AMC 10
MOCK TEST 5
Solution Book

Geometry

ThrivingScholars

1. Which of the following shows the digit 6 after it has been rotated clockwise through 135°?

A  B  C  D  E 

Solution **D**

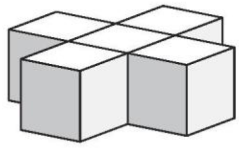
Because 135° = 90° + 45°, a rotation through 135° clockwise is equivalent to a rotation clockwise through 90°, which is a quarter turn, followed by a rotation clockwise through 45°, which is one-eighth of a complete turn.

Therefore, as the diagram shows, a rotation through 135° clockwise, results in the configuration given as option D.

2. A shape is made from five unit cubes, as shown.

What is the surface area of the shape?

A 22 B 24 C 26 D 28 E 30



SOLUTION **A**

The surface of the shape is made up of two faces of the central cube, and five faces of each of the four other cubes.

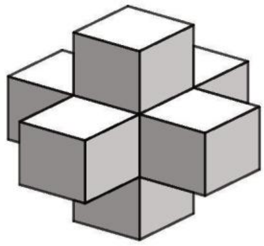
Therefore the surface is made up of $2 + 4 \times 5 = 22$ square faces each of size 1×1.

Hence the surface area of the shape is 22.

FOR INVESTIGATION

The shape shown is made by adding two unit cubes to the shape of this question. It is made from seven unit cubes.

What is the surface area of the shape?



3. The sizes of the three angles in a triangle, in degrees, are x, $7x$ and x^2.

What is the size of the largest angle?

A $10°$ B $18°$ C $100°$ D $120°$ E $121°$

SOLUTION **C**

The sum of the angles in a triangle is $180°$. Therefore $x + 7x + x^2 = 180$. Hence $x^2 + 8x - 180 = 0$. By factorizing the left-hand side of this equation, we deduce that $(x + 18)(x - 10) = 0$.

This implies that either $x = 10$ or $x = -18$. Because $x°$ is one of the angles of the triangle, $x > 0$. We conclude that $x = 10$.

It follows that the angles of the triangle are $10°$, $70°$ and $100°$. Hence the largest angle is $100°$.

FOR INVESTIGATION

The angles of a triangle, in degrees, are $x^2 + 9x$, x^2 and $9x$.

What is the size of the smallest angle?

The angles of a triangle, in degrees, are x^3, $14x^2$ and $9x$, where x is an integer.

Show that this triangle is isosceles.

4. Alex draws a scalene triangle. One of the angles is $80°$.

Which of the following could be the difference between the other two angles in Alex's triangle?

A $0°$ B $60°$ C $80°$ D $100°$ E $120°$

Solution **C**

All the side lengths of a scalene triangle are different, therefore all the angles are different.

It follows that the difference between two of the angles cannot be $0°$. This rules out option A.

The sum of the angles in a triangle is $180°$. Therefore, as one of the angles in Alex's triangle is $80°$, the sum of the other two angles is $100°$. It follows that the difference between these angles is less than $100°$. This rules out options D and E.

If the two angles with sum $100°$ have a difference of $60°$, these angles would be $80°$ and $20°$. So the triangle would have two angles of $80°$, which is impossible for a scalene triangle. This rules out option B.

Therefore the correct option is C, since it is the only one we have not eliminated.

Note

In the context of the SMC it is adequate to stop here. Once four of the options have been eliminated, the remaining option must be correct.

However, for a full solution you would need to show that it is possible for the difference between the other two angles to be $80°$. You are asked to do this in Exercise 4.1, below.

For investigation

What are the angles of a scalene triangle in which one angle is $80°$ and the difference between the other two angles is $80°$?

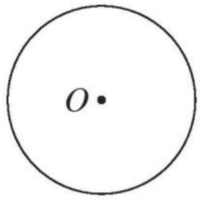
Show that, as stated in the solution, if two angles have sum $100°$ and difference $60°$, then the two angles are $80°$ and $20°$.

The argument used in the solution above is based on the fact that

> In a scalene triangle (that is, a triangle where all the side lengths are different) all the angles are different.

Prove that this is correct.

5. Three points, P, Q and R are placed on the circumference of a circle with centre O. The arc lengths PQ, QR and RP are in the ratio 1 : 2 : 3.

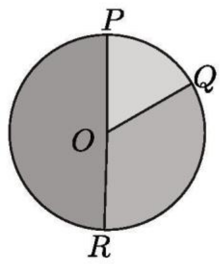


In what ratio are the areas of the sectors POQ, QOR and ROP?

A 1 : 1 : 1 B 1 : 2 : 3 C $1 : \pi : \pi^2$
D 1 : 4 : 9 E 1 : 8 : 27

Solution **B**

The key facts that we need to use here are



(a) the ratio of the length of an arc to the length of the circumference is the same as the ratio of the angle that the arc subtends at the centre of the circle to the angle in a complete revolution,

and

(b) the ratio of the area of a sector to the area of the circle is the same as the ratio of the angle in the sector to the angle in a complete revolution.

It follows from (a) that the ratios of the arc lengths of PQ, QR and RP are the same as the ratios of the angles that the arcs subtend at the centre of the circle.

Therefore these angles are in the ratio 1 : 2 : 3.

Similarly, it follows from (b) that the ratios of the areas of the sectors POQ, QOR and ROP are the same as the ratios of the angles in the sectors.

Therefore the areas of the sectors POQ, QOR and ROP are in the ratio 1 : 2 : 3.

Commentary

From the basic facts (a) and (b) we can deduce that

(c) The ratio of the length of an arc of a circle to the circumference of a circle is equal to the ratio of the area the arc subtends at the centre of the circle to the area of the circle.

An alternative method would have been to base the solution on this fact.

For investigation

Suppose that the circle has radius 3 and the arc lengths PQ, QR and RP are in the ratio 2 : 3 : 4.

What is the area of the sector QOR?

6. One face of a solid polyhedron is an octagon.

What is the smallest possible number of edges the solid could have?

A 9 B 10 C 12 D 16 E 24

SOLUTION **D**

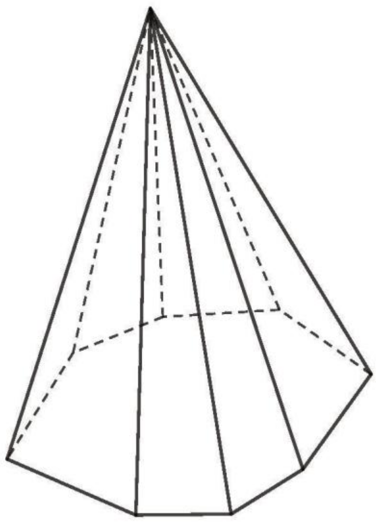
The octagonal face of the polyhedron has 8 edges.

Each vertex of this face must be joined by an edge to a vertex not in this face. So there must be at least 8 more edges.

Hence the polyhedron has at least $8 + 8 = 16$ edges.

If all the vertices of the octagonal face are joined by an edge to the same vertex, we obtain a pyramid with an octagonal base that has 16 edges.

Therefore the smallest number of edges that the solid could have is 16.



FOR INVESTIGATION

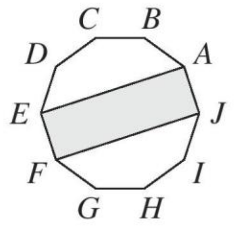
Three of the faces of a polyhedron are squares.

What is the smallest number of vertices that this polyhedron could have?

7. The diagram shows a rectangle $AEFJ$ inside a regular decagon $ABCDEFGHIJ$.

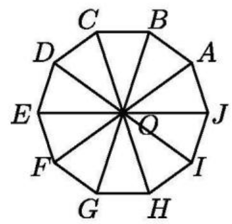
What is the ratio of the area of the rectangle to the area of the decagon?

A $2:5$ B $1:4$ C $3:5$ D $3:10$ E $3:20$



Solution **A**

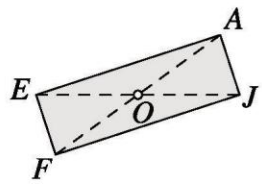
Let O be the centre of the regular decagon. The decagon is divided into ten congruent isosceles triangles by the radii joining O to each of the vertices of the decagon.The triangles AOJ and EOF are two of these ten congruent triangles.



Because O is the centre of the rectangle $AEFJ$ and the diagonals of a rectangle split its area into four equal areas, the triangles EOA and FOJ each have the same area as triangles AOJ and EOF.

Therefore the area of $AEFJ$ is equal to the area of 4 of the 10 congruent triangles that make up the decagon.

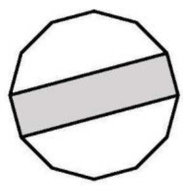
Hence the ratio of the area of $AEFJ$ to the area of the decagon $ABCDEFGHI$ is $4:10$ which simplifies to $2:5$.



For investigation

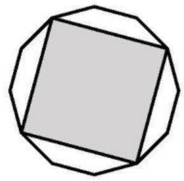
- Prove the following geometrical facts that the above solution tacitly assumes. [A full solution would need to include these proofs.]
  (a) The decagon has a centre, that is, there is circle which goes through all its vertices. (The centre of this circle is the centre of the decagon.)
  (b) The centre of the regular decagon is also the centre of the rectangle $AEFJ$.
  (c) The triangle EOA has the same area as the triangle AOJ.
- The diagram shows a rectangle whose vertices are two pairs of adjacent vertices of a regular dodecagon.

  What is the ratio of the area of the rectangle to the area of the dodecagon?



- Generalize the result of the question and the above problem to the case of a rectangle whose vertices are two pairs of adjacent vertices of a regular polygon with $2n$ sides, where n is a positive integer.
- The diagram shows a square inscribed in a regular dodecagon.

  What is the ratio of the area of the square to the area of the dodecagon?



8. A rectangle is divided into three smaller congruent rectangles as shown.

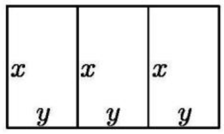
Each smaller rectangle is similar to the large rectangle.

In each of the four rectangles, what is the ratio of the length of a longer side to that of a shorter side?

A $2\sqrt{3} : 1$ B $3 : 1$ C $2 : 1$ D $\sqrt{3} : 1$ E $\sqrt{2} : 1$

Solution **D**

We suppose that the length of the longer sides of the three smaller rectangles is x and the length of their shorter sides is y.



It follows that the longer sides of the large rectangle have length $3y$, and that its shorter sides have length x.

Because the smaller rectangles are similar to the larger rectangle $\frac{x}{y} = \frac{3y}{x}$. Therefore $\frac{x^2}{y^2} = \frac{3}{1}$.

Hence $\frac{x}{y} = \frac{\sqrt{3}}{1}$.

It follows that the ratio of the length of a longer side to that of a shorter side in all the rectangles is $\sqrt{3} : 1$.

For investigation

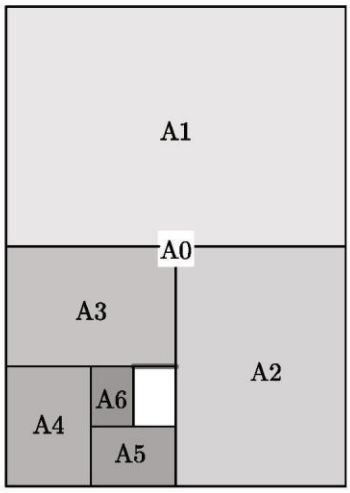
The A series of paper sizes, (A0, A1, A2, A3, . . .), is defined as follows.

The largest size is A0.

Two A1 sized sheets of paper are obtained by cutting an A0 sheet in half along a line parallel to its shorter edges.

Two A2 sized sheets of paper are obtained by cutting an A1 sheet in half in a similar way, and so on, as shown in the figure.

The shapes of all these sheets of paper are similar rectangles.



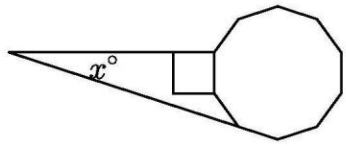
(a) What is the ratio of the length of a longer side to the length of the shorter side in all these rectangles?

(b) An A0 sheet of paper has area $1\,\text{m}^2$. What are lengths, to the nearest cm, of the longer and shorter sides of an A0 sheet of paper?

(c) The most commonly used of these sizes is A4. What are the lengths, to the nearest cm, of the longer and shorter sides of an A4 sheet of paper?

(d) Standard quality paper weighs $80\,\text{g}/\text{m}^2$. What is the weight of one standard quality sheet of A4 paper?

9. The diagram shows a square and a regular decagon that share an edge. One side of the square is extended to meet an extended edge of the decagon.

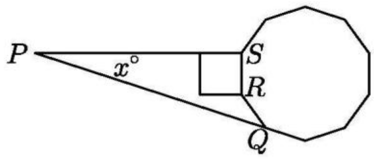


What is the value of x?

A 15 B 18 C 21 D 24 E 27

SOLUTION **B**

We consider the quadrilateral $PQRS$ as shown in the figure on the right. Because $PQRS$ is a quadrilateral its angles have total 360°.



Because it is the exterior angle of a decagon, $\angle RQP = \frac{1}{10} \times 360° = 36°$.

It follows that the interior angle of a decagon is $(180 - 36)° = 144°$. Hence the reflex angle SRQ is $(360 - 144)° = 216°$. The angle PSR is an angle of the square and hence is 90°.

Therefore, we have

$$x° + 36° + 216° + 90° = 360°.$$

It follows that

$$x° = (360 - 342)° = 18°.$$

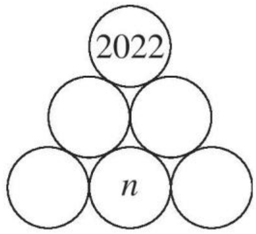
FOR INVESTIGATION

- We have used here the fact that the exterior angle of a regular polygon with n sides is $\frac{1}{n} \times 360°$. Explain why this is true.
- Prove that the sum of the angles of a quadrilateral is 360°.

10. In the number triangle shown, each disc is to be filled with a positive integer. Each disc in the top or middle row contains the number which is the product of the two numbers immediately below.

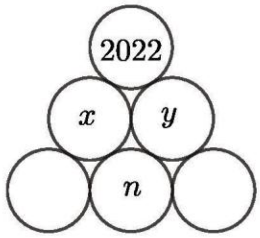
What is the value of n?

A 1   B 2   C 3   D 6   E 33



SOLUTION **A**

We let x and y be the positive integers in the discs in the middle row, as shown.



Since these are the products of the two numbers immediately below, n is a factor of both x and y. Hence n^2 is a factor of xy.

Now $xy = 2022$. It follows that n^2 is a factor of 2022.

The prime factorization of 2022 is

$$2022 = 2 \times 3 \times 337.$$

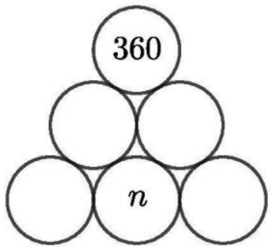
It follows that 1^2 is the only square of a positive integer that is a factor of 2022.

We deduce that $n = 1$.

FOR INVESTIGATION

- Check that $2 \times 3 \times 337$ is the factorization of 2022 into prime factors.
- In the number triangle shown on the right, each disc is to be filled with a positive integer. Each disc in the top or middle row contains the number which is the product of the two numbers immediately below.

  What are the possible values of n?



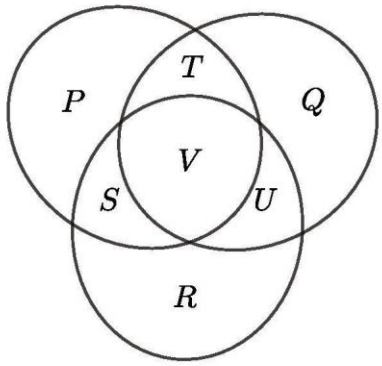
11. Three rugs have a combined area of $90\,\text{m}^2$. When they are laid down to cover completely a floor of area $60\,\text{m}^2$, the area which is covered by exactly two layers of rug is $12\,\text{m}^2$.

What is the area of floor covered by exactly three layers of rug?

A $2\,\text{m}^2$ B $6\,\text{m}^2$ C $9\,\text{m}^2$ D $10\,\text{m}^2$ E $12\,\text{m}^2$

Solution **C**

The diagram shows the three overlapping rugs.



We let the areas of the different regions, in m^2, be P, Q, R, S, T, U and V, as shown.

The three rugs have areas $P+S+T+V$, $Q+T+U+V$ and $R+S+U+V$. Therefore, because the three rugs have a combined area of $90\,\text{m}^2$,

$$(P+S+T+V)+(Q+T+U+V)+(R+S+U+V)=90.$$

This equation may be rearranged to give

$$(P+Q+R)+2(S+T+U)+3V=90. \qquad (1)$$

Because the floor has an area of $60\,\text{m}^2$,

$$(P+Q+R)+(S+T+U)+V=60. \qquad (2)$$

By subtracting equation (2) from equation (1), we obtain

$$(S+T+U)+2V=30. \qquad (3)$$

Because the area covered by exactly two layers of rug is $12\,\text{m}^2$,

$$S+T+U=12. \qquad (4)$$

By subtracting equation (4) from equation (3), we have $2V=18$. Therefore $V=9$.

Therefore the area of floor covered by exactly three layers of rug is $9\,\text{m}^2$.

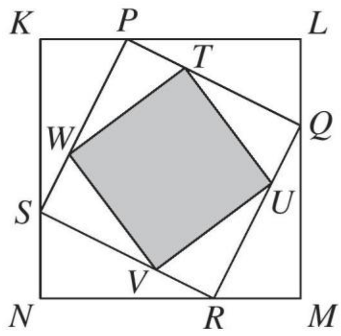
For investigation

What is the area of the floor that is covered by exactly one rug?

12. The diagram shows a square, $KLMN$. A second square $PQRS$ is drawn inside it, as shown in the diagram, where P divides the side KL in the ratio 1 : 2. Similarly, a third square $TUVW$ is drawn inside $PQRS$ with T dividing PQ in the ratio 1 : 2.

What fraction of the area of $KLMN$ is shaded?

A $\frac{25}{81}$ B $\frac{16}{49}$ C $\frac{4}{9}$ D $\frac{40}{81}$ E $\frac{2}{3}$



SOLUTION **A**

Since P divides the side KL in the ratio 1 : 2, we choose units so that KL has length 3. Hence KP has length 1, and PL has length 2. We note also that it follows that the square $KLMN$ has area 3^2, that is, 9.

We leave it to the reader to show that the triangles PKS, QLP, RMQ and SNR are congruent.

It follows, in particular, that $SK = PL$. Therefore SK has length 2.

Using Pythagoras' Theorem, applied to the right-angled triangle PKS, we have $PS^2 = 1^2 + 2^2 = 5$. Therefore the square $PQRS$ has area 5.

Therefore the area of the square $PQRS$ as a fraction of the area of the square $KLMN$ is $\frac{5}{9}$.

It follows, similarly, that the area of the square $TUVW$ as a fraction of the area of the square $PQRS$ is also $\frac{5}{9}$.

It follows that the area of the shaded square $TUVW$ is $\frac{5}{9} \times \frac{5}{9} \times$ the area of the square $KLMN$.

Since $\frac{5}{9} \times \frac{5}{9} = \frac{25}{81}$, it follows that the fraction of the area of $KLMN$ that is shaded is $\frac{25}{81}$.

FOR INVESTIGATION

- Show that the triangles PKS, QLP, RMQ and SNR are congruent.
- Suppose that P divides the side KL in the ratio 1 : 3, and that T divides PQ in the ratio 1 : 3.

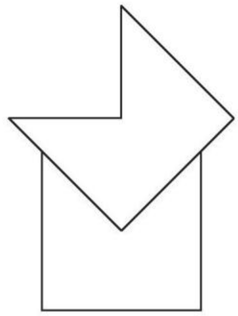
  In this case, what fraction of the area of $KLMN$ is shaded?
- Suppose that P divides the side KL in the ratio $p : q$, and that T divides PQ in the ratio $r : s$.

  In this case, what fraction of the area of $KLMN$ is shaded?

13. Two congruent pentagons are each formed by removing a right-angled isosceles triangle from a square of side-length 1. The two pentagons are then fitted together as shown.

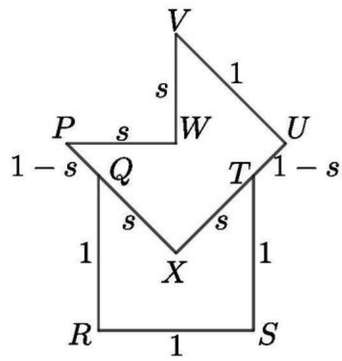
What is the length of the perimeter of the octagon formed?

A 4 B $4+2\sqrt{2}$ C 5 D $6-2\sqrt{2}$ E 6



Solution **E**

Let P, Q, R, S, T, U, V and W be the vertices of the octagon, as shown in the diagram, and let X be the vertex as shown.



The sides PX, XU, UV, QR, RS and ST all have length 1.

Let PW have length s. Then, because PWV is a right-angled isosceles triangle, VW also has length s.

Because the pentagons $PXUVW$ and $QRSTX$ are congruent, QX and TX also have length s.

It follows that PQ and TU each have length $1-s$.

We can now deduce the the length of the perimeter of the octagon $PQRSTUVW$ is

$$(1-s)+1+1+1+(1-s)+1+s+s=6.$$

Commentary

Note that in order to work out the length of the perimeter we did not need to know the value of s. It is, however, not difficult to find the value of s. You are asked to do this in Problem 11.1.

For investigation

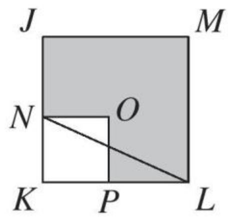
Find the length of QX.

Find the area of the octagon $PQRSTUVW$.

14. The diagram shows two squares, $JKLM$ and $NKPO$.
The length of NL is $10\,\text{cm}$. The shaded region has area $62\,\text{cm}^2$.

What is the length of KN in cm?

A 3   B $\sqrt{18}$ C $\sqrt{19}$ D $\sqrt{22}$ E 5



Solution **C**

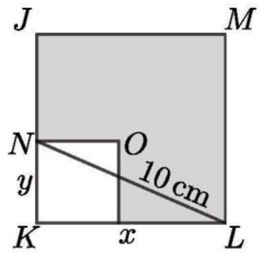
Suppose that the side length of the square $JKLM$ is $x\,\text{cm}$ and the side length of the square $NKPO$ is $y\,\text{cm}$.

The area of the shaded region is the area of $JKLM$ minus the area of $NKPO$. Therefore

$$x^2 - y^2 = 62. \qquad (1)$$

Also, by Pythagoras' Theorem $KL^2 + NK^2 = NL^2$. That is,

$$x^2 + y^2 = 10^2. \qquad (2)$$



By subtracting equation (1) from equation (2), we obtain

$$2y^2 = 10^2 - 62 = 100 - 62 = 38.$$

Therefore $y^2 = 19$. Hence $y = \sqrt{19}$.

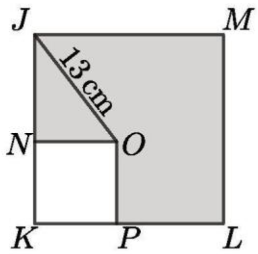
For investigation

$JKLM$ and $KPON$ are squares.

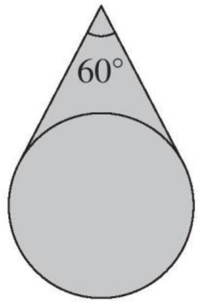
The square $JKLM$ has area $289\ \text{cm}^2$.

JO has length $13\,\text{cm}$.

What are the possibilities for the area of the square $KPON$?



15. The shaded area shown in the diagram consists of the interior of a circle of radius 3 together with the area between the circle and two tangents to the circle. The angle between the tangents at the point where they meet is 60°.



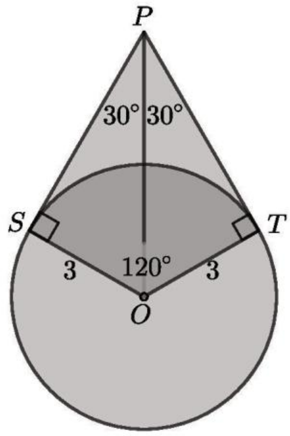
What is the shaded area?

A $6\pi + 9\sqrt{3}$ B $15\sqrt{3}$ C 9π

D $9\pi + 4\sqrt{3}$ E $6\pi + \frac{9\sqrt{3}}{4}$

SOLUTION **A**

Let O be the centre of the circle and let PS and PT be the tangents to the circle, as shown.



In the triangles PSO and PTO we have $\angle PSO = \angle PTO = 90°$ because PS and PT are tangents to the circle; $SO = TO$, because they are radii of the same circle; and the side PO is common.

Therefore, the two triangles are congruent (RHS). Hence $\angle SPO = \angle TPO = 30°$.

It follows that $\dfrac{OS}{PS} = \tan 30° = \dfrac{1}{\sqrt{3}}$.

Hence $PS = OS\sqrt{3} = 3\sqrt{3}$.

Using the formula area $= \frac{1}{2}(\text{base} \times \text{height})$, for the area of a triangle, it follows that the area of the triangle PSO is $\frac{1}{2}(OS \times PS) = \frac{1}{2}(3 \times 3\sqrt{3}) = \frac{1}{2}(9\sqrt{3})$.

This is also the area of the congruent triangle PTO. Therefore the sum of the areas of these two triangles is $9\sqrt{3}$.

The total shaded area is the sum of the areas of these two triangle plus the area of that part of the circle that lies outside the two triangles. Because $\angle SOT = 120°$, the part of the circle outside the two triangles makes up two-thirds of the circle and hence its area is given by

$$\tfrac{2}{3}(\pi \times 3^2) = 6\pi.$$

Hence the shaded area is $6\pi + 9\sqrt{3}$.

FOR INVESTIGATION

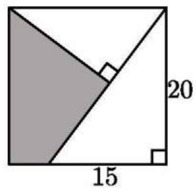
- Explain why $\angle SOT = 120°$.
- Explain why it follows that the area of the part of the circle outside the two triangles is two-thirds of the total area of the circle.
- The solution above uses the fact that $\tan 30° = \dfrac{1}{\sqrt{3}}$.

  Explain why this is correct.
- What is the shaded area in the case where $\angle SPT = 120°$?

16. The diagram shows two right-angled triangles inside a square. The perpendicular edges of the larger triangle have lengths 15 and 20.

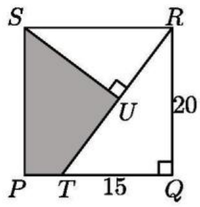
What is the area of the shaded quadrilateral?

A 142 B 146 C 150 D 154 E 158



SOLUTION **D**

We let the vertices of the square be P, Q, R and S, and the points T and U be as shown.



We note first that, by Pythagoras' Theorem, applied to the right-angled triangle TQR,

$$RT^2 = 15^2 + 20^2 = 225 + 400 = 625 = 25^2.$$

It follows that $RT = 25$.

Because SR is parallel to PQ, the alternate angles, $\angle SRU$ and $\angle QTU$ are equal. Therefore, the right-angled triangles SUR and RQT are similar. Therefore their corresponding sides are in proportion. Hence

$$\frac{SR}{RT} = \frac{SU}{RQ} = \frac{RU}{TQ}.$$

Now $SR = RQ = 20$. Hence

$$\frac{20}{25} = \frac{SU}{20} = \frac{RU}{15}.$$

It follows that $SU = 16$ and $RU = 12$.

The area of the shaded quadrilateral is the area of the square $PQRS$ less the areas of the triangles SUR and RQT. It follows that the area of the shaded quadrilateral is

$$20^2 - \tfrac{1}{2}(16 \times 12) - \tfrac{1}{2}(20 \times 15) = 400 - 96 - 150 = 154.$$

NOTE

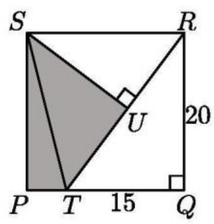
Another way to find the length of RT is to note that the lengths of TQ and QR are in the ratio $3 : 4$, and that therefore the right-angled triangle TQR is a $(3, 4, 5)$ triangle scaled by the factor 5. Hence the length of RT is $5 \times 5 = 25$.

Then, because the triangle RUS is similar to triangle TQR and has a hypotenuse of length 20, it follows that RUS is a $(3, 4, 5)$ triangle scaled by the factor 4. Hence RU has length $4 \times 3 = 12$ and SU has length $4 \times 4 = 16$.

FOR INVESTIGATION

Calculate the area of the triangle SPT and the area of the triangle TUS.

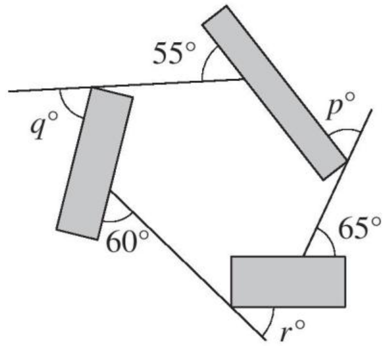
Check that the sum of these areas is 154.



17. The diagram shows three rectangles and three straight lines.

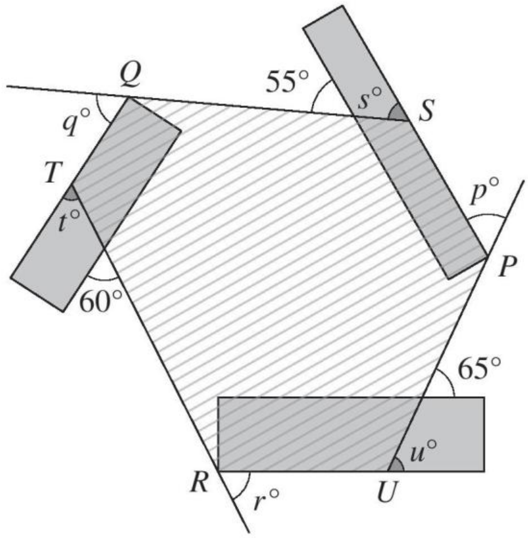
What is the value of $p + q + r$?

A 135 B 180 C 210 D 225 E 270



SOLUTION **B**

Let P, Q and R be the points shown in the diagram where the rectangles touch the straight lines. Let the straight lines when extended meet the rectangles at the points S, T and U, as shown.



Then $PSQTRU$ is a hexagon. The external angles of this hexagon at the vertices P, Q and R are p°, q°, and r°, respectively, as given. Let the external angles at S, T and U be s°, t° and u°, as shown.

The external angles of a hexagon have sum 360°. Therefore $p + q + r + s + t + u = 360$.

The opposite sides of a rectangle are parallel. When lines are parallel the corresponding angles are equal. Therefore $s = 55$, $t = 60$ and $u = 65$. Hence $s + t + u = 55 + 60 + 65 = 180$.

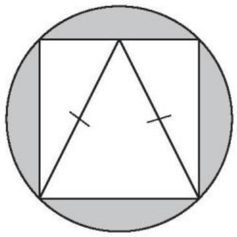
It follows that $p + q + r = (p + q + r + s + t + u) - (s + t + u) = 360 - 180 = 180$.

FOR INVESTIGATION

Explain why the sum of the external angles of a hexagon, and, indeed, of any polygon, is 360°.

18. A square is inscribed in a circle of radius 1. An isosceles triangle is inscribed in the square as shown.

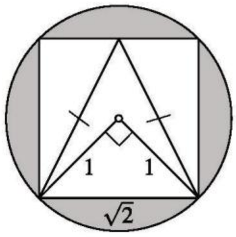
What is the ratio of the area of this triangle to the area of the shaded region?



A $\pi : \sqrt{2}$ B $\pi : 1$ C $1 : 4$ D $1 : \pi - 2$ E $2 : \pi$

Solution **D**

Let the sides of the square have length s. We see from the diagram that, by Pythagoras' theorem, $s^2 = 1^2 + 1^2$ and hence $s = \sqrt{2}$.



The area of the circle is $\pi \times 1^2$, which equals π. The area of the square is $\sqrt{2} \times \sqrt{2}$, which equals 2. Hence the area of the shaded region is $\pi - 2$.

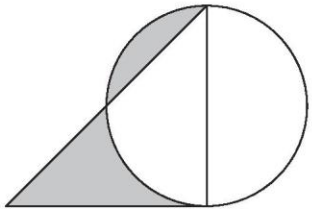
The base of the isosceles triangle is one of the sides of the square and so has length $\sqrt{2}$. The perpendicular height of the isosceles triangle is also $\sqrt{2}$. The area of the triangle is therefore $\frac{1}{2}(\sqrt{2} \times \sqrt{2})$ which equals 1.

Hence the ratio of the area of the triangle to the area of the shaded region is $1 : \pi - 2$.

For investigation

Show that the area of the isosceles triangle is half the area of the square.

19. A circle of radius r and a right-angled isosceles triangle are drawn such that one of the shorter sides of the triangle is a diameter of the circle.

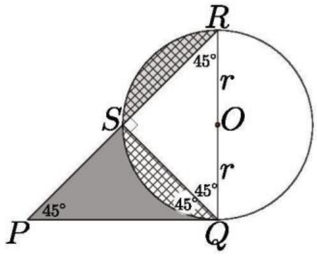


What is the shaded area?

A $\sqrt{2}r$ B r^2 C $2\pi r$ D $\frac{\pi r^2}{4}$
E $(\sqrt{2}-1)\pi r^2$

Solution **B**

Let O be the centre of the circle and let P, Q, R and S be the points as shown in the diagram.



Because PQR is a right-angled isosceles triangle, $\angle PRQ = \angle RPQ = 45°$, and $PQ = QR = 2r$.

Because the angle in a semicircle is a right angle [this is Thales' theorem], $\angle RSQ = 90°$. Therefore, because the sum of the angles in a triangle is 180°, we have $\angle RQS = 45°$.

We therefore have $\angle SRQ = 45° = \angle SQR$. It follows that $SQ = SR$.

Because $SQ = SR$ the segments of the circle cut off by these lines, shown as hatched in the diagram, are congruent. Hence they have the same area.

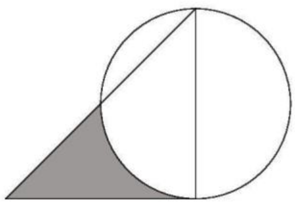
It follows that the shaded area is the same as the area of the triangle PQS.

PQS is a right-angled isosceles triangle with hypotenuse PQ of length $2r$. Therefore the triangle PQS has area r^2.

Therefore the shaded area is r^2.

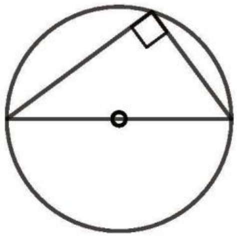
For investigation

- Explain why from the fact that the hypotenuse of the triangle PQS has length $2r$, it follows that the area of the triangle is r^2.
- In the diagram on the right there is a circle of radius r and a right-angled isosceles triangle. One of the shorter sides of the triangle is a diameter of the circle.

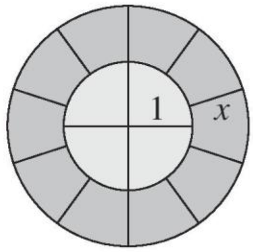
  

  What is the shaded area?
- Give a proof of Thales' theorem:

  The angle in a semicircle is a right angle.



20. The diagram shows two concentric circles divided by radial lines into 14 pieces of equal area. The radius of the smaller circle is 1.



What is the length, x, of an outer radial line?

A $\sqrt{14} - 1$ B $\sqrt{14} - 2$ C $\dfrac{\sqrt{14}}{2} - 1$

D $\dfrac{\sqrt{14}}{2} - 2$ E $\dfrac{\sqrt{14} - 1}{2}$

Solution **C**

The outer circle has radius $1 + x$. Hence its area is $\pi(1 + x)^2$.

The inner circle has radius 1, and hence has area $\pi(1^2) = \pi$. Hence each of the four inner pieces has area $\dfrac{\pi}{4}$.

Because all the 14 pieces have the same area, the area of the outer circle is 14 times the area of one of the inner pieces. Therefore

$$\pi(1 + x)^2 = 14\left(\frac{\pi}{4}\right).$$

This simplifies to give

$$(1 + x)^2 = \frac{14}{4}.$$

Hence, as $1 + x > 0$,

$$1 + x = \sqrt{\frac{14}{4}} = \frac{\sqrt{14}}{2}.$$

We conclude that

$$x = \frac{\sqrt{14}}{2} - 1.$$

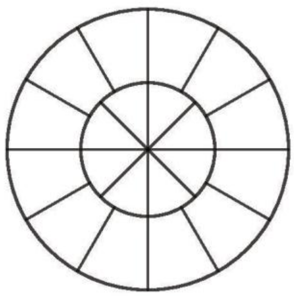
For investigation

The diagram on the right shows two concentric circles divided by radial lines into 20 pieces of equal area.

The inner circle has radius r.

The outer circle has radius s.

Find the ratio $r : s$.



21. P, Q and R are the three angles of a triangle, when each has been rounded to the nearest degree.

Which of the following is the complete list of possible values of $P + Q + R$?

A $179°$, $180°$ or $181°$ B $180°$, $181°$ or $182°$ C $178°$, $179°$ or $180°$
D $180°$ E $178°$, $179°$, $180°$, $181°$ or $182°$

SOLUTION **A**

We suppose that the actual angles of the triangle are P', Q' and R', which are rounded to P, Q and R, respectively.

The sum of the angles of a triangle is $180°$ and therefore

$$P' + Q' + R' = 180°.$$

When an angle is rounded up to the nearest degree, it is increased by at most $0.5°$; when it is rounded down to the nearest degree, it is decreased by at most $0.5°$. Therefore,

$$\begin{aligned} P' - 0.5° &\le P \le P' + 0.5°, \\ Q' - 0.5° &\le Q \le Q' + 0.5°, \\ \text{and} \quad R' - 0.5° &\le R \le R' + 0.5°. \end{aligned}$$

Adding these inequalities gives

$$P' + Q' + R' - 1.5° \le P + Q + R \le P' + Q' + R' + 1.5°.$$

Therefore, as $P' + Q' + R' = 180°$,

$$178.5° \le P + Q + R \le 181.5°.$$

Each of P, Q and R is an integer number of degrees. Hence $P + Q + R$ is also an integer number of degrees. It follows that the only possible values of $P + Q + R$ are $179°$, $180°$ and $181°$.

To complete the solution we show that each of these possible values actually occurs for some triangle. This is shown by the following examples.

$P' = 60.3°$, $Q' = 60.3°$, $R' = 59.4°$ gives $P = 60°$, $Q = 60°$, $R = 59°$. Hence $P + Q + R = 179°$.

$P' = 60°$, $Q' = 60°$, $R' = 60°$ gives $P = 60°$, $Q = 60°$, $R = 60°$. Hence $P + Q + R = 180°$.

$P' = 59.7°$, $Q' = 59.7°$, $R' = 60.6°$ gives $P = 60°$, $Q = 60°$, $R = 61°$. Hence $P + Q + R = 181°$.

We deduce that $179°$, $180°$, $181°$ is a complete list of the possible values of $P + Q + R$.

FOR INVESTIGATION

Give an example of a triangle whose angles are all different non-integer numbers of degrees, but whose rounded angles have sum $180°$.

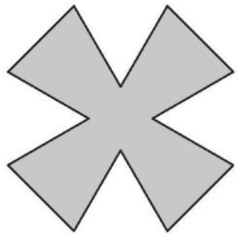
P, Q, R and S are the four angles of a quadrilateral, when each has been rounded to the nearest degree. Give a list of all the values that $P + Q + R + S$ can take.

The angles of a polygon with n vertices are each rounded to the nearest integer. Give, in terms of n, a list of the values that the sum of these rounded angles can take.

22. The shape shown is made by removing four equilateral triangles with side-length 1 from a regular octagon with side-length 1.

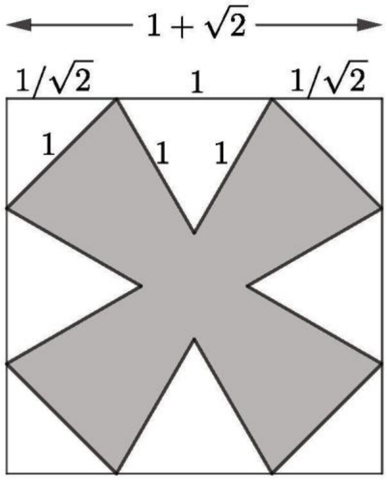
What is the area of the shape?

A $2 - 2\sqrt{2} + \sqrt{3}$ B $2 + 2\sqrt{2} - \sqrt{3}$ C $2 + 2\sqrt{2} + \sqrt{3}$
D $3 - 2\sqrt{2} - \sqrt{3}$ E $2 - 2\sqrt{2} - \sqrt{3}$



Solution **B**

As the diagram on the right shows, a regular octagon with side length 1 may be obtained by cutting four triangular corners from the square.



Each of these corners is an isosceles right-angled triangles with a hypotenuse of length 1. It follows, by Pythagoras' Theorem that the other two sides of these triangles have length $1/\sqrt{2}$.

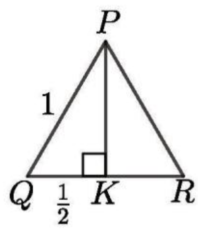
It follows that the side length of the square is given by $1/\sqrt{2} + 1 + 1/\sqrt{2} = 1 + \sqrt{2}$. Therefore the area of the square is $(1 + \sqrt{2})^2 = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}$.

The four triangular corners fit together to make a square of side length 1 and hence area 1.

Therefore the area of the octagon is $(3 + 2\sqrt{2}) - 1 = 2 + 2\sqrt{2}$.

From this area we need to subtract the area of the four equilateral triangles that are removed from the octagon to make the shaded shape.

We let PQR be an equilateral triangle with side length 1, and let PK be the perpendicular from P to QR as shown.



It can be checked that K is the midpoint of QR (see Problem 17.1) and hence that QK has length $\frac{1}{2}$. By Pythagoras' Theorem applied to the right-angled triangle PQK we have $QK^2 + PK^2 = PQ^2$. Therefore $PK^2 = PQ^2 - QK^2 = 1 - \frac{1}{2}^2 = \frac{3}{4}$.

Therefore $PK = \frac{1}{2}\sqrt{3}$.

We can now deduce that the area of the triangle PQR is $\frac{1}{2}(QR \times PK) = \frac{1}{2}(1 \times \frac{1}{2}\sqrt{3}) = \frac{1}{4}\sqrt{3}$.

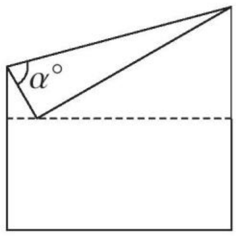
It now follows that the area of the shape is

$$2 + 2\sqrt{2} - 4 \times \frac{1}{4}\sqrt{3} = 2 + 2\sqrt{2} - \sqrt{3}.$$

For investigation

- Prove that the triangles PKQ and PKR are congruent. Deduce that the length of QK is $\frac{1}{2}$.
- Use the $\frac{1}{2}ab\sin C$ formula for the area of a triangle to confirm that the area of an equilateral triangle with side length 1 is $\frac{1}{4}\sqrt{3}$.

23. Alison folds a square piece of paper in half along the dashed line shown in the diagram. After opening the paper out again, she then folds one of the corners onto the dashed line.

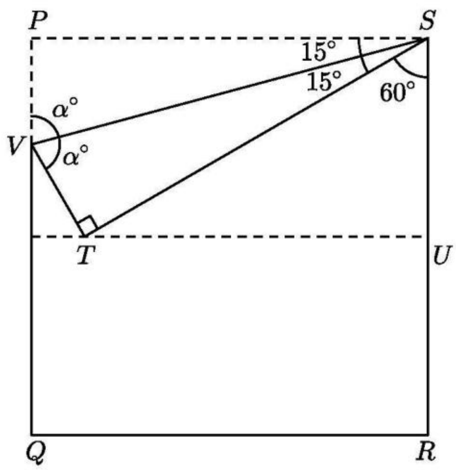


What is the value of α?

A 45 B 60 C 65 D 70 E 75

Solution **E**

Let the vertices of the square be P, Q, R and S, as shown. Let T be the position to which P is folded, and let U and V be the points shown in the diagram.



Because after the fold the triangle PSV coincides with the triangle TSV, these triangles are congruent. In particular $TS = PS$ and $\angle PSV = \angle TSV$.

Therefore SUT is a triangle with a right angle at U and in which $SU = \frac{1}{2}SR = \frac{1}{2}PS = \frac{1}{2}TS$. It follows that [see Exercise 12.4] $\angle TSU = 60°$.

Because $\angle USP = 90°$ and $\angle PSV = \angle TSV$, it follows that $\angle TSV = \frac{1}{2}(90 - 60)° = 15°$.

Therefore, because the sum of the angles in the triangle TSV is 180°, we have

$$\alpha + 15 + 90 = 180,$$

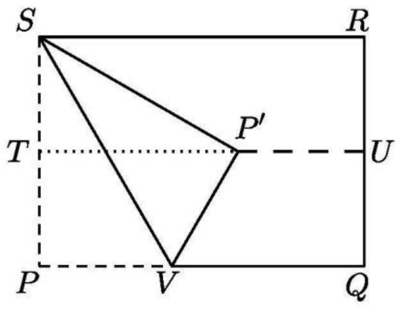
and therefore $\alpha = 75$.

For investigation

Suppose that $PQRS$ is a rectangular piece of paper, in which PQ is longer than QR.

The paper is folded in half along the line TU, and then unfolded.

Next, the paper is folded along the line SV through S so that the corner P ends up at the point P' on the first fold line TU.

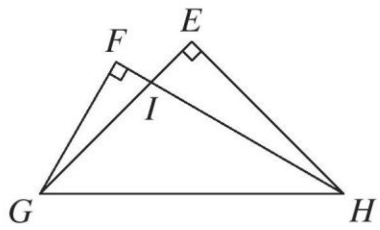


(a) Prove that $\angle SVP' = 60°$.

(b) Show how a rectangular piece of paper $PQRS$ may be folded to make an equilateral triangle, provided that the ratio $PQ : QR$ is sufficiently large.

24. The diagram shows two overlapping triangles: triangle FGH with interior angles $60°$, $30°$ and $90°$ and triangle EGH which is a right-angled isosceles triangle.

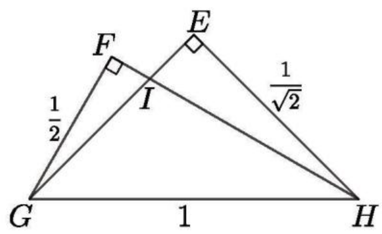
What is the ratio of the area of triangle IFG to the area of triangle IEH?



A $1:1$ B $1:\sqrt{2}$ C $1:\sqrt{3}$ D $1:2$ E $1:3$

Solution **D**

We suppose that we have chosen units so that the length of GH is 1.



Because FGH is a $60°, 30°, 90°$ triangle, it follows that FG has length $\frac{1}{2}$.

Because EGH is a right-angled isosceles triangle it also follows that EH has length $\frac{1}{\sqrt{2}}$.

In the triangles IFG and IEH we have

$$\angle GFI = \angle HEI = 90°$$

and

$$\angle GIF = \angle HIE \text{ (vertically opposite angles).}$$

Because the sum of the angles in both these triangles is $180°$, it follows that

$$\angle FGI = \angle EHI.$$

Therefore the triangles IFG and IEH are similar.

The ratio of the areas of similar triangles equals the ratio of the squares of the lengths of corresponding sides. Therefore

$$\begin{aligned}\text{area of } IFG : \text{area of } IEH &= FG^2 : EH^2 \\ &= \left(\tfrac{1}{2}\right)^2 : \left(\tfrac{1}{\sqrt{2}}\right)^2 \\ &= \tfrac{1}{4} : \tfrac{1}{2} \\ &= 1 : 2.\end{aligned}$$

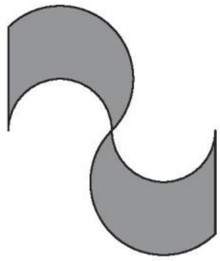
For investigation

- Explain why, given that GH has length 1, FG has length $\frac{1}{2}$ and EH has length $\frac{1}{\sqrt{2}}$.
- Explain why the ratio of the areas of similar triangles equals the ratio of the squares of the lengths of corresponding sides.
- (a) Given that GH has length 1, find the area of the triangle GIH.

  (b) Given that GH has length 1, find the areas of the triangles IFG and IEH.

  (c) Hence verify that the ratio of the areas of the triangles IFG and IEH is $1:2$.

25. The perimeter of a logo is created from two vertical straight edges, two small semicircles with horizontal diameters and two large semicircles. Both of the straight edges and the diameters of the small semicircles have length 2. The logo has rotational symmetry as shown.

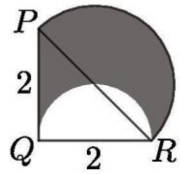


What is the shaded area?

A 4 B $4-\pi$ C 8 D $4+\pi$ E 12

SOLUTION **D**

The diagram on the right shows the top half of the shaded region.



This region is made up of the right-angled triangle PQR and the semicircle with diameter PR, but with the semicircle with QR as diameter removed.

The area of the triangle PQR is $\frac{1}{2}(PQ \times QR) = \frac{1}{2}(2 \times 2) = 2$.

By Pythagoras' Theorem, applied to the right-angled triangle PQR, we have $PR^2 = 2^2 + 2^2 = 8$. Therefore $PR = 2\sqrt{2}$.

Hence the semicircle with diameter PR has radius $\sqrt{2}$. Hence the area of this semicircle is $\frac{1}{2}\left(\pi(\sqrt{2})^2\right) = \pi$.

The semicircle with QR as diameter has radius 1, and therefore its area is $\frac{1}{2}(\pi 1^2) = \frac{1}{2}\pi$.

It follows that the area of the shaded region in the diagram above is

$$2 + \pi - \tfrac{1}{2}\pi = 2 + \tfrac{1}{2}\pi.$$

This area is half of the shaded area in the logo.

Therefore the shaded area in the logo is $2 \times (2 + \frac{1}{2}\pi) = 4 + \pi$.

FOR INVESTIGATION

Find the length of the perimeter of the shaded area.