direction shown in the figure on the right above.

In this figure we have also shown the direction in which the arrow points on all the other edges during this process, apart from its initial position.

It will be seen that in this process the arrow has been turned through $2\frac{1}{2}$ complete revolutions. Therefore the total angle it has turned through is $2\frac{1}{2} \times 360°$, that is, through $900°$. In the process the arrow been rotated 9 times through the angle $x°$.

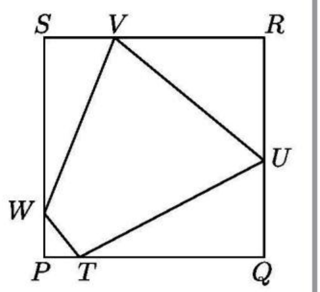
Therefore $9x = 900$ and hence $x = 100$.

18. The diagram shows a square $PQRS$. Points T, U, V and W lie on the edges of the square, as shown, such that $PT = 1$, $QU = 2$, $RV = 3$ and $SW = 4$.

The area of $TUVW$ is half that of $PQRS$.

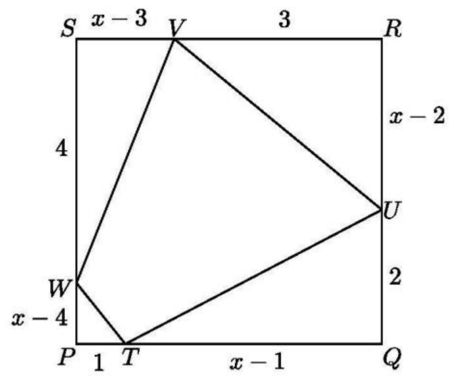
What is the length of PQ?

A 5 B 6 C 7 D 8 E 9



Solution **B**

We let the side length of the square $PQRS$ be x. Then the lengths of TQ, UR, VS and WP are $x - 1$, $x - 2$, $x - 3$ and $x - 4$, respectively.



The area of the square $PQRS$ is x^2. The area of $TUVW$ is a half of this. It follows that the sum of the areas of the triangles PTW, TQU, URV and VSW is also a half of the area of the square, and hence this sum equals $\frac{1}{2}x^2$.

The area of a triangle is half the product of its base and its height.

We therefore have

$$\tfrac{1}{2}(1 \times (x - 4)) + \tfrac{1}{2}(2 \times (x - 1)) + \tfrac{1}{2}(3 \times (x - 2)) + \tfrac{1}{2}(4 \times (x - 3)) = \tfrac{1}{2}x^2.$$

This equation simplifies to give

$$10x - 24 = x^2,$$

and therefore

$$x^2 - 10x + 24 = 0.$$

The left-hand side of the last equation factorizes to give

$$(x - 4)(x - 6) = 0.$$

Hence $x = 6$ or $x = 4$.

For there to be four triangles, as shown in the diagram, $x > 4$. We therefore deduce that $x = 6$.

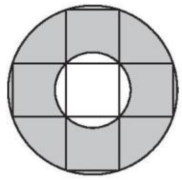
So the length of PQ is 6.

For investigation

Suppose that, as in the question $PT = 1$, $QU = 2$, $RV = 3$ and $SW = 4$, but the area of $TUVW$ is two-thirds of the area of $PQRS$.

What is the length of PQ in this case?

19. Five congruent squares, each of side $2a$, are placed edge to edge. Two circles with the same centre are drawn through the vertices as shown.

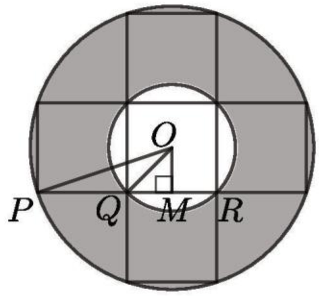


What is the area of the region between the two circles?

A $2\pi a^2$ B $4\pi a^2$ C $6\pi a^2$ D $8\pi a^2$ E $10\pi a^2$

SOLUTION **D**

We let O be the centre of the two circles and P, Q and R be the vertices shown in the diagram.



We also let M be the midpoint of the side QR of the central square.

Then OM is perpendicular to QR and $OM = QM$. [You are asked to prove this in Problem 13.1.]

Since each square has side length $2a$, $OM = QM = a$ and $PM = PQ + QM = 2a + a = 3a$.

Therefore, by applying Pythagoras' Theorem to the right-angled triangle QMO, we have

$$QO^2 = QM^2 + OM^2 = a^2 + a^2 = 2a^2.$$

QO is the radius of the smaller circle. Hence the area of this circle is $\pi(QO^2) = \pi(2a^2) = 2\pi a^2$.

Similarly. by applying Pythagoras' Theorem to the right-angled triangle PMO, we have

$$PO^2 = PM^2 + OM^2 = (3a)^2 + a^2 = 9a^2 + a^2 = 10a^2.$$

Hence the area of the larger circle is $10\pi a^2$.

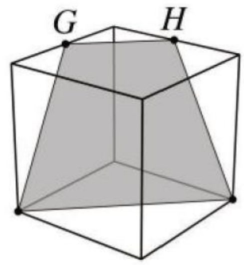
The area of the region between the circles is the difference of their areas. Hence this area is $10\pi a^2 - 2\pi a^2 = 8\pi a^2$.

FOR INVESTIGATION

Prove that OM is perpendicular to QR and $OM = QM$.

What is the area of the region inside each of the outer squares but not in the smaller circle?

20. G and H are midpoints of two adjacent edges of a cube. A trapezium-shaped cross-section is formed cutting through G, H and two further vertices, as shown. The edge-length of the cube is $2\sqrt{2}$.

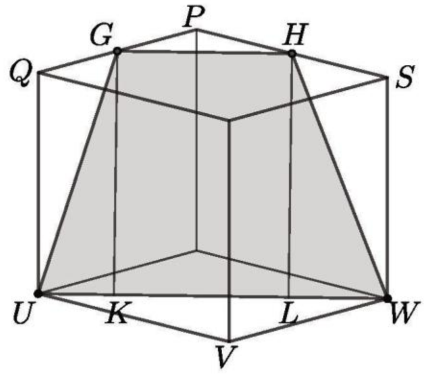


What is the area of the trapezium?

A 9 B 8 C $4\sqrt{5}$ D $4\sqrt{3}$ E $4\sqrt{2}$

SOLUTION **A**

We let P, Q, S, U, V and W be vertices of the cube, as shown in the diagram, and K, L be the feet of the perpendiculars from G, H, respectively, to UW.



We apply Pythagoras' Theorem to the right-angled triangle UVW. This gives $UW^2 = UV^2 + WV^2 = (2\sqrt{2})^2 + (2\sqrt{2})^2 = 8 + 8 = 16$. Hence $UW = 4$.

G is the midpoint of the edge PQ. Hence $PG = \sqrt{2}$. Similarly, $PH = \sqrt{2}$. Therefore, applying Pythagoras' Theorem to the right-angled triangle GPH gives $GH^2 = PG^2+PH^2 = (\sqrt{2})^2+(\sqrt{2})^2 = 2+2 = 4$. Hence $GH = 2$.

Also, applying Pythagoras' Theorem to the right-angled triangle GQU gives $GU^2 = GQ^2+UQ^2 = (\sqrt{2})^2 + (2\sqrt{2})^2 = 2 + 8 = 10$. Hence $GU = \sqrt{10}$.

Since $GKLH$ is a rectangle, $KL = GH = 2$. Therefore $UK + WL = UW - KL = 4 - 2 = 2$. By symmetry, $UK = WL$. Hence $UK = WL = 1$.

Applying Pythagoras' Theorem to the right-angled triangle GKU, gives $GK^2 = GU^2 - UK^2 = (\sqrt{10})^2 - 1^2 = 10 - 1 = 9$. Hence $GK = 3$.

We now use the formula $\frac{1}{2}(a + b)h$ for the area of a trapezium whose parallel sides have lengths a and b and which has height h. [You are asked to prove this formula in Problem 16.1.]

It follows from this formula that

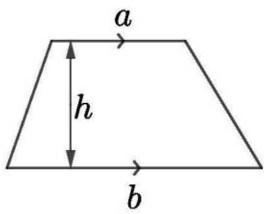
$$\text{area of the trapezium } GUWH = \tfrac{1}{2}(UW + GH)GK = \tfrac{1}{2}(4 + 2) \times 3 = 9.$$

FOR INVESTIGATION

Prove that the formula

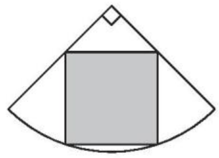
$$\tfrac{1}{2}(a + b)h$$

for the area of a trapezium is correct.



21. A square is inscribed inside a quadrant of a circle. The circle has radius 10.

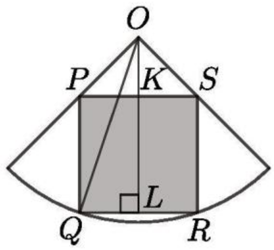
What is the area of the square?



A $25\sqrt{2}$ B 36 C 12π D 40 E $30\sqrt{2}$

SOLUTION **D**

We let O be the centre of the circle, and P, Q, R and S be the vertices of the square, as shown. We let OL be the perpendicular from O to QR, and K be the point where this perpendicular meets PS.



We let s be the side length of the square. It may be checked (see Problem 20.1) that the triangles OLQ and OLR are congruent. It follows that $QL = \frac{1}{2}QR = \frac{1}{2}s = PK$.

It may be checked that OKP is a right-angled isosceles triangle (see Problem 20.2). Therefore $OK = PK = \frac{1}{2}s$.

It follows that in the right-angled triangle OLQ we have $QL = \frac{1}{2}s$, $OL = OK + KL = \frac{1}{2}s + s = \frac{3}{2}s$ and $OQ = 10$.

Therefore, by Pythagoras' Theorem

$$(\tfrac{1}{2}s)^2 + (\tfrac{3}{2}s)^2 = 10^2.$$

Hence

$$\tfrac{1}{4}s^2 + \tfrac{9}{4}s^2 = 100.$$

That is,

$$\tfrac{5}{2}s^2 = 100.$$

Hence

$$s^2 = \tfrac{2}{5} \times 100 = 40.$$

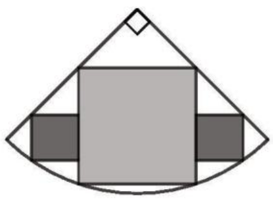
It follows that the area of the square is 40.

FOR INVESTIGATION

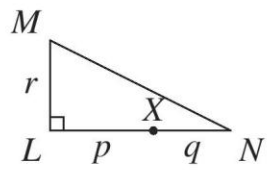
- Show that the triangles OLQ and OLR are congruent.
- Show that OKP is a right-angled isosceles triangle.
- The diagram on the right shows one larger square and two smaller squares inscribed in the quadrant of a circle.

  The circle has radius 10.

  Find the side length of the smaller squares.



22. Triangle LMN represents a right-angled field with $LM = r$, $LX = p$ and $XN = q$. Jenny and Vicky walk at the same speed in opposite directions along the edge of the field, starting at X at the same time. Their first meeting is at M.



Which of these expressions gives q in terms of p and r?

A $\frac{p}{2} + r$ B $\sqrt{p^2 + r^2} + \frac{p}{2}$ C $\frac{pr}{2p + r}$ D $\frac{p}{2}$ E 1

Solution **C**

Because Jenny and Vicky meet after walking at the same speed, they walk the same distance. Therefore

$$XL + LM = XN + NM. \qquad (1)$$

The question tells us that $XL = p$, $LM = r$ and $XN = q$, To find NM, we apply Pythagoras' Theorem to the right-angled triangle LMN. This gives

$$NM^2 = LN^2 + LM^2 = (p + q)^2 + r^2.$$

It follows that $NM = \sqrt{(p + q)^2 + r^2}$. Substituting these values in equation (1) gives

$$p + r = q + \sqrt{(p + q)^2 + r^2}.$$

Hence

$$p + r - q = \sqrt{(p + q)^2 + r^2}. \qquad (2)$$

By squaring both sides of equation (2) we obtain

$$(p + r - q)^2 = (p + q)^2 + r^2.$$

It follows that

$$p^2 + r^2 + q^2 + 2pr - 2pq - 2rq = p^2 + 2pq + q^2 + r^2. \qquad (3)$$

Equation (3) may be rearranged to give

$$4pq + 2rq = 2pr$$

from which it follows that

$$q(2p + r) = pr.$$

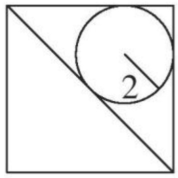
Therefore

$$q = \frac{pr}{2p + r}.$$

For investigation

- Check that $(p + r - q)^2 = p^2 + r^2 + q^2 + 2pr - 2pq - 2rq$.
- Suppose that in this problem, $LM = LN$. In this case what is the ratio $LX : XN$?

23. The diagram shows a square, one of its diagonals and a circle. The circle touches the diagonal and two sides of the square. The circle has radius 2.

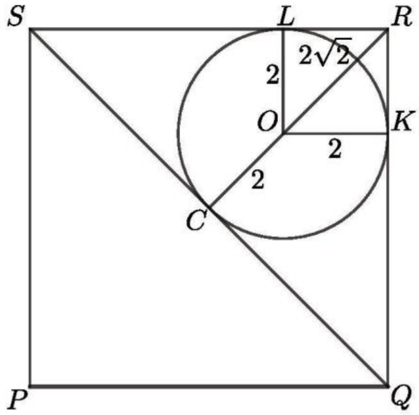


What is the length of the side of the square?

A $4 + 2\sqrt{2}$ B $8 - \sqrt{3}$ C $2 + 2\sqrt{2}$ D $8 - 2\sqrt{2}$ E $4 + 2\sqrt{3}$

SOLUTION **A**

We let O be the centre of the circle, and C be the centre of the square. The other points are labelled as shown in the diagram. Since both the circle and the square are symmetrical about diagonal PR of the square, it follows that the circle touches the diagonal QS at C.



Since the circle touches the sides QR and RS of the square, the radii OK and OL are perpendicular to QR and RS, respectively. Therefore $OKRL$ is a square with side length 2.

By Pythagoras' Theorem applied to the right angled triangle OKR, we have $RO^2 = OK^2 + KR^2 = 2^2 + 2^2 = 8$.

It follows that $OR = \sqrt{8} = 2\sqrt{2}$. Hence $RC = RO + OC = 2\sqrt{2} + 2$.

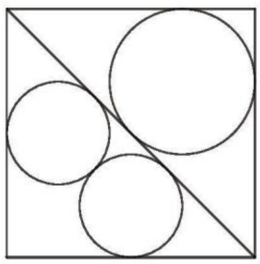
Because C is the centre of the square, $CR = CQ$. Because QS is a tangent to the circle, $\angle RCQ = 90°$. Hence RCQ is a right-angled isosceles triangle. Therefore its sides have lengths in the ratio $1 : 1 : \sqrt{2}$.

It follows that the length of the side of the square is given by

$$QR = \sqrt{2}(2\sqrt{2} + 2) = 4 + 2\sqrt{2}.$$

FOR INVESTIGATION

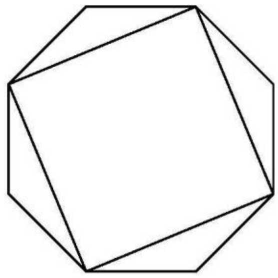
The diagram shows a square, one of its diagonals and three circles.



The largest circle touches the diagonal and two sides of the square. The smaller circles touch the diagonal, one side of the square and each other, as shown.

What is the ratio of the area of the largest circle to the total area of the two smaller circles?

24. The diagram shows a regular octagon and a square formed by drawing four diagonals of the octagon. The edges of the square have length 1.

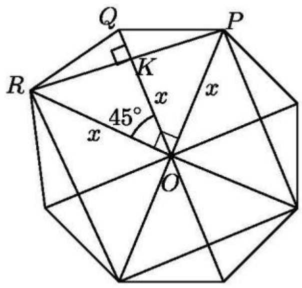


What is the area of the octagon?

A $\frac{\sqrt{6}}{2}$ B $\frac{4}{3}$ C $\frac{7}{5}$ D $\sqrt{2}$ E $\frac{3}{2}$

Solution **D**

Let O be the centre of the regular octagon, and let P, Q and R be adjacent vertices of the octagon as shown in the figure on the right. Let K be the point where OQ meets PR.



Let x be distance of O from the vertices of the octagon.

Since the edges of the square have length 1, $PR = 1$. By Pythagoras' Theorem applied to the right-angled triangle PRO, we have $x^2 + x^2 = 1^2$. Therefore $x^2 = \frac{1}{2}$ and hence $x = \frac{1}{\sqrt{2}}$.

The triangle ROQ has a base OQ of length x, that is $\frac{1}{\sqrt{2}}$, and height RK of length $\frac{1}{2}$. Therefore the area of the triangle ROQ is $\frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{2}$, which equals $\frac{1}{4\sqrt{2}}$.

The octagon is made up of 8 triangles each congruent to triangle ROQ.

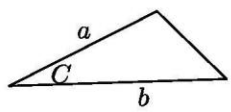
Therefore the area of the octagon is given by $8 \times \frac{1}{4\sqrt{2}} = \sqrt{2}$.

For investigation

The solution above assumes that $\angle POR = 90°$, $\angle RKO = 90°$ and $RK = \frac{1}{2}$. Explain why these statements are true.

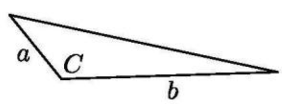
An alternative method for finding the area of the triangle ROQ is to use the "$\frac{1}{2}ab \sin C$" formula for the area of a triangle. Show how this also gives $\frac{1}{4\sqrt{2}}$ for the area of triangle ROQ.

(a) Show how the formula $\frac{1}{2}ab \sin C$ for the area of a triangle which has sides of lengths a and b, with included angle C, may be deduced from the fact that the area of a triangle is given by the formula

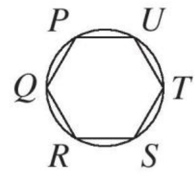


$$\text{area} = \tfrac{1}{2}(\text{base} \times \text{height}).$$

(b) Does your argument cover the case where C is an obtuse angle as well as the case where it is an acute angle?



25. A regular hexagon $PQRSTU$ is inscribed in a circle of radius 5. A point X on the circumference of the circle is connected to the vertices of the hexagon to form six chords XP, XQ, XR, XS, XT and XU.



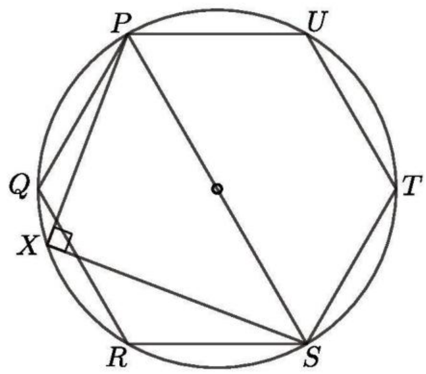
What is the value of $XP^2 + XQ^2 + XR^2 + XS^2 + XT^2 + XU^2$?

A 150 B 216 C 256 D 300 E 360

Solution **D**

The circle has radius 5. Hence it has diameter 10.

PS is a diameter of the circle. Hence, by the *Angle in a Semicircle* theorem (otherwise known as *Thales'* theorem), $\angle PXS = 90°$.



Therefore, by Pythagoras' Theorem

$$XP^2 + XS^2 = PS^2 = 10^2 = 100.$$

Similarly, $XQ^2 + XT^2 = 100$ and $XR^2 + XU^2 = 100$.

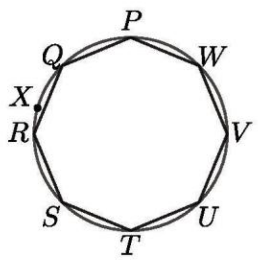
It follows that

$$XP^2 + XQ^2 + XR^2 + XS^2 + XT^2 + XU^2 = 100 + 100 + 100 = 300.$$

For investigation

A regular octagon $PQRSTUVW$ is inscribed in a circle of radius r.

A point X on the circumference of the circle is connected to the vertices of the octagon to form eight chords, XP, XQ, XR, XS, XT, XU, XV and XW.



Find the value of

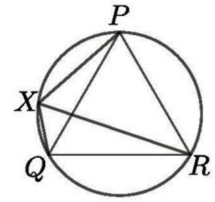
$$XP^2 + XQ^2 + XR^2 + XS^2 + XT^2 + XU^2 + XV^2 + XW^2$$

in terms of r.

An equilateral triangle PQR is inscribed in a circle of radius r.

The point X is on the circumference of the circle between P and Q, as shown in the diagram.

X is connected to the vertices of the triangle to form three chords, XP, XQ and XR.



(a) Find the value of $XP^2 + XQ^2 + XR^2$ in terms of r.

(b) Prove that $XP + XQ = XR$. [This is van Schooten's Theorem. Search the web to find a proof. This theorem can be generalized to any regular polygon with an odd number of sides, but the proof of this is not easy.]