TMUA TEST 1
Solution Book

Paper 1 Styled

- Algebra and Functions
- Sequences And Series
- Functions

ThrivingScholars

Question 1

The expansion of $(a - bx)^c$ is:

$$4 - px + 108x^2 - qx^3 + rx^4$$

where a, b, c, p, q, r are **positive real constants**.

> **Find the value of $p + q + r$.**

Options:

A. $81 - 84\sqrt{2}$
B. $81 + 132\sqrt{2}$
C. $132\sqrt{2} - 81$
D. $81 + 84\sqrt{2}$

For the expansion to give a quartic polynomial, $c = 4$ is necessary.
Looking at the constant term of $(a - bx)^4$ shows that $a^4 = 4$. Since a is positive and real, $a = \sqrt{2}$ is necessary.
The x^2 term in the expansion of $\left(\sqrt{2} - bx\right)^4$ is $6 \times \left(\sqrt{2}\right)^2 \times (-bx)^2 \equiv 12b^2x^2$, so $12b^2 = 108 \Leftrightarrow b^2 = 9 \Leftrightarrow b = 3$, since b is positive. (Seeing the possible answers and the appearance everywhere of $81 = 3^4$, does suggest the possibility of $b = 3$.)
The expansion is $\left(\sqrt{2} - 3x\right)^4 \equiv \left(\sqrt{2}\right)^4 + 4 \times \left(\sqrt{2}\right)^3 \times (-3x) + 6 \times \left(\sqrt{2}\right)^2 \times (-3x)^2 + 4 \times \sqrt{2} \times (-3x)^3 + (-3x)^4 \equiv 4 - 24\sqrt{2}x + 108x^2 - 108\sqrt{2}x^3 + 81x^4$.
From this, $p + q + r = 24\sqrt{2} + 108\sqrt{2} + 81 = 132\sqrt{2} + 81$
The answer is B.

Question 2

The coefficient of x^{11} in the expansion of

$$(2+x^2+x^3)^8$$

is equal to **28 times** the coefficient of x^2 in

$$(2+ax)^6$$

> **Find all the possible values of the constant a.**

Options:

A. $\pm 2\sqrt{7}$

B. $\pm \frac{1}{2}$

C. $\pm \frac{1}{4}$

D. ± 1

Thinking of the ways in which x^{11} terms can be obtained from $(2+x^2+x^3)^8$:
$(x^3)^3(x^2)^1 \times 2^4 \equiv 16x^{11}$ happens $\binom{8}{3}\binom{5}{1} = \frac{8\times7\times6}{3!} \times 5 = 56 \times 5 = 280$ times;
$(x^3)^1(x^2)^4 \times 2^3 \equiv 8x^{11}$ happens $\binom{8}{1}\binom{7}{4} = 8 \times \frac{7\times6\times5}{3!} = 8 \times 35 = 280$ times.
It follow that the coefficient of x^{11} is $280 \times 16 + 280 \times 8 = 280 \times 24$.

The coefficient of x^2 in $(2+ax)^6$ is $\binom{6}{2} \times 2^4 \times a^2 = \frac{6\times5}{2!} \times 16 \times a^2 = 240a^2$.

So $280 \times 24 = 28 \times 240a^2 \Leftrightarrow 10 = 10a^2 \Leftrightarrow a^2 = 1 \Leftrightarrow a = \pm 1$.

The answer is D

Question 3

The coefficient of x^2 in the expansion of

$$(2+bx)^4$$

is **2 times** the coefficient of x^3 in

$$(1+bx)^6$$

Given that $b \neq 0$,

> **What is the value of b?**

Options:

A. $\frac{8}{15}$

B. $\frac{3}{5}$

C. $\frac{3}{20}$

D. $\frac{6}{5}$

The x^2 term in the expansion of $(2+bx)^4$ is $6 \times 2^2 \times (bx)^2$, so the coefficient is $24b^2$.
The x^3 term in the expansion of $(1+bx)^6$ is $\binom{6}{3} \times 1^3 \times (bx)^3$, so the coefficient is $\frac{6\times5\times4}{3!}b^3 = 20b^3$.
Therefore, $24b^2 = 2 \times 20b^3 = 40b^3 \Leftrightarrow 3b^2 = 5b^3 \Leftrightarrow 0 = b^2(5b-3) \Leftrightarrow b = 0$ or $b = \frac{3}{5}$.
If $b = 0$, then both polynomials become constants (16 and 1, respectively) and there is no solution to the question.
The answer is B.

Question 4

Find the coefficient of:

$$(1-x)^0-(1-x)^1+(1-x)^2-(1-x)^3+(1-x)^4-\cdots+(1-x)^{50}$$

> **What is the value of the coefficient?**

Options:

A. 51

B. –51

C. –25

D. 25

For any positive integer, n, the expansion of $(1-x)^n$ begins $1-nx+\cdots$, and $(1-x)^0\equiv 1$ (assuming $x\neq 0$).
It follows that the x term in the above expression is $-(-x)+(-2x)-(-3x)+(-4x)-\cdots+(-50x)$, and that this has coefficient $1-2+3-4+\cdots-50$. This sum is 25 groups of $n-(n+1)=-1$ where $n=1,3,\ldots,49$. Therefore the coefficient of x is $25\times(-1)=-25$. The answer is C.

Question 5

Find the value of the expression:

$$\sqrt{16+8\sqrt{3}+3}-\sqrt{12-4\sqrt{3}+1}$$

What is the value of the expression?

Options:

A. $\sqrt{6}+12\sqrt{3}$

B. $\sqrt{3}-5$

C. 3

D. $5-\sqrt{3}$

Although there are other, much longer, available methods for answering this question, the examiners do drop a hint that there must be an easier approach on this occasion. In particular, for example, why do they write $16+8\sqrt{3}+3$ instead of $19+8\sqrt{3}$? This must be done for a reason. If you can quickly spot a value that squares to give $16+8\sqrt{3}+3$, then square rooting simply reverses the process. Considering $(a+b)^2 \equiv a^2+2ab+b^2$:

$16=4^2$ and $3=\left(\sqrt{3}\right)^2$, so it is worth checking the expansion of $\left(4\pm\sqrt{3}\right)^2 \equiv 16\pm 8\sqrt{3}+3$... you are in luck!
$12=\left(2\sqrt{3}\right)^2$ and $1=1^2$, so it is worth checking the expansion of $\left(2\sqrt{3}\pm 1\right)^2 \equiv 12\pm 4\sqrt{3}+1$... you are in luck, again, and grateful not to have 'piled in' to this question!

Therefore, $\sqrt{16+8\sqrt{3}+3}-\sqrt{12-4\sqrt{3}+1}=\sqrt{\left(4+\sqrt{3}\right)^2}-\sqrt{\left(2\sqrt{3}-1\right)^2}=4+\sqrt{3}-\left(2\sqrt{3}-1\right)=5-\sqrt{3}$.

The answer is D.

Question 6

The circles with equations:

$$(x-r)^2 + (y-2)^2 = r^2 \quad \text{and} \quad (x+r)^2 + (y+r)^2 = 4r^2$$

touch precisely once when:

Options:

A. $r = \sqrt{3} - 1$

B. $r = \frac{8}{5}$

C. $r = \frac{2}{1+\sqrt{5}}$

D. $r = \frac{1+\sqrt{5}}{2}$

Neither circle contains the centre of the other circle and so the circles only touch when the centres are distance $3r$ apart (the sum of their radii). It simplifies the algebra to work with the square of the distance between the centres $(r, 2)$ and $(-r, -r)$. Solving $(2r)^2 + (2+r)^2 = 9r^2$ gives $r = \frac{1 \pm \sqrt{5}}{2}$. Since $r > 0$, the correct answer is D.

Question 7

Find the value(s) of a such that the turning point of the parabola

$$y = x^2 + 2ax + 1$$

is **closest to the origin**.

> **What is the value of a?**

Options:

A. $a = \sqrt{2}$

B. $a = \pm\sqrt{2}$

C. $a = \frac{\sqrt{2}}{2}$

D. $a = \pm\frac{1}{\sqrt{2}}$

$x^2 + 2ax + 1 = (x + a)^2 + 1 - a^2$ so the turning point has coordinates $(-a, 1 - a^2)$. Letting d denote the distance from the turning point to the origin gives $d^2 = a^2 + (1 - a^2)^2 = a^4 - a^2 + 1 = \left(a^2 - \frac{1}{2}\right)^2 + \frac{3}{4}$. So the required values of a are $\pm\frac{1}{\sqrt{2}}$. The correct answer is D.

Question 8

The line $y = 2x + c$ is such that it intersects the circle

$$x^2 + y^2 = 9$$

at two points A and B. Let M be the **midpoint of the chord** AB of the circle.

> **Find the equation of the locus of** M as c varies between $-3\sqrt{5}$ and $3\sqrt{5}$.

Options:

A. $x^2 + y^2 = 2$

B. $y = -\frac{1}{2}x$

C. $y = \frac{1}{2}x$

D. $y = x^2$

. The line $y = 2x + c$ and the circle $x^2 + y^2 = 9$ intersect when $x^2 + (2x + c)^2 = 9$ giving $5x^2 + 4cx + (c^2 - 9) = 0$, so $x = \frac{-4c \pm \sqrt{180 - 4c^2}}{10}$ (note that there are 2 points of intersection provided that $180 - 4c^2 > 0$, and hence $-4\sqrt{5} < c < 4\sqrt{5}$). By considering the means of the coordinates of A and B you obtain $M(-\frac{2c}{5}, \frac{c}{5})$ (note that when computing the mean of the x-coordinates of A and B the discriminant cancels, and the mean of the y-coordinates of A and B is given by $2x + c$ with $x = -\frac{2c}{5}$). By inspection, as c varies, the coordinates of M satisfy $y = -\frac{x}{2}$. The correct answer is B.

Question 9

The tangent to the circle

$$x^2 + y^2 + 6x + 2y + 2 = 0$$

at the point $(-5, 1)$ passes through the point $(-3, 3)$.

The **other tangent** to the circle that passes through this point touches the circle at the point:

Options:

A. $(-3,\ 2\sqrt{2})$

B. $(-3,\ 1)$

C. $(-1,\ 2\sqrt{2})$

D. $(-1,\ 1)$

The centre-radius form for the circle is $(x + 3)^2 + (y + 1)^2 = 8$, so the circle's centre, $(-3, -1)$, lies vertically below the point $(-3, 3)$. By symmetry, the other tangent meets the circle at the point $(-1, 1)$. The correct answer is D.

Question 10

The circle

$$x^2 + y^2 + 2ax + 2by = c$$

encloses the point $(1,\ 1)$ **precisely when**:

Options:

A. $c > 2(1 - a - b)$

B. $a^2 + b^2 < 1$

C. $c > 2(a + b + 1)$

D. $a^2 + 2ab + b^2 < c$

$$x^2 + y^2 + 2ax + 2by = c \Rightarrow (x + a)^2 + (y + b)^2 = c + a^2 + b^2$$

The point $(1, 1)$ lies within the circle if and only if:

$$(1 + a)^2 + (1 + b)^2 < c + a^2 + b^2$$

$$\Rightarrow 2 + 2a + 2b < c \Rightarrow c > 2(1 + a + b)$$

The correct answer is **(c)**.

Question 11

Which of the following is a **tangent** to the circle

$$(x-2)^2+(y-3)^2=4 \ ?$$

Options:

A. $x=3+\sqrt{2}$

B. $y=5+\sqrt{2}-x$

C. $y=2(2-\sqrt{2})$

D. $y=x+1-2\sqrt{2}$

. A quick sketch rules out (b) and (d) since the vertical and horizontal tangents are $x=4$ and $y=5$ respectively. The point $(2,3+\sqrt{2})$ lies inside the circle (since $(x-2)^2+(y-3)^2=2$) and hence (c) can be eliminated. T[illegible] d) [illegible] are no obvious interior points for (a) and (e) so use direct substitution and check for a ze[illegible]scriminant. Considering (a) first, you obtain $(x-2)^2+(5-x)^2=4$ which rearranges to $2x^2-14x+25=0$ with discriminant -4, so the line does not intersect the circle. Considering (e) first, you obtain $(x-2)^2+\left(x-2-2\sqrt{2}\right)^2=4$ which rearranges to give $2x^2-\left(8+4\sqrt{2}\right)x+8\sqrt{2}+12=0$ with discriminant 0. The correct answer is (d)

Question 12

The reflection of the point

$$(a,\ a)$$

in the line

$$y = \frac{1}{a}x \quad \text{where } a \neq 0$$

is:

Options:

A.

$$\left(\frac{a(a-1)^2}{a^2+1}, \frac{a^2(a^2+3)}{a^2+1}\right)$$

B.

$$\left(\frac{a(a^2+2a-1)}{a^2+1}, \frac{-a(a^2-2a-1)}{a^2+1}\right)$$

C.

$$\left(\frac{a(a-1)^2}{a^2+1}, \frac{-a(a-1)^2}{a^2+1}\right)$$

D.

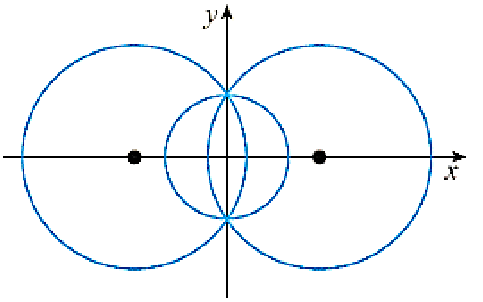
$$\left(\frac{a(a+1)^2}{a^2+1}, \frac{a^2(a+1)^2}{a^2+1}\right)$$

When $a = 1$ you need the image of the point $(1,1)$ when reflected in the line $y = x$ (which is $(1,1)$). Only option (b) gives this point so the rest can be eliminated. The correct answer is (b).

Question 13

The circle C has equation

$$x^2 + y^2 = 1$$



and is intersected at $(0, 1)$ and $(0, -1)$ by two circles of radius r:

- One with center at $(a, 0)$
- The other with center at $(-a, 0)$

(This is shown in the diagram.)

The value of a that results in circle C having **three distinct regions of equal area** satisfies the equation:

A.

$$\frac{\pi(a^2+1)\arctan\left(\frac{1}{a}\right)}{90} - 6a = \pi$$

B.

$$\frac{\pi(a^2+1)\arctan\left(\frac{1}{a}\right)}{30} - 6a = \pi$$

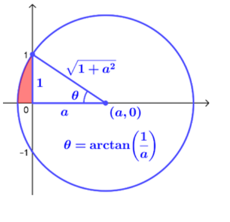
C.

$$\pi(a^2+1)\arctan\left(\frac{1}{a}\right) - 6a = \pi$$

D.

$$\frac{(a^2+1)\arctan\left(\frac{1}{a}\right)}{30} = 1$$

The area of the shaded region is $\frac{(1+a^2)\pi\arctan\left(\frac{1}{a}\right)}{360} - \frac{a}{2}$. Setting this equal to $\frac{\pi}{12}$ (so that the area enclosed by the overlapping circles is $\frac{\pi}{3}$) gives $\frac{\pi(a^2+1)\arctan\left(\frac{1}{a}\right)}{30} - 6a = \pi$. The correct answer is (b).



Question 14

The function f is defined on the **positive integers** as follows:

$$f(1) = 2 \quad \text{and for } n \geq 1:$$

$$f(n+1) = \begin{cases} 5f(n) + 1 & \text{if } f(n) \text{ is odd} \\ \frac{1}{2}f(n) & \text{if } f(n) \text{ is even} \end{cases}$$

> What is the value of

$$\sum_{r=1}^{100} f(r)?$$

Options:

A. 535

B. 546

C. 560

D. 563

$f(1) = 2, f(2) = 1, f(3) = 6, f(4) = 3, f(5) = 16, f(6) = 8, f(7) = 4, f(8) = 2$

Since $f(8) = f(1) = 2$, the values will repeat in sets of the seven values $2, 1, 6, 3, 16, 8, 4$

$\frac{100}{7} = 14\frac{2}{7}$ so there will be 14 complete sets followed by $2, 1$

$$\sum_{r=1}^{100} f(r) = 14(2 + 1 + 6 + 3 + 16 + 8 + 4) + 1 + 2 = 14 \times 40 + 3$$

The answer is D

Question 15

When

$$2x^2 + 3x - 15$$

is multiplied by $(ax - 4)$, and the resulting product is **divided by** $x + 2$, the **remainder is 182**.

> **What is the value of a?**

Options:

A. –9

B. –5

C. 5

D. 9

Left $f(x) = (ax - 4)(2x^2 + 3x - 15)$
$f(-2) = 182$
$(-2x - 4)(8 - 6 - 15) = 182$
$26(a + 2) = 182$
$a + 2 = 7$
$a = 5$
The correct answer is C.

Question 16

Find the **non-zero solution** to the equation:

$$\frac{3^{(16^x)}}{81^{(4^x)}} = \frac{1}{27}$$

Options:

A. $\log_4 3$

B. $2 \log_4 3$

C. 1

D. 2

$$\frac{3^{16^x}}{81^{4^x}} = \frac{1}{27}$$

$$\frac{3^{(4^2)^x}}{(3^4)^{4^x}} = 3^{-3}$$

$$\frac{3^{4^{2x}}}{3^{4^{x+1}}} = 3^{-3}$$

$$3^{(4^{2x}-4^{x+1})} = 3^{-3}$$

$$4^{2x} - 4^{x+1} = -3$$

$$4^{2x} - 4 \times 4^x + 3 = 0$$

Let $a = 4^x$

$$a^2 - 4a + 3 = 0$$

$$(a-1)(a-3) = 0$$

$a = 1 \iff 4^x = 1 \iff x = 0$ but non-zero solution required

$a = 3 \iff 4^x = 3 \iff x = \log_4 3$

The correct answer is A

Question 17

Let n be a positive integer.
$x^2 + 2$ is a factor of

$$(x^4 - 5)^n - (x^2 + 1)^{n+1} + (x^2 + 4)^n (3 + x^2)^n$$

for:

A. odd n

B. even n

C. no n

D. all n

E. $n = 1$

Answer: E. $n = 1$

If $x^2 + 2$ is a factor of the polynomial $F(x)$, then $F(x) = 0$ whenever $x^2 = -2$.

Let $x^2 = -2$. Then $x^4 = (x^2)^2 = 4$. Substitute into each term:

1. $(x^4 - 5)^n = (4 - 5)^n = (-1)^n$
2. $-(x^2 + 1)^{n+1} = -(-2 + 1)^{n+1} = -(-1)^{n+1} = (-1)^n$
3. $(x^2 + 4)^n (3 + x^2)^n = (-2 + 4)^n (3 - 2)^n = (2)^n (1)^n = 2^n$

So

$$F(x) = (-1)^n + (-1)^n + 2^n = 2(-1)^n + 2^n.$$

For divisibility, we need $F(x) = 0$:

$$2(-1)^n + 2^n = 0 \quad \Longrightarrow \quad (-1)^n + 2^{n-1} = 0 \quad \Longrightarrow \quad 2^{n-1} = -(-1)^n.$$

Now $2^{n-1} > 0$, so the right-hand side must be positive:

$$-(-1)^n > 0 \implies (-1)^n = -1 \implies n \text{ is odd.}$$

If n is odd, the equation becomes

$$2^{n-1} = 1 \implies n - 1 = 0 \implies n = 1.$$

Thus $x^2 + 2$ is a factor **only when** $n = 1$.

Question 18

Let a and b be **non-zero integers**.

When the polynomial

$$x^2 - 2ax - a^2$$

is divided by $x - b$, the **remainder is 1**.

Also, the polynomial

$$4bx^2 - 6x + 6$$

has $2x - a$ as a **factor**.

> It follows that $a + b$ equals:

Options:

A. –5

B. –3

C. –1

D. 0

$x^2 - 2ax - a^2$ divided by $x - b$ remainder 1
$4bx^2 - 6x - 10$ divided by $2x - a$ remainder 0
Using the remainder theorem
$x = b$ in $x^2 - 2ax - a^2$ gives $b^2 - 2ab - a^2 = 1$ A
Using the factor theorem
$x = \frac{a}{2}$ in $4bx^2 - 6x - 10$ gives $a^2b - 3a - 10 = 0$ B
From A
$a^2 + 2ab + 1 - b^2 = 0$
Completing the square for a gives
$(a + b)^2 + 1 - 2b^2 = 0$
$(a + b)^2 = 2b^2 - 1$
From the options, if $a + b = \pm 5$ (options (a) and (e)) then
$2b^2 - 1 = 25$ but this means that b is not an integer as specified in the question so (a) and (e) can be eliminated.
If $a + b = -3$ then $2b^2 - 1 = 9$ and again, b is not an integer so (b) can be eliminated
If $a + b = -1$ then $2b^2 - 1 = 1$ giving $b = \pm 1$ since $a + b = -1$, b must be $+1$ otherwise $a = 0$. So it is possible that (c) is correct with $b = 1$ and $a = -2$
If $a + b = 0$ then $2b^2 - 1 = 0$ and again, b is not an integer.
(c) is the correct answer.

Question 19

The polynomial $P_n(x)$ is defined by:

$$P_n(x) = (x - 2n + 1) + (2x - 2n + 3) + (3x - 2n + 5) + \cdots + (nx - 1)$$

Given that $n \geq 2$,

> What is the **remainder** when $P_n(x)$ is divided by $P_{n-1}(x)$?

Options:

A. $\dfrac{n^2 - 1}{2}$

B. n

C. -1

D. 1

The answer can be found by using a simple value for n
If $n = 3$ then
$p_3(x) = (x-5) + (2x-3) + (3x-1) = 6x - 9$
$p_2(x) = (x-3) + (2x-1) = 3x - 4$
Using the remainder theorem with $x = \frac{4}{3}$
$p_3\left(\frac{4}{3}\right) = 6 \times \frac{4}{3} - 9 = 8 - 9 = -1$
It looks like (c) is the correct answer. To confirm this the other values can be checked
$n = 3$ (a) $\frac{n^2-1}{2} = \frac{9-1}{2} = 4,$ (b) $n = 3,$ (d) $1 \neq -1,$ (e) $\frac{n}{2} = \frac{3}{2}$
The correct answer is (c)

Question 20

Given that x, y, z are **positive real numbers**, the equations

$$4\log_z x = y, \quad \log_x z = y, \quad x + \log_y z = 0$$

> Which of the following is true?

Options:

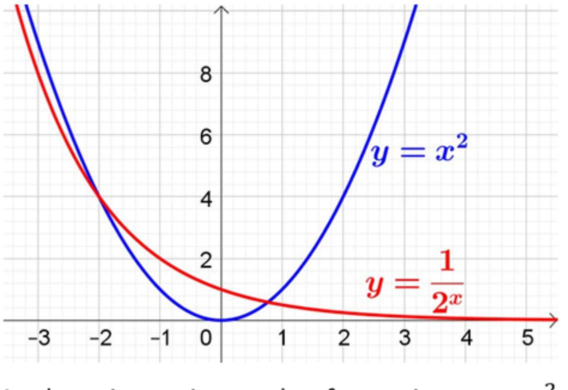
A. Have a unique solution for x, but not for y and z

B. Have unique solutions for x and y, but infinitely many solutions for z

C. Have unique solutions for x and z, but infinitely many solutions for y

D. Have a unique solution for x, y, z

From $4\log_z x = y,\ x^4 = z^y$ A
From $\log_x z = y,\ z = x^y$ B
From $x + \log_y z = 0,\ z = y^{-x}$ C
Since x, y and z all feature as the base of a logarithm, all three must be positive.
Substituting B into A gives $x^4 = (x^y)^y$ so $x^{y^2} = x^4$ and $y^2 = 4$ so $y = 2$ since $y > 0$
A becomes $x^4 = z^2$, B becomes $z = x^2$ and C becomes $z = 2^{-x}$ i.e. $z = \frac{1}{2^x}$
Substituting $z = \frac{1}{2^x}$ into $z = x^2$ gives $x^2 = \frac{1}{2^x}$
To see if this gives unique values for x, the graphs of $y = \frac{1}{2^x}$ and $y = x^2$ can be sketched



There is one solution for $x > 0$ (negative values can be ignored since you know that $x > 0$)

As there is a unique value for x, since $z = x^2$, there is also a unique value for z.
x, y and z all have unique solutions.
The correct answer is D