ESAT PRACTICE TEST SERIES

ESAT Mock Test 5

Mathematics 1 · Physics · Mathematics 2

- 27 multiple-choice questions per module
- 40 minutes per module
- Non-calculator
- Hard ESAT-style problem set with full worked solutions

SOLUTION BOOK

ThrivingScholars

ANSWER KEY

ESAT Mock Test 5 — Mathematics 1

Twenty-seven challenging Mathematics 1 questions with stronger distractors, exact reasoning and linked multi-step ideas.

Mathematics 1 answer key

1 C	2 B	3 D	4 A	5 E	6 C	7 B	8 D	9 A
10 E	11 C	12 B	13 D	14 A	15 E	16 B	17 C	18 D
19 A	20 E	21 C	22 B	23 D	24 A	25 E	26 C	27 B

ESAT MOCK TEST 5

Mathematics 1

27 hard ESAT-style questions with complete worked solutions.

M1-01

Laws of indices and exact simplification

Answer C

Question 1

Simplify fully

$$\frac{\left(50x^{3/2}y^{-1}\right)^2\left(2x^{-1}y^{5/2}\right)^3}{\left(20x^{1/2}y\right)^2},$$

where x and y are positive.

A $\dfrac{10y^3\sqrt{y}}{x}$

B $\dfrac{50y^3}{x}$

C $\dfrac{50y^3\sqrt{y}}{x}$

D $\dfrac{50\sqrt{y}}{x}$

E $\dfrac{250y^3\sqrt{y}}{x}$

WORKED SOLUTION

$\left(50x^{3/2}y^{-1}\right)^2 = 2500x^3y^{-2}$ and $\left(2x^{-1}y^{5/2}\right)^3 = 8x^{-3}y^{15/2}$.

So the numerator is $2500x^3y^{-2} \cdot 8x^{-3}y^{15/2} = 20000y^{11/2}$.

The denominator is $\left(20x^{1/2}y\right)^2 = 400xy^2$.

Therefore

$$\frac{20000y^{11/2}}{400xy^2} = \frac{50y^{7/2}}{x} = \boxed{\frac{50y^3\sqrt{y}}{x}}.$$

M1-02

Successive percentage change and comparison

Answer **B**

Question 2

The price of item P is reduced by 20%, and then a sales tax of 5% is applied to the reduced price.

The price of item Q is increased by 10%, and then reduced by 10%.

After these two-step changes, the final prices of P and Q are equal.

What was the ratio of the original price of P to the original price of Q?

A $28 : 33$

B $33 : 28$

C $11 : 10$

D $6 : 5$

E $99 : 80$

WORKED SOLUTION

Final price of P is $0.8 \times 1.05P = 0.84P$.

Final price of Q is $1.1 \times 0.9Q = 0.99Q$.

So $0.84P = 0.99Q$, giving

$$\frac{P}{Q} = \frac{0.99}{0.84} = \frac{99}{84} = \frac{33}{28}.$$

M1-03

Algebraic fractions and hidden substitution

Answer D

Question 3

For real x, define

$$E = \frac{\dfrac{1}{x-3} - \dfrac{1}{x+2}}{\dfrac{1}{x-3} + \dfrac{1}{x+2}}.$$

Given that $x > 3$ and

$$E + \frac{1}{E} = \frac{13}{2},$$

what is the value of x?

A $\frac{11}{2}$

B 8

C 12

D $\frac{31}{2}$

E 18

WORKED SOLUTION

Simplify first:

$$E = \frac{\frac{(x+2)-(x-3)}{(x-3)(x+2)}}{\frac{(x+2)+(x-3)}{(x-3)(x+2)}} = \frac{5}{2x-1}.$$

Let $t = E$. Then $t + \dfrac{1}{t} = \dfrac{13}{2}$, so

$$2t^2 - 13t + 2 = 0 = (2t-1)(t-6).$$

Hence $t = 6$ or $t = \dfrac{1}{6}$. Since $x > 3$, we have $0 < E < 1$, so $E = \dfrac{1}{6}$.

Therefore

$$\frac{5}{2x-1} = \frac{1}{6} \Rightarrow 2x - 1 = 30 \Rightarrow \boxed{x = \frac{31}{2}}.$$

M1-04

Inequalities with domain restrictions

Answer **A**

Question 4

How many integer values of x satisfy all three conditions?

- $-5 \le x \le 5$
- $\dfrac{x^2+3x-10}{x^2-1}$ is defined
- $\dfrac{x^2+3x-10}{x^2-1} \le 1$

A 6 ✓

B 5

C 4

D 7

E 8

WORKED SOLUTION

Bring everything to one side:

$$\frac{x^2+3x-10}{x^2-1} - 1 \le 0 \Rightarrow \frac{3x-9}{(x-1)(x+1)} \le 0.$$

So we solve

$$\frac{x-3}{(x-1)(x+1)} \le 0, \qquad x \ne \pm 1.$$

A sign chart gives $(-\infty, -1) \cup (1, 3]$.

Intersecting with $-5 \le x \le 5$ gives $-5, -4, -3, -2, 2, 3$, so there are $\boxed{6}$ values.

M1-05

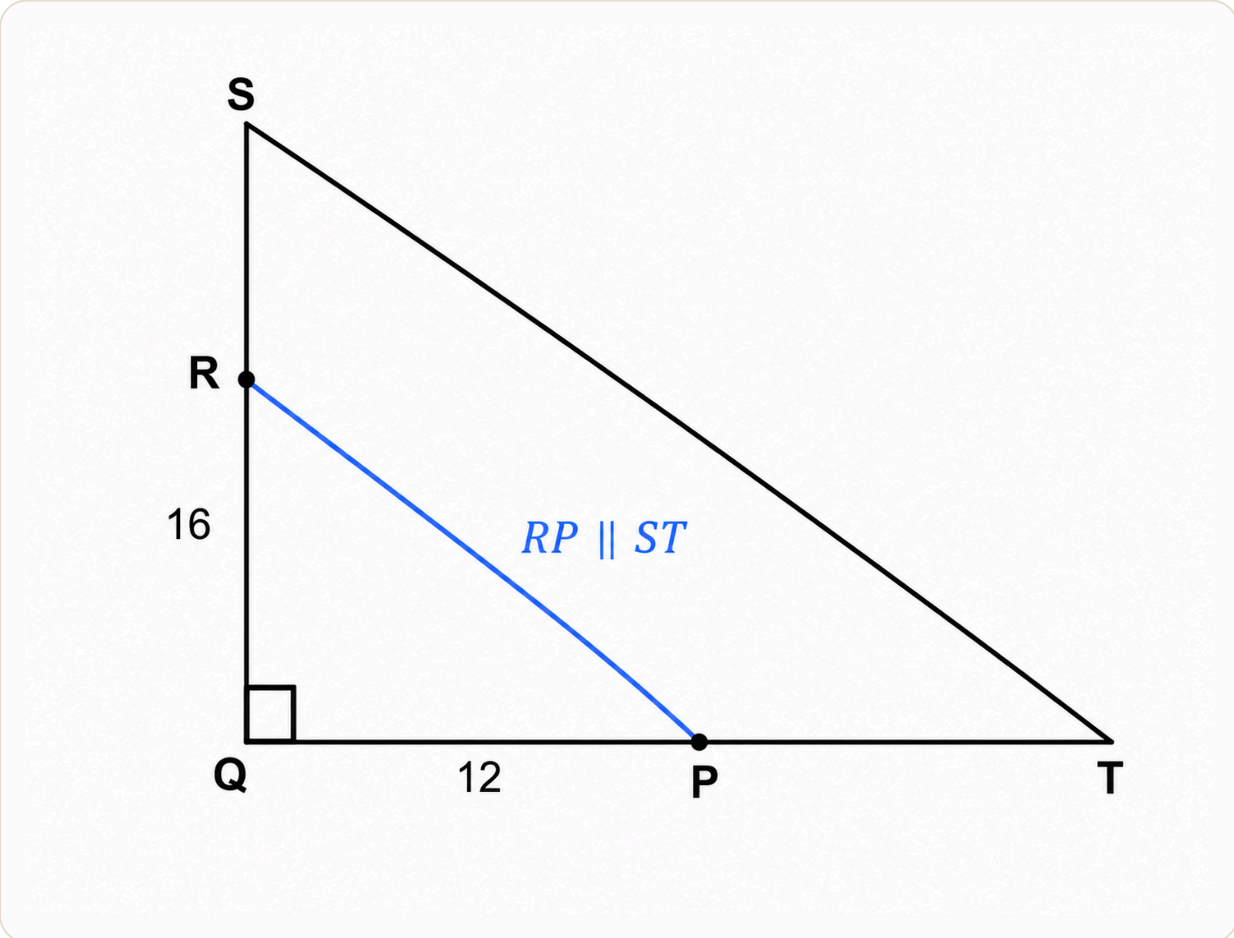
Similarity and area scaling in a right triangle

Answer E

Question 5

Triangle SQT is right-angled at Q, with $SQ = 16$ cm and $QT = 12$ cm.

Point R lies on SQ such that $SR : RQ = 1 : 3$. Through R, a line parallel to ST meets QT at P.



What is the area of quadrilateral $SRPT$?

A $24\ \text{cm}^2$

B $30\ \text{cm}^2$

C $36\ \text{cm}^2$

D $40\ \text{cm}^2$

E $42\ \text{cm}^2$

WORKED SOLUTION

The area of the whole triangle is $\frac{1}{2} \cdot 16 \cdot 12 = 96$.

Since $SR : RQ = 1 : 3$, we get $RQ = \frac{3}{4} \cdot 16 = 12$.

Because $RP \parallel ST$, triangles QRP and QST are similar with scale factor $\frac{QR}{QS} = \frac{12}{16} = \frac{3}{4}$.

So $QP = \frac{3}{4} \cdot 12 = 9$. Then area of triangle QRP is $\frac{1}{2} \cdot 12 \cdot 9 = 54$.

Hence quadrilateral $SRPT$ has area $96 - 54 = \boxed{42\ \mathrm{cm}^2}$.

M1-06

Arithmetic sequences and common terms

Answer C

Question 6

The nth term of sequence T is $4n - 3$.

The nth term of sequence V is $7n + p$, where p is an integer and $0 \leq p < 7$.

The sequences have consecutive common terms 53 and 81.

What is the value of p?

A 1

B 2

C 4

D 5

E 6

WORKED SOLUTION

Using the common term 53,

$$53 = 7n + p.$$

Since $53 \equiv 4 \pmod 7$, and $0 \leq p < 7$, we must have $\boxed{p = 4}$.

M1-07

Cylinders with swapped dimensions

Answer B

Question 7

Two solid cylinders P and Q are defined using positive numbers x and y, where $x > y$.

- Cylinder P has diameter x and height y.
- Cylinder Q has diameter y and height x.

The total surface area of P is $\frac{8}{5}$ of the total surface area of Q.

The volume of P exceeds the volume of Q by $32\pi\ \mathrm{cm}^3$.

What is the value of $x + y$?

A 10

B 12

C 14

D 16

E 18

WORKED SOLUTION

For a cylinder of diameter d and height h, total surface area is $2\pi(\frac{d}{2})^2 + 2\pi(\frac{d}{2})h$.

Thus

$$A_P = \pi\left(\frac{x^2}{2} + xy\right), \qquad A_Q = \pi\left(\frac{y^2}{2} + xy\right).$$

Set $x = ky$. Then

$$\frac{k^2 + 2k}{1 + 2k} = \frac{8}{5}.$$

So $5k^2 - 6k - 8 = 0$, giving the positive root $k = 2$. Hence $x = 2y$.

Now

$$V_P - V_Q = \frac{\pi}{4}(x^2y - y^2x) = \frac{\pi}{4}xy(x - y) = 32\pi.$$

Substitute $x = 2y$:

$$\frac{\pi}{4}(2y)(y)(y) = 32\pi \Rightarrow \frac{y^3}{2} = 32 \Rightarrow y^3 = 64 \Rightarrow y = 4.$$

Hence $x = 8$, so $x + y = \boxed{12}$.

M1-08

Ratio change after equal spending

Answer D

Question 8

Pat and Alex have a combined total of $£84$. The ratio of Pat's money to Alex's money is $5:2$.

They each spend the same amount on sweets.

After this, Pat receives $£6$ and Alex receives $£1$. The ratio of their new amounts is then $3:1$.

How much did each of them spend?

A $£3$

B $£4$

C $£4.25$

D $£4.50$

E $£6$

WORKED SOLUTION

Initially the amounts are 60 and 24.

If each spends s, then afterwards they have $60-s$ and $24-s$. After receiving the extra money, they have $66-s$ and $25-s$.

So

$$66-s=3(25-s) \Rightarrow 66-s=75-3s \Rightarrow 2s=9 \Rightarrow s=4.5.$$

Each spent $\boxed{£4.50}$.

M1-09

Coordinate geometry of a tangent

Answer A

Question 9

The circle $x^2 + y^2 = 25$ has centre at the origin.

A tangent to the circle passes through the point $(1, 7)$ and has positive gradient.

What is the equation of the tangent?

- **A** $3x - 4y + 25 = 0$
- **B** $4x - 3y + 25 = 0$
- **C** $3x + 4y - 25 = 0$
- **D** $4x + 3y - 25 = 0$
- **E** $y = \frac{4}{3}x + \frac{17}{3}$

WORKED SOLUTION

Let the tangent be $y = mx + c$. Since it passes through $(1, 7)$, we have $c = 7 - m$.

The distance from the origin to the tangent must equal the radius 5:

$$\frac{|c|}{\sqrt{m^2 + 1}} = 5.$$

So

$$(7 - m)^2 = 25(m^2 + 1) \Rightarrow 12m^2 + 7m - 12 = 0 = (4m - 3)(3m + 4).$$

The positive gradient is $m = \frac{3}{4}$. Then $c = 7 - \frac{3}{4} = \frac{25}{4}$.

Hence

$$y = \frac{3}{4}x + \frac{25}{4} \iff \boxed{3x - 4y + 25 = 0}.$$

M1-10

Conditional probability with non-standard dice

Answer **E**

Question 10

Two fair dice X and Y are available.

- The faces on X are labelled $1, 1, 2, 3, 4, 5$.
- The faces on Y are labelled $2, 3, 3, 4, 5, 6$.

One of the dice is chosen at random and rolled twice.

Given that the total shown is 7, what is the probability that the chosen die was X?

A $\frac{1}{3}$

B $\frac{3}{8}$

C $\frac{1}{2}$

D $\frac{3}{5}$

E $\frac{2}{5}$

WORKED SOLUTION

For die X, the ordered outcomes summing to 7 are $(2, 5), (5, 2), (3, 4), (4, 3)$: 4 outcomes out of 36. So $P(7 \mid X) = \dfrac{1}{9}$.

For die Y, 3 appears twice, so the ordered outcomes summing to 7 are 6 in total. Hence $P(7 \mid Y) = \dfrac{1}{6}$.

With equal prior probability of choosing each die,

$$P(X \mid 7) = \frac{\frac{1}{2} \cdot \frac{1}{9}}{\frac{1}{2} \cdot \frac{1}{9} + \frac{1}{2} \cdot \frac{1}{6}} = \frac{\frac{1}{9}}{\frac{1}{9} + \frac{1}{6}} = \frac{2}{5}.$$

M1-11

Exponential equation via substitution

Answer C

Question 11

The equation $2^x + 2^{2-x} = 5$ has two real solutions.

What is the larger possible value of x?

A 0

B 1

C 2

D 3

E 4

WORKED SOLUTION

Let $t = 2^x$. Then $2^{2-x} = \frac{4}{t}$, so

$$t + \frac{4}{t} = 5 \Rightarrow t^2 - 5t + 4 = 0 = (t-1)(t-4).$$

Thus $t = 1$ or $t = 4$, so $x = 0$ or $x = 2$. The larger value is $\boxed{2}$.

M1-12

Rearranging to make a variable the subject

Answer B

Question 12

Suppose x and y satisfy

$$x = y + \frac{4}{y},$$

where $y > 0$ and $x > 4$.

Which of the following is the larger possible value of y in terms of x?

A $\dfrac{x - \sqrt{x^2 - 16}}{2}$

B $\dfrac{x + \sqrt{x^2 - 16}}{2}$

C $\dfrac{x + \sqrt{x^2 + 16}}{2}$

D $\dfrac{2}{x + \sqrt{x^2 - 16}}$

E $\dfrac{4}{x + \sqrt{x^2 - 16}}$

WORKED SOLUTION

Multiply by y:

$$xy = y^2 + 4 \Rightarrow y^2 - xy + 4 = 0.$$

Using the quadratic formula,

$$y = \frac{x \pm \sqrt{x^2 - 16}}{2}.$$

The larger possible value uses the positive sign, so

$$\boxed{y = \frac{x + \sqrt{x^2 - 16}}{2}}.$$

M1-13

Surface area comparison of a cylinder and a cube

Answer D

Question 13

A solid cylinder has radius r cm and height h cm.

A cube has side length $\sqrt{\pi}\,r$ cm.

The total surface area of the cylinder is equal to four times the total surface area of the cube.

What is $\frac{h}{r}$?

A 7

B 9

C 10

D 11

E 12

WORKED SOLUTION

The cylinder's total surface area is $2\pi r^2 + 2\pi rh = 2\pi r(r+h)$.

The cube's total surface area is $6(\sqrt{\pi}r)^2 = 6\pi r^2$.

Given the cylinder equals four times the cube,

$$2\pi r(r+h) = 24\pi r^2.$$

Cancel $2\pi r$:

$$r + h = 12r \Rightarrow h = 11r.$$

Therefore $\frac{h}{r} = \boxed{11}$.

M1-14

Mean and median with an ordered list

Answer A

Question 14

Consider the list

$$1,\ 2,\ x,\ x+1,\ 2x-1,\ 15,$$

where $2 < x < 8$.

The mean of the six numbers is one more than the median.

What is the value of x?

A $\frac{9}{2}$

B 4

C 5

D $\frac{11}{2}$

E 6

WORKED SOLUTION

Since $x > 2$, the list is already ordered. So the median is

$$\frac{x+(x+1)}{2} = x + \frac{1}{2}.$$

The mean is one more than this, so the mean is $x + \frac{3}{2}$.

The mean is also

$$\frac{1+2+x+(x+1)+(2x-1)+15}{6} = \frac{4x+18}{6}.$$

Therefore

$$\frac{4x+18}{6} = x + \frac{3}{2} \Rightarrow 4x+18 = 6x+9 \Rightarrow 2x = 9 \Rightarrow \boxed{x = \frac{9}{2}}.$$

M1-15

Speed, time and average-rate reasoning

Answer **E**

Question 15

A cyclist rides from A to B uphill at $12\ \mathrm{km\ h^{-1}}$, immediately turns round, and returns from B to A along the same route at $18\ \mathrm{km\ h^{-1}}$.

She spends exactly 6 minutes stationary at B.

The total time for the complete trip is 46 minutes.

What is the distance from A to B?

- **A** $3.6\ \mathrm{km}$
- **B** $4.0\ \mathrm{km}$
- **C** $4.2\ \mathrm{km}$
- **D** $4.5\ \mathrm{km}$
- **E** $4.8\ \mathrm{km}$

WORKED SOLUTION

Excluding the 6-minute stop, the travel time is 40 minutes = $\frac{2}{3}$ hours.

If the one-way distance is d km, then

$$\frac{d}{12} + \frac{d}{18} = \frac{2}{3}.$$

Since $\frac{1}{12} + \frac{1}{18} = \frac{5}{36}$,

$$d \cdot \frac{5}{36} = \frac{2}{3} \Rightarrow d = \frac{2}{3} \cdot \frac{36}{5} = \frac{24}{5} = 4.8.$$

So the distance is $\boxed{4.8\ \mathrm{km}}$.

M1-16

Exact trigonometric values

Answer **B**

Question 16

Given that

$$y = \frac{2\sin 15^\circ}{\cos 30^\circ},$$

what is the value of y^2?

A $\dfrac{4-2\sqrt{3}}{3}$

B $\dfrac{8-4\sqrt{3}}{3}$

C $\dfrac{8+4\sqrt{3}}{3}$

D $2-\sqrt{3}$

E $4-2\sqrt{3}$

WORKED SOLUTION

Use

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4}, \qquad \cos 30^\circ = \frac{\sqrt{3}}{2}.$$

Hence

$$y = \frac{2\cdot\frac{\sqrt{6}-\sqrt{2}}{4}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{3}}.$$

Therefore

$$y^2 = \frac{(\sqrt{6}-\sqrt{2})^2}{3} = \frac{6+2-2\sqrt{12}}{3} = \frac{8-4\sqrt{3}}{3}.$$

M1-17

Inverse proportion in standard form

Answer C

Question 17

The variable t is inversely proportional to the square of w.

When $w = 2 \times 10^3$, the value of t is 4.5×10^{-5}.

What is the value of w when $t = 8 \times 10^{-6}$?

- **A** 7.5×10^3
- **B** $3\sqrt{10} \times 10^2$
- **C** $1.5\sqrt{10} \times 10^3$
- **D** $2\sqrt{10} \times 10^3$
- **E** 4.5×10^3

WORKED SOLUTION

Since $t = \dfrac{k}{w^2}$, use the first condition:

$$k = (4.5 \times 10^{-5})(2 \times 10^3)^2 = (4.5 \times 10^{-5})(4 \times 10^6) = 180.$$

Now solve

$$8 \times 10^{-6} = \frac{180}{w^2} \Rightarrow w^2 = 2.25 \times 10^7.$$

So

$$w = \sqrt{2.25 \times 10^7} = 1.5\sqrt{10} \times 10^3.$$

M1-18

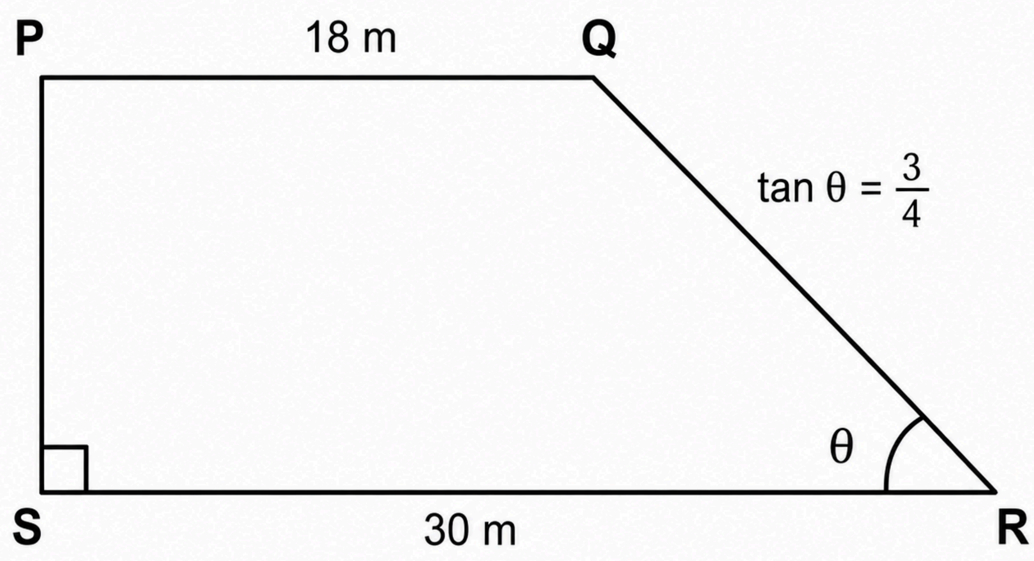
Trapezium area using trigonometry

Answer D

Question 18

$PQRS$ is a trapezium in which $PQ \parallel SR$ and $PS \perp SR$.

Also, $PQ = 18\ \mathrm{m}$, $SR = 30\ \mathrm{m}$, and $\tan \angle QRS = \dfrac{3}{4}$.



What is the area of the trapezium?

- **A** $180\ \mathrm{m}^2$
- **B** $192\ \mathrm{m}^2$
- **C** $204\ \mathrm{m}^2$
- **D** $216\ \mathrm{m}^2$
- **E** $240\ \mathrm{m}^2$

WORKED SOLUTION

The difference between the parallel sides is $30 - 18 = 12$, which is the horizontal run of the sloping side.

If the height is h, then

$$\tan \angle QRS = \frac{h}{12} = \frac{3}{4} \Rightarrow h = 9.$$

So the area is

$$\frac{1}{2}(18+30)(9) = 24 \cdot 9 = \boxed{216\ \mathrm{m}^2}.$$

M1-19

Quadratic roots and discriminant reasoning

Answer A

Question 19

The equation $x^2 + bx + 8 = 0$ has two real roots whose positive difference is 2. The sum of the roots is negative.

What is the value of b?

- A 6
- B 5
- C 4
- D -6
- E -4

WORKED SOLUTION

If the roots are r and s, then $(r-s)^2 = (r+s)^2 - 4rs$.

Here $r+s=-b$ and $rs=8$. Since the difference is 2,

$$4 = b^2 - 32 \Rightarrow b^2 = 36.$$

So $b = \pm 6$. The sum of the roots is negative, so $-b < 0 \Rightarrow b > 0$. Hence $\boxed{b=6}$.

M1-20

Algebraic geometry with area change

Answer E

Question 20

A rectangle has length $(2x - 1)$ cm and width $(x + 1)$ cm.

A larger rectangle is made by adding 3 cm to both the length and the width.

The larger rectangle has twice the area of the original rectangle.

What is the perimeter of the original rectangle?

- **A** 24 cm
- **B** 26 cm
- **C** 28 cm
- **D** 29 cm
- **E** 30 cm

WORKED SOLUTION

Original area is $(2x - 1)(x + 1)$. New dimensions are $2x + 2$ and $x + 4$, so the new area is $(2x + 2)(x + 4)$.

Given new area is twice old area,

$$(2x + 2)(x + 4) = 2(2x - 1)(x + 1).$$

This simplifies to

$$x^2 - 4x - 5 = 0 = (x - 5)(x + 1).$$

So $x = 5$. The original dimensions are 9 and 6, so the perimeter is $2(9 + 6) = \boxed{30\text{ cm}}$.

M1-21

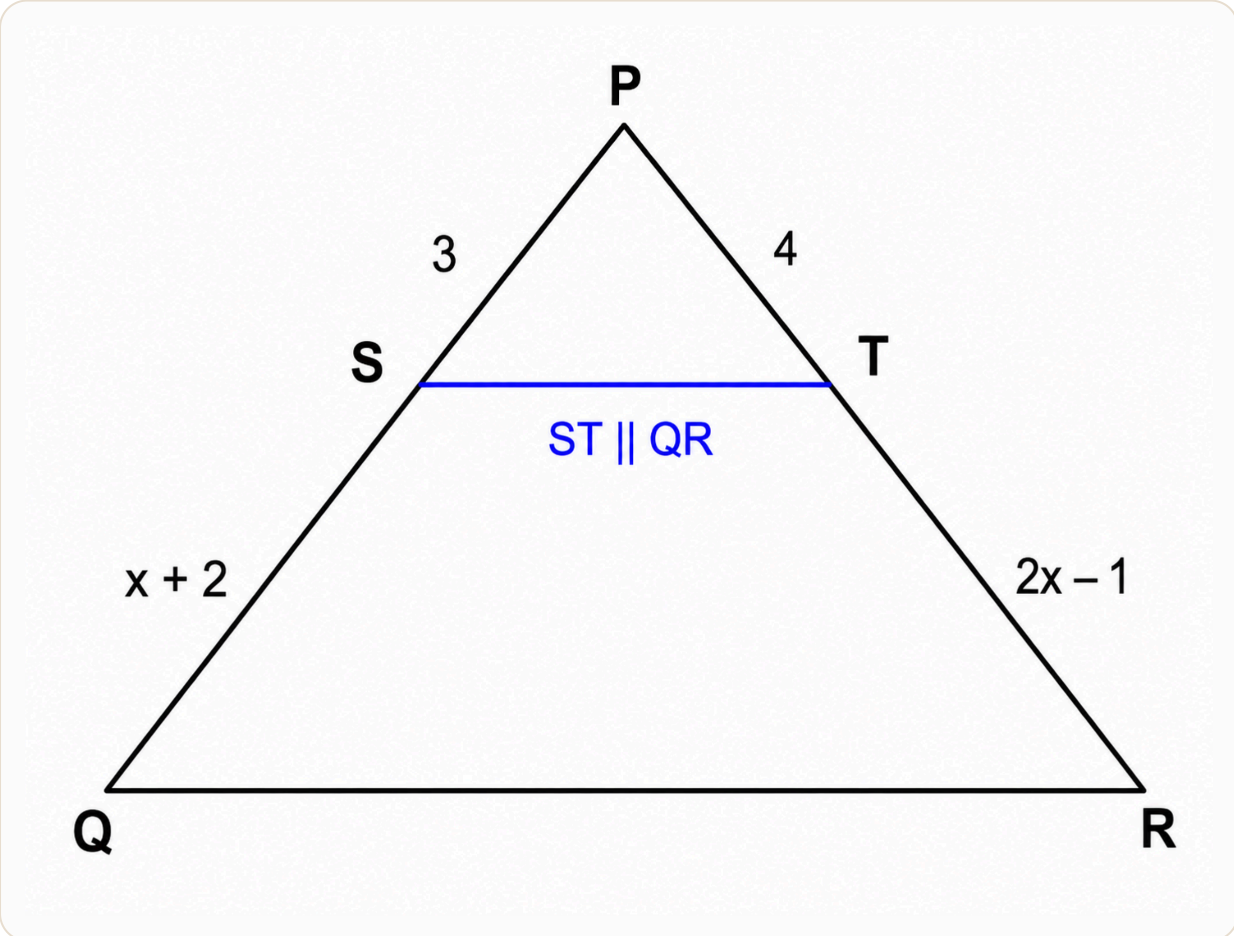
Similarity and area scaling

Answer C

Question 21

In triangle PQR, point S lies on PQ and point T lies on PR. The line ST is parallel to QR.

The lengths are labelled as follows: $PS = 3 \text{ cm}$, $SQ = (x+2) \text{ cm}$, $PT = 4 \text{ cm}$, $TR = (2x-1) \text{ cm}$.



Given that the area of triangle PST is 12 cm^2, what is the area of trapezium $STQR$?

A 96 cm^2

B 120 cm^2

C 135 cm^2

D 144 cm^2

E 147 cm^2

WORKED SOLUTION

Since $ST \parallel QR$, triangles PST and PQR are similar, so

$$\frac{PS}{PQ} = \frac{PT}{PR}.$$

Now $PQ = 3 + (x+2) = x+5$ and $PR = 4 + (2x-1) = 2x+3$. Therefore

$$\frac{3}{x+5} = \frac{4}{2x+3}.$$

So $6x+9=4x+20$, giving $x=\dfrac{11}{2}$.

Then

$$\frac{PS}{PQ}=\frac{3}{21/2}=\frac{2}{7}.$$

Hence the area scale factor is $\left(\dfrac{2}{7}\right)^2=\dfrac{4}{49}$.

So if area $PST=12$, then area $PQR=12\cdot\dfrac{49}{4}=147$.

Thus the area of trapezium $STQR$ is $147-12=\boxed{135\text{ cm}^2}$.

M1-22

Probability with two colours

Answer B

Question 22

Charlie has a bowl containing red sweets and green sweets only. There are 9 more red sweets than green sweets.

Two sweets are chosen at random without replacement.

The probability that the two sweets are the same colour is $\frac{4}{7}$.

How many sweets are there in the bowl altogether?

A 18

B 21

C 24

D 27

E 30

WORKED SOLUTION

Let the number of green sweets be g. Then the number of red sweets is $g+9$, so total sweets are $2g+9$.

The probability condition is

$$\frac{\binom{g+9}{2}+\binom{g}{2}}{\binom{2g+9}{2}}=\frac{4}{7}.$$

This simplifies to

$$(g-6)(g+18)=0.$$

So $g=6$. Then red sweets $=15$, giving total sweets $=21$.

M1-23

Successive translations and vector equations

Answer D

Question 23

The point $(-1, 5)$ is translated by two successive vectors.

The first translation is by $\begin{pmatrix} 3p \\ 2-q \end{pmatrix}$, and the second is by $\begin{pmatrix} q+1 \\ p-3 \end{pmatrix}$.

The final image is $(8, 4)$.

What is the value of $p^2 + q^2$?

- **A** 4
- **B** 5
- **C** 6
- **D** 8
- **E** 10

WORKED SOLUTION

The total translation vector is

$$\begin{pmatrix} 3p+q+1 \\ p-q-1 \end{pmatrix}.$$

Since $(-1, 5) \to (8, 4)$, the net translation is $\begin{pmatrix} 9 \\ -1 \end{pmatrix}$.

So

$$3p + q = 8, \qquad p - q = 0.$$

Thus $p = q$, so $4p = 8 \Rightarrow p = 2$, hence $q = 2$.

Therefore $p^2 + q^2 = 4 + 4 = \boxed{8}$.

M1-24

Quadratic graphs and minimum values

Answer **A**

Question 24

Consider the family of graphs

$$y = ax^2 + 2(a-1)x + a + 3.$$

What is the complete range of values of a for which the minimum point of the graph lies above the x-axis?

A $a > \frac{1}{5}$

B $a > 0$

C $a \geq \frac{1}{5}$

D $a < 0$

E $0 < a < \frac{1}{5}$

WORKED SOLUTION

For there to be a minimum, we need $a > 0$.

For $y = ax^2 + bx + c$, the minimum value is $c - \frac{b^2}{4a}$.

Here $b = 2(a-1)$ and $c = a + 3$, so the minimum value is

$$a + 3 - \frac{(2(a-1))^2}{4a} = a + 3 - \frac{(a-1)^2}{a} = 5 - \frac{1}{a}.$$

We need this to be positive:

$$5 - \frac{1}{a} > 0.$$

Since $a > 0$, this gives $5a > 1 \Rightarrow a > \frac{1}{5}$.

M1-25

Counting constrained selections

Answer **E**

Question 25

An online company sells storage containers. The following items are available:

Capacity of container	Number available
2 litres	2
3 litres	3
5 litres	4

A customer chooses a selection of containers with total capacity 23 litres.

Different selections are distinguished only by how many containers of each size are chosen.

If all possible selections are equally likely, what is the probability that the chosen selection contains at least one 2-litre container?

A $\frac{1}{3}$

B $\frac{1}{2}$

C $\frac{3}{5}$

D $\frac{3}{4}$

E $\frac{2}{3}$

WORKED SOLUTION

Let a, b, c be the numbers of 2-, 3- and 5-litre containers. Then

$$2a + 3b + 5c = 23,$$

with $0 \le a \le 2$, $0 \le b \le 3$, $0 \le c \le 4$.

The valid selections are

- $(0, 1, 4)$
- $(1, 2, 3)$
- $(2, 3, 2)$

Two of the three contain at least one 2-litre container, so the probability is $\boxed{\frac{2}{3}}$.

M1-26

Similarity and three-dimensional scaling

Answer C

Question 26

The space diagonal of a cuboid is $7\sqrt{13}$ cm.

A similar cuboid has volume 8 times the volume of the original cuboid.

What is the space diagonal of the larger cuboid?

- **A** $7\sqrt{26}$ cm
- **B** $14\sqrt{7}$ cm
- **C** $14\sqrt{13}$ cm
- **D** $21\sqrt{13}$ cm
- **E** $28\sqrt{13}$ cm

WORKED SOLUTION

A volume scale factor of 8 means a linear scale factor of $\sqrt[3]{8} = 2$.

So the new diagonal is

$$2 \cdot 7\sqrt{13} = \boxed{14\sqrt{13}\ \text{cm}}.$$

M1-27 Relative speed in a race **Answer B**

Question 27

Alex, Cameron and Sam are running a $400\ \mathrm{m}$ race. They each run at a different constant speed.

When Alex finishes, Cameron has run $300\ \mathrm{m}$ and Sam has run $240\ \mathrm{m}$.

Alex beats Sam by $33\frac{1}{3}$ seconds.

What is Cameron's speed?

A $5\ \mathrm{m\,s^{-1}}$

B $6\ \mathrm{m\,s^{-1}}$

C $6.4\ \mathrm{m\,s^{-1}}$

D $7\ \mathrm{m\,s^{-1}}$

E $8\ \mathrm{m\,s^{-1}}$

WORKED SOLUTION

Let Alex's speed be A, Cameron's speed be C, and Sam's speed be S.

When Alex finishes, the elapsed time is $\frac{400}{A}$. In that time, Cameron runs 300 m, so

$$C \cdot \frac{400}{A} = 300 \Rightarrow \frac{C}{A} = \frac{3}{4}.$$

Similarly, Sam runs 240 m, so

$$S \cdot \frac{400}{A} = 240 \Rightarrow \frac{S}{A} = \frac{3}{5}.$$

Thus $S = \frac{3}{5}A$.

Alex beats Sam by $\frac{100}{3}$ s, so

$$\frac{400}{S} - \frac{400}{A} = \frac{100}{3}.$$

Substitute $S = \frac{3}{5}A$:

$$\frac{400}{3A/5} - \frac{400}{A} = \frac{100}{3} \Rightarrow \frac{2000}{3A} - \frac{400}{A} = \frac{100}{3} \Rightarrow \frac{800}{3A} = \frac{100}{3}.$$

So $A = 8\ \mathrm{m\,s^{-1}}$. Hence

$$C = \frac{3}{4}A = 6\ \mathrm{m\,s^{-1}}.$$

Therefore Cameron's speed is $\boxed{6 \mathrm{~m~s}^{-1}}$.

ANSWER KEY

ESAT Mock Test 5 — Physics

Twenty-seven challenging Physics questions using coherent physical models, mathematical reasoning and diagnostic distractors.

Physics answer key

1 D	2 D	3 C	4 C	5 B	6 D	7 D	8 C	9 D
10 C	11 E	12 D	13 C	14 C	15 D	16 D	17 E	18 D
19 C	20 C	21 B	22 B	23 D	24 D	25 B	26 C	27 C

ESAT MOCK TEST 5

Physics

27 hard ESAT-style questions with complete worked solutions and embedded diagrams where required.

P-01

Charge, current, power and heating

Answer D

Question 1

A resistor has a constant potential difference of $9.0\ \mathrm{V}$ across it. A total charge of $180\ \mathrm{C}$ passes through it in $4.00\ \mathrm{min}$.

The resistor is then left connected for a total of $6.00\ \mathrm{min}$. Assume all the energy dissipated warms a $0.250\ \mathrm{kg}$ metal block of specific heat capacity $400\ \mathrm{J\ kg^{-1}K^{-1}}$, and that there are no losses.

What is the temperature rise of the block?

- **A** $12.0\ \mathrm{K}$
- **B** $18.0\ \mathrm{K}$
- **C** $21.6\ \mathrm{K}$
- **D** $24.3\ \mathrm{K}$
- **E** $27.0\ \mathrm{K}$

WORKED SOLUTION

The current is

$$I = \frac{Q}{t} = \frac{180}{240} = 0.75\ \mathrm{A}.$$

So the power is

$$P = VI = 9.0 \times 0.75 = 6.75\ \mathrm{W}.$$

Over $6.00\ \mathrm{min} = 360\ \mathrm{s}$, the energy transferred is

$$E = Pt = 6.75 \times 360 = 2430\ \mathrm{J}.$$

Hence

$$\Delta T = \frac{E}{mc} = \frac{2430}{0.250 \times 400} = 24.3\ \mathrm{K}.$$

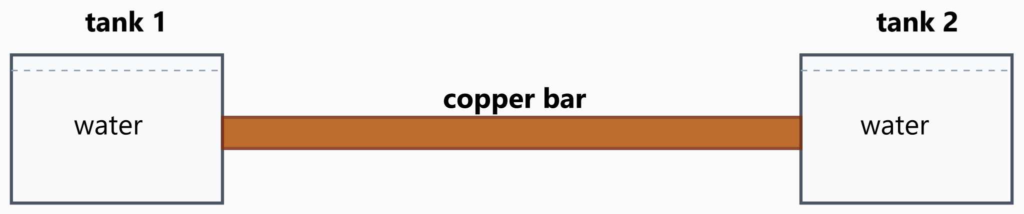
P-02

Thermal conduction and scaling

Answer D

Question 2

Two large copper tanks of water are connected by a solid cylindrical copper bar. Initially the water temperatures are $20^\circ\mathrm{C}$ and $80^\circ\mathrm{C}$, and the bar has length L and diameter d.



The system is modified so that the temperatures become $35^\circ\mathrm{C}$ and $95^\circ\mathrm{C}$, the bar length becomes $0.75L$, and the diameter becomes $1.5d$.

By what factor does the rate of conduction of thermal energy along the bar change?

A 0.75

B 1.5

C 2.25

D 3.0

E 4.0

WORKED SOLUTION

For conduction,

$$P \propto \frac{A\Delta T}{L}.$$

The temperature difference stays $60^\circ\mathrm{C}$, so ΔT is unchanged. The area factor is

$$\left(\frac{1.5d}{d}\right)^2 = 2.25.$$

The length factor is

$$\frac{L}{0.75L} = \frac{4}{3}.$$

Hence the overall factor is

$$2.25 \times \frac{4}{3} = 3.0.$$

P-03

Forces, towing and tension

Answer C

Question 3

A car of mass 1400 kg tows a caravan of mass 1000 kg along a horizontal road at constant speed. The driving force from the car's engine is then increased by 3000 N.

At one instant during the resulting acceleration, the resistive force on the car has increased by 200 N and the resistive force on the caravan has increased by 400 N, compared with their values at constant speed.

By how much has the tension in the tow bar increased compared with its value at constant speed?

- **A** 600 N
- **B** 1000 N
- **C** 1400 N
- **D** 1600 N
- **E** 1800 N

WORKED SOLUTION

For the combined car–caravan system, the increase in the forward resultant is

$$3000 - 200 - 400 = 2400 \text{ N}.$$

The total mass is 2400 kg, so the acceleration is

$$a = \frac{2400}{2400} = 1.0 \text{ m s}^{-2}.$$

Let T_0 be the original tow-bar tension and R_0 the original resistive force on the caravan. At constant speed, $T_0 = R_0$.

During the acceleration, let the new tension be T_1. The caravan equation is

$$T_1 - (R_0 + 400) = 1000(1.0).$$

Hence $T_1 - R_0 = 1400 \text{ N}$. Since $T_0 = R_0$,

$$T_1 - T_0 = \boxed{1400 \text{ N}}.$$

P-04

Impulse and pressure

Answer C

Question 4

A small piece of space debris of mass $0.10\ \mathrm{kg}$ strikes the International Space Station at a relative speed of $1.5 \times 10^4\ \mathrm{m\,s^{-1}}$. The debris comes to rest relative to the station in $0.010\ \mathrm{s}$.

Assume the average contact area during the impact is $2.0\ \mathrm{cm}^2$.

What is the average pressure exerted on the station during the impact?

- **A** $7.5 \times 10^6\ \mathrm{Pa}$
- **B** $7.5 \times 10^7\ \mathrm{Pa}$
- **C** $7.5 \times 10^8\ \mathrm{Pa}$
- **D** $1.5 \times 10^8\ \mathrm{Pa}$
- **E** $1.5 \times 10^9\ \mathrm{Pa}$

WORKED SOLUTION

The average force magnitude is

$$F = \frac{\Delta p}{\Delta t} = \frac{0.10 \times 1.5 \times 10^4}{0.010} = 1.5 \times 10^5\ \mathrm{N}.$$

The contact area is

$$2.0\ \mathrm{cm}^2 = 2.0 \times 10^{-4}\ \mathrm{m}^2.$$

So the pressure is

$$p = \frac{F}{A} = \frac{1.5 \times 10^5}{2.0 \times 10^{-4}} = 7.5 \times 10^8\ \mathrm{Pa}.$$

P-05

Light, Doppler shift and photon energy

Answer **B**

Question 5

A star is moving away from a space telescope. The star emits light of wavelength $500\ \mathrm{nm}$, but the telescope detects it with wavelength $502\ \mathrm{nm}$.

What is the percentage decrease in the detected frequency, and hence in the energy of each detected photon?

- **A** 0.20%
- **B** 0.40%
- **C** 0.80%
- **D** 1.0%
- **E** 2.0%

WORKED SOLUTION

In vacuum, $c = f\lambda$, so $f \propto \dfrac{1}{\lambda}$.

The fractional decrease in frequency is therefore approximately the same as the fractional increase in wavelength:

$$\frac{502-500}{502} \approx \frac{2}{502} \approx 0.00398.$$

This is about 0.40%. Since photon energy is $E = hf$, the percentage decrease in photon energy is the same.

P-06

Efficiency, power and useful work

Answer D

Question 6

An electric motor is supplied with power at $0.800\ \mathrm{kW}$. Its efficiency is 60%, and it runs for half an hour while lifting a load vertically at constant speed.

Ignoring any change in the motor itself, what maximum mass can be lifted through a height of $180\ \mathrm{m}$? Take $g = 10\ \mathrm{N\ kg^{-1}}$.

- **A** $120\ \mathrm{kg}$
- **B** $240\ \mathrm{kg}$
- **C** $360\ \mathrm{kg}$
- **D** $480\ \mathrm{kg}$
- **E** $720\ \mathrm{kg}$

WORKED SOLUTION

Input power is $800\ \mathrm{W}$, so useful power is

$$0.60 \times 800 = 480\ \mathrm{W}.$$

Over $1800\ \mathrm{s}$, the useful energy is

$$E = 480 \times 1800 = 864000\ \mathrm{J}.$$

This equals the gain in gravitational potential energy:

$$mgh = 864000.$$

Hence

$$m = \frac{864000}{10 \times 180} = 480\ \mathrm{kg}.$$

P-07

Ideal gases and density

Answer D

Question 7

Air is trapped in a cylinder by a freely moving piston. Initially the trapped air has density ρ at temperature $300\ \mathrm{K}$.

The external load on the piston is adjusted so that the pressure of the trapped air becomes 20% greater than its initial pressure. The load is then kept unchanged, so the pressure remains at this value while the trapped air cools to $270\ \mathrm{K}$. No gas escapes.

Assume the air behaves as an ideal gas. What is the final density of the trapped air?

A 1.09ρ

B 1.20ρ

C 1.25ρ

D $\frac{4}{3}\rho$

E 1.44ρ

WORKED SOLUTION

For a fixed mass of an ideal gas,

$$p = \rho RT \quad \Rightarrow \quad \rho \propto \frac{p}{T}.$$

The final pressure is 1.20 times the initial pressure, while the temperature changes from $300\ \mathrm{K}$ to $270\ \mathrm{K}$. Therefore

$$\frac{\rho_f}{\rho_i} = \frac{1.20}{270/300} = 1.20 \times \frac{300}{270} = \frac{4}{3}.$$

Hence $\boxed{\rho_f = \frac{4}{3}\rho}$.

P-08

Current, magnetic force and geometry

Answer C

Question 8

A steady current flows in a conducting wire. A charge of $48\ \mathrm{C}$ passes a point on the wire in $2.0\ \mathrm{min}$.

A straight $36\ \mathrm{cm}$ section of the wire lies in a uniform magnetic field of strength $0.20\ \mathrm{T}$. This section makes an angle of 30° to the direction of the field.

What is the magnitude of the magnetic force on this section?

- A $2.88 \times 10^{-3}\ \mathrm{N}$
- B $7.2 \times 10^{-3}\ \mathrm{N}$
- C $1.44 \times 10^{-2}\ \mathrm{N}$
- D $2.88 \times 10^{-2}\ \mathrm{N}$
- E $5.76 \times 10^{-2}\ \mathrm{N}$

WORKED SOLUTION

The current is

$$I = \frac{Q}{t} = \frac{48}{120} = 0.40\ \mathrm{A}.$$

The magnetic force on a wire is

$$F = BIL\sin\theta.$$

So

$$F = 0.20 \times 0.40 \times 0.36 \times \sin 30^\circ = 0.20 \times 0.40 \times 0.36 \times 0.5 = 1.44 \times 10^{-2}\ \mathrm{N}.$$

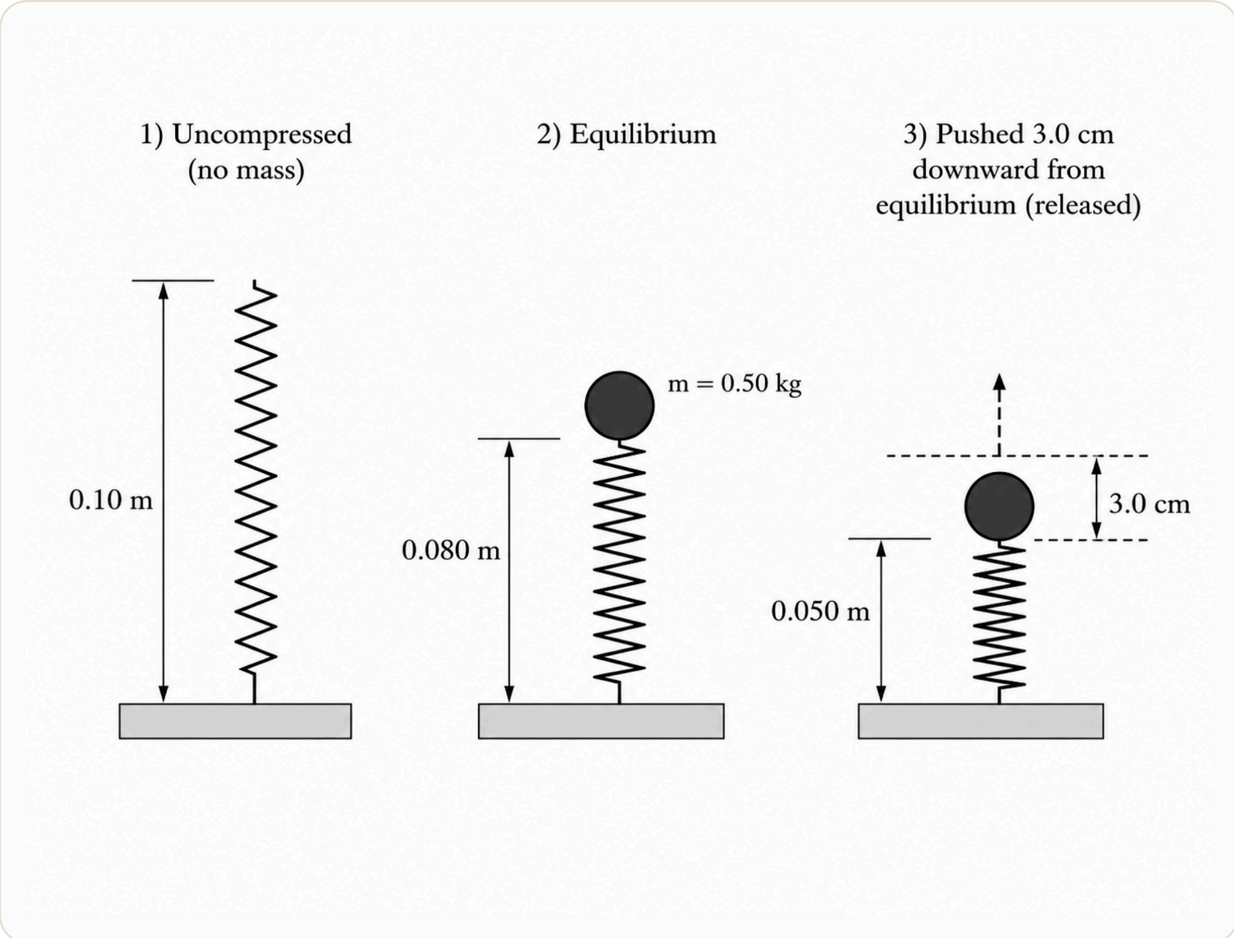
P-09

Springs, equilibrium and oscillation energy

Answer D

Question 9

A light spring has uncompressed length $0.10\ \mathrm{m}$. When an object of mass $0.50\ \mathrm{kg}$ rests on top of the spring in equilibrium, the spring length is $0.080\ \mathrm{m}$.



The mass is then pushed a further $3.0\ \mathrm{cm}$ downward from equilibrium and released, moving on a frictionless vertical guide.

What is the speed of the mass as it passes back through its equilibrium position?

Take $g = 10\ \mathrm{N\ kg^{-1}}$.

- **A** $0.30\ \mathrm{m\ s^{-1}}$
- **B** $0.47\ \mathrm{m\ s^{-1}}$
- **C** $0.60\ \mathrm{m\ s^{-1}}$
- **D** $0.67\ \mathrm{m\ s^{-1}}$
- **E** $0.95\ \mathrm{m\ s^{-1}}$

WORKED SOLUTION

At equilibrium, the spring is compressed by $0.020\ \mathrm{m}$, so

$$kx = mg \quad \Rightarrow \quad k = \frac{0.50 \times 10}{0.020} = 250\ \mathrm{N\ m^{-1}}.$$

If the mass is displaced by an additional $a = 0.030\ \mathrm{m}$ from equilibrium, then at the equilibrium position all of the extra elastic/gravitational energy has become kinetic energy:

$$\frac{1}{2}ka^2 = \frac{1}{2}mv^2.$$

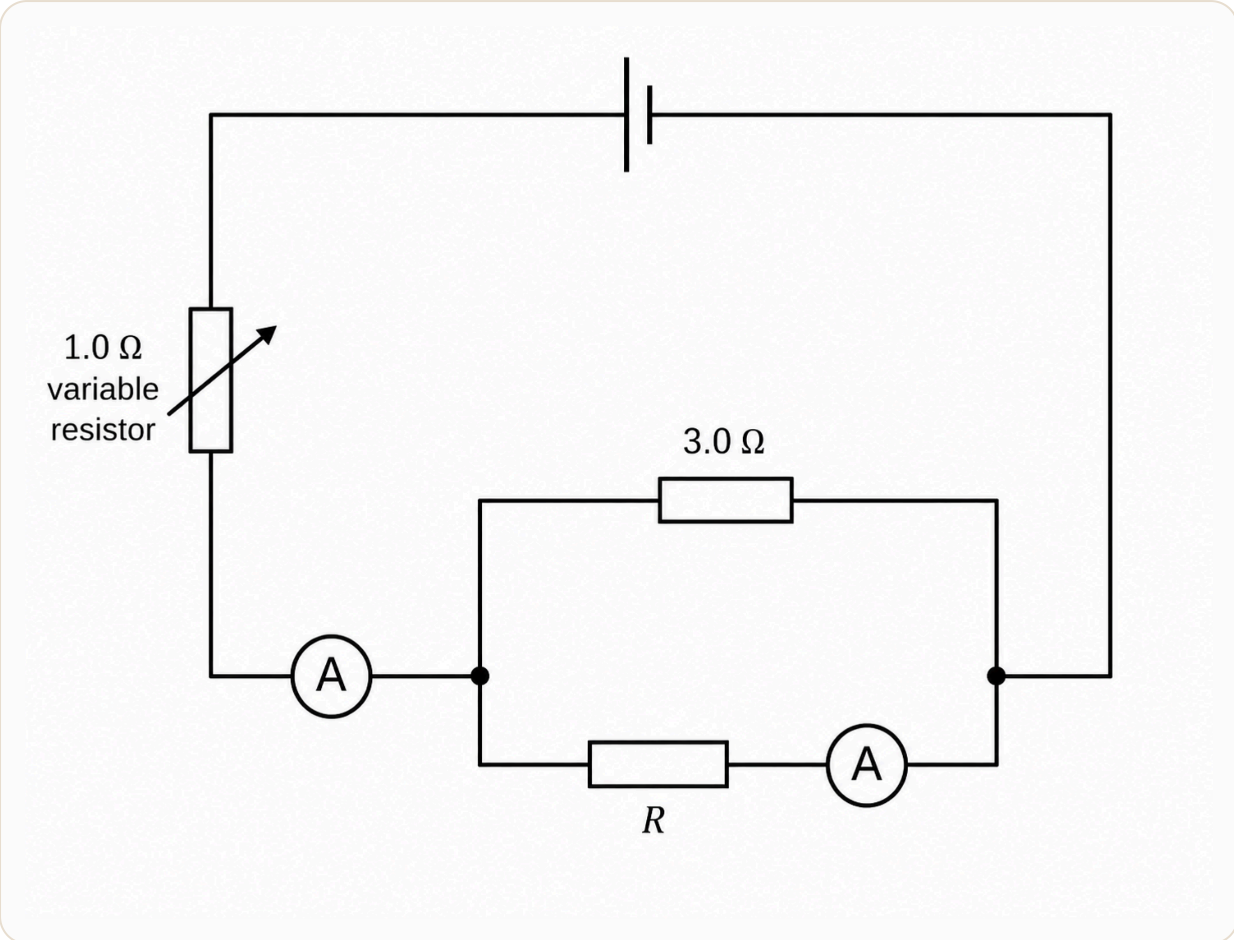
Hence

$$v = a\sqrt{\frac{k}{m}} = 0.030\sqrt{\frac{250}{0.50}} = 0.030\sqrt{500} \approx 0.67\ \mathrm{m\,s^{-1}}.$$

P-10 | Circuits, current splitting and emf | **Answer C**

Question 10

The circuit includes a variable resistor in series with a parallel network containing a $3.0\,\Omega$ resistor and a resistor R. The two ammeter readings are shown.



The variable resistor has resistance $1.0\,\Omega$. Assume ideal ammeters and negligible internal resistance of the supply.

What is the emf of the supply?

- **A** $4.5\ \mathrm{V}$
- **B** $6.6\ \mathrm{V}$
- **C** $7.8\ \mathrm{V}$
- **D** $9.0\ \mathrm{V}$
- **E** $12.3\ \mathrm{V}$

WORKED SOLUTION

The right ammeter is in the R branch, so the current in R is $1.8\ \mathrm{A}$.

The current in the $3.0\,\Omega$ branch is

$$3.3 - 1.8 = 1.5\ \mathrm{A}.$$

So the p.d. across the parallel network is

$$V_{\parallel} = 1.5 \times 3.0 = 4.5\ \mathrm{V}.$$

The current through the series variable resistor is the total current $3.3\ \mathrm{A}$, so its p.d. is

$$V_{\text{var}} = 3.3 \times 1.0 = 3.3\ \mathrm{V}.$$

Hence the supply emf is

$$\mathcal{E} = 4.5 + 3.3 = 7.8\ \mathrm{V}.$$

P-11

Nuclides and algebraic reasoning

Answer E

Question 11

Element Q has several isotopes. Consider the two isotopes

$$ {}^{3x-7}_{x}Q \qquad \text{and} \qquad {}^{2x+3}_{x}Q. $$

The first contains 6 more neutrons than the second.

Another isotope of Q is

$$ {}^{3x+1}_{x}Q. $$

How many neutrons does this third isotope contain?

- A 19
- B 21
- C 25
- D 29
- E 33

WORKED SOLUTION

The neutron numbers are:

first isotope: $(3x-7)-x=2x-7$

second isotope: $(2x+3)-x=x+3$

Given that the first has 6 more neutrons,

$$ (2x-7)-(x+3)=6. $$

So

$$ x-10=6 \Rightarrow x=16. $$

For the third isotope, the number of neutrons is

$$ (3x+1)-x=2x+1=2(16)+1=33. $$

P-12

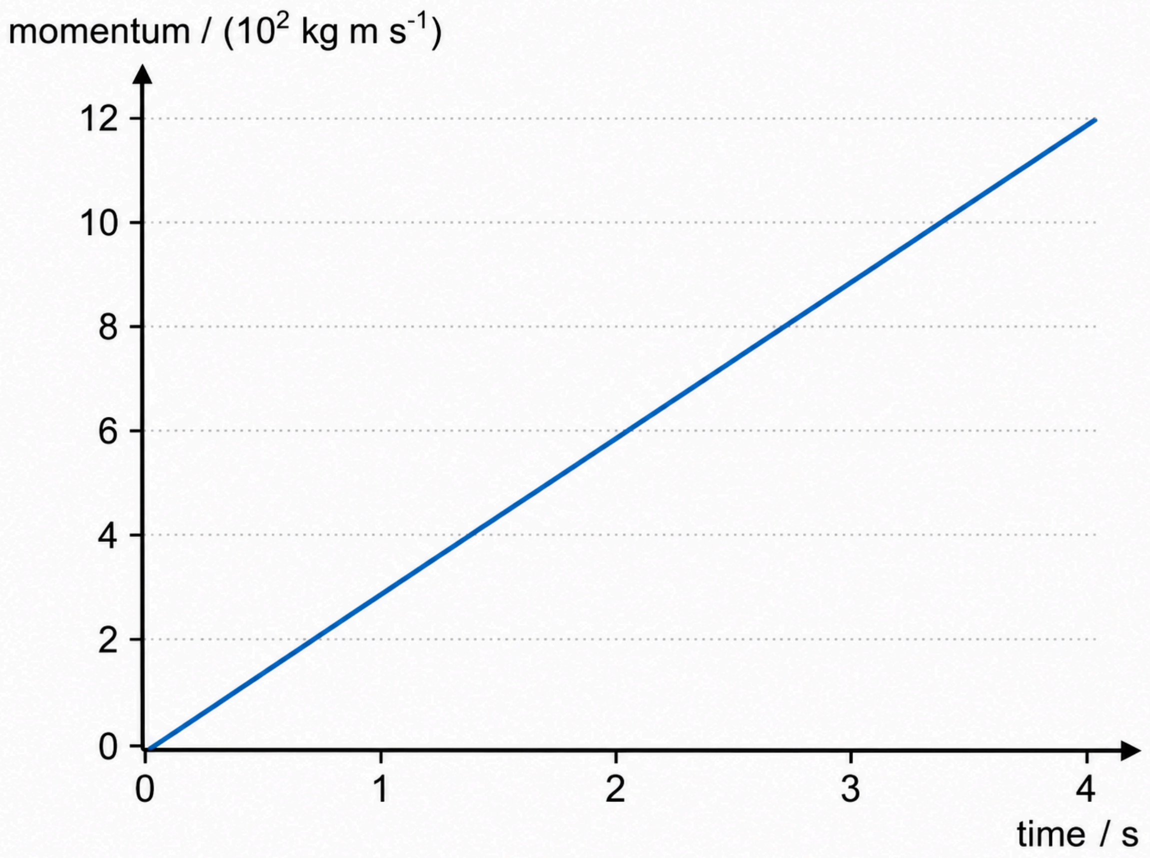
Momentum graphs, friction and power

Answer D

Question 12

A motorised trolley of mass 2.4×10^3 kg accelerates from rest along a horizontal track. The friction force is constant at 1.8×10^3 N.

The graph shows how the momentum of the trolley varies with time.



What is the power delivered by the trolley's motor at $t = 4.0$ s?

- **A** 1.5×10^2 W
- **B** 3.0×10^2 W
- **C** 6.3×10^2 W
- **D** 1.05×10^3 W
- **E** 2.1×10^3 W

WORKED SOLUTION

The gradient of a momentum–time graph gives the resultant force. From the graph, the momentum reaches 12×10^2 kg m s^{-1} at 4.0 s, so

$$F_{\text{net}} = \frac{12 \times 10^2}{4} = 3.0 \times 10^2 \text{ N}.$$

The motor must provide both the resultant force and the force needed to overcome friction:

$$F_{\text{motor}} = 3.0 \times 10^2 + 1.8 \times 10^3 = 2.1 \times 10^3\ \text{N}.$$

At $t = 4.0$ s,

$$v = \frac{p}{m} = \frac{1.2 \times 10^3}{2.4 \times 10^3} = 0.50\ \text{m s}^{-1}.$$

Therefore the power delivered is

$$P = F_{\text{motor}} v = (2.1 \times 10^3)(0.50) = \boxed{1.05 \times 10^3\ \text{W}}.$$

P-13

Work–energy and resistive forces

Answer C

Question 13

A block of mass $6.0\ \mathrm{kg}$ is pushed along a rough horizontal surface by a constant force of $8.0\ \mathrm{N}$. It accelerates uniformly from rest and reaches speed $2.0\ \mathrm{m\ s^{-1}}$ after $4.0\ \mathrm{s}$.

What fraction of the work done by the pushing force during this $4.0\ \mathrm{s}$ is converted into thermal energy against resistive forces?

A $\dfrac{3}{8}$

B $\dfrac{1}{2}$

C $\dfrac{5}{8}$

D $\dfrac{3}{4}$

E $\dfrac{5}{6}$

WORKED SOLUTION

The acceleration is

$$a = \frac{2.0}{4.0} = 0.50\ \mathrm{m\ s^{-2}}.$$

The resultant force is therefore

$$F_{\text{net}} = ma = 6.0 \times 0.50 = 3.0\ \mathrm{N}.$$

So the resistive force is

$$8.0 - 3.0 = 5.0\ \mathrm{N}.$$

The displacement in $4.0\ \mathrm{s}$ is

$$s = \frac{1}{2}at^2 = \frac{1}{2}(0.50)(4.0)^2 = 4.0\ \mathrm{m}.$$

Work done by the pushing force:

$$W_{\text{push}} = 8.0 \times 4.0 = 32\ \mathrm{J}.$$

Thermal energy against resistance:

$$W_{\text{res}} = 5.0 \times 4.0 = 20\ \mathrm{J}.$$

Hence the fraction is

$$\frac{20}{32} = \frac{5}{8}.$$

P-14

Specific heat capacity and gradients

Answer C

Question 14

A metal block of mass M is heated at a constant rate P. The graph of temperature rise ΔT against time t is a straight line of gradient k.

A second block is made of the same material, has mass $3M$, and is heated at constant rate $2P$.

What is the gradient of its ΔT-against-t graph?

A $\frac{k}{6}$

B $\frac{k}{3}$

C $\frac{2k}{3}$

D $\frac{3k}{2}$

E $6k$

WORKED SOLUTION

For the first block,

$$P = Mc\frac{dT}{dt}.$$

Hence

$$k = \frac{P}{Mc}.$$

For the second block, the gradient is

$$k_2 = \frac{2P}{3Mc} = \frac{2}{3}\frac{P}{Mc} = \frac{2k}{3}.$$

P-15

Transformers, efficiency and heating

Answer **D**

Question 15

A non-ideal transformer has 100 turns on the primary coil and 25 turns on the secondary coil. It is supplied with $3.0\ \mathrm{kW}$ of electrical power at a current of $12.5\ \mathrm{A}$.

The output voltage is the same as for an ideal transformer, but the current in the secondary coil is $40\ \mathrm{A}$.

The transformer runs for $15\ \mathrm{min}$. Assume all power lost by inefficiency remains in the transformer, whose thermal capacity is $2.0 \times 10^4\ \mathrm{J\ K^{-1}}$.

What is the temperature rise of the transformer?

- **A** $12\ \mathrm{K}$
- **B** $18\ \mathrm{K}$
- **C** $24\ \mathrm{K}$
- **D** $27\ \mathrm{K}$
- **E** $30\ \mathrm{K}$

WORKED SOLUTION

Input voltage is

$$V_p = \frac{P_{in}}{I_p} = \frac{3000}{12.5} = 240\ \mathrm{V}.$$

For the turns ratio,

$$\frac{V_s}{V_p} = \frac{25}{100} = \frac{1}{4},$$

so $V_s = 60\ \mathrm{V}$.

Thus the output power is

$$P_{out} = V_s I_s = 60 \times 40 = 2400\ \mathrm{W}.$$

Power lost is

$$P_{loss} = 3000 - 2400 = 600\ \mathrm{W}.$$

In $900\ \mathrm{s}$, the thermal energy gained is

$$E = 600 \times 900 = 5.4 \times 10^5\ \mathrm{J}.$$

So the temperature rise is

$$\Delta T = \frac{E}{C} = \frac{5.4 \times 10^5}{2.0 \times 10^4} = 27\ \mathrm{K}.$$

P-16

Series circuits and power scaling

Answer D

Question 16

A set of decorative lights consists of 20 identical lamps connected in series to a d.c. supply of constant voltage. The total power transferred by all 20 lamps together is P.

One lamp fails so that it becomes short-circuited and has zero resistance. The remaining 19 lamps are still lit.

By what factor does the power transferred to *each remaining lamp* increase?

A $\frac{19}{20}$

B $\frac{20}{19}$

C $\frac{361}{400}$

D $\frac{400}{361}$

E $\frac{400}{19}$

WORKED SOLUTION

If each lamp has resistance R, then the original total resistance is $20R$, so current is

$$I = \frac{V}{20R}.$$

After one lamp is short-circuited, the total resistance is $19R$, so current becomes

$$I' = \frac{V}{19R}.$$

Power in each functioning lamp is proportional to I^2R. Therefore the factor increase is

$$\left(\frac{I'}{I}\right)^2 = \left(\frac{V/(19R)}{V/(20R)}\right)^2 = \left(\frac{20}{19}\right)^2 = \frac{400}{361}.$$

P-17

Electromagnetic spectrum

Answer E

Question 17

The wavelength range of visible light is about $400\ \mathrm{nm}$ to $700\ \mathrm{nm}$. Light of frequency $6.0 \times 10^{14}\ \mathrm{Hz}$ is green. Microwaves used in cooking have wavelength $12\ \mathrm{cm}$.

Which statements are correct?

1. Light of frequency $7.5 \times 10^{14}\ \mathrm{Hz}$ is red.
2. Microwaves used in cooking have frequency $2.5 \times 10^{9}\ \mathrm{Hz}$.
3. Electromagnetic radiation of frequency $3.0 \times 10^{13}\ \mathrm{Hz}$ can be used for thermal imaging of buildings.

Take $c = 3.0 \times 10^{8}\ \mathrm{m\ s^{-1}}$.

A none of them

B 1 only

C 2 only

D 3 only

E 2 and 3 only

WORKED SOLUTION

Statement 1 is false: a higher visible frequency corresponds to bluer/violet light, not red.

For statement 2,

$$f = \frac{c}{\lambda} = \frac{3.0 \times 10^{8}}{0.12} = 2.5 \times 10^{9}\ \mathrm{Hz},$$

so statement 2 is true.

Statement 3 is also true: $3.0 \times 10^{13}\ \mathrm{Hz}$ lies in the infrared region, which is used in thermal imaging.

P-18

Series circuits and filament lamps

Answer D

Question 18

A $6.0\ \mathrm{V}$ battery is connected in series with an $8.0\ \Omega$ resistor, a filament lamp and an ammeter. The ammeter reading is $0.25\ \mathrm{A}$.

Which pair gives the potential difference across the lamp and the resistance of the lamp at this operating point?

- **A** $2.0\ \mathrm{V},\ 8.0\ \Omega$
- **B** $4.0\ \mathrm{V},\ 8.0\ \Omega$
- **C** $4.0\ \mathrm{V},\ 16\ \Omega$ and power $0.50\ \mathrm{W}$
- **D** $4.0\ \mathrm{V},\ 16\ \Omega$ and power $1.0\ \mathrm{W}$
- **E** $6.0\ \mathrm{V},\ 24\ \Omega$

WORKED SOLUTION

The resistor has p.d.

$$V_R = IR = 0.25 \times 8.0 = 2.0\ \mathrm{V}.$$

So the lamp has

$$V_L = 6.0 - 2.0 = 4.0\ \mathrm{V}.$$

Its resistance at this operating point is

$$R_L = \frac{V_L}{I} = \frac{4.0}{0.25} = 16\ \Omega.$$

Its power is

$$P_L = V_L I = 4.0 \times 0.25 = 1.0\ \mathrm{W}.$$

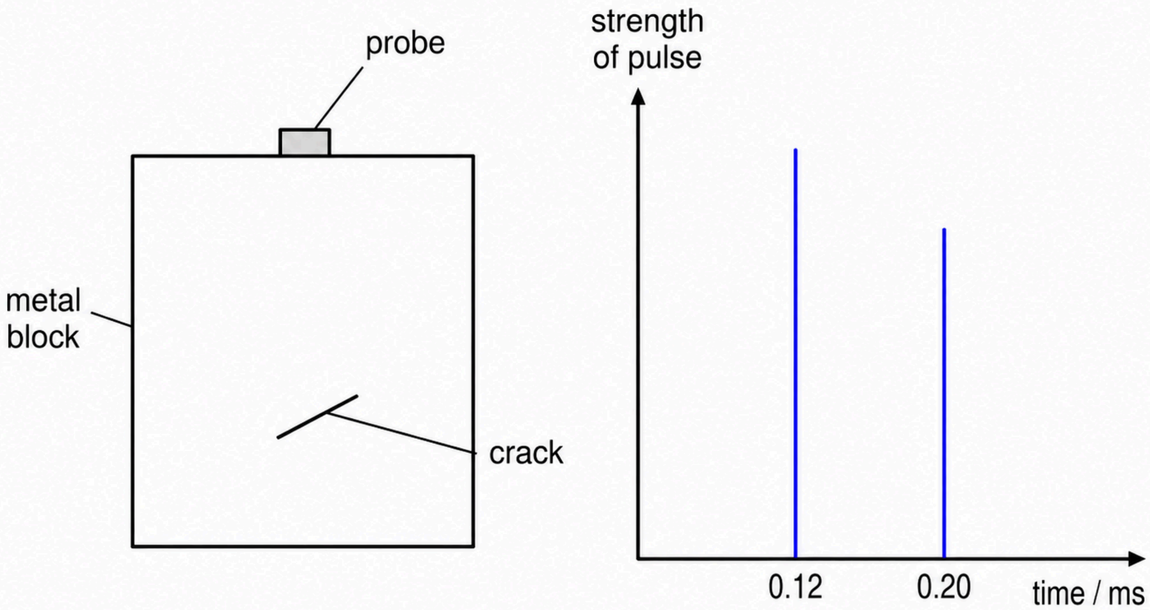
P-19

Ultrasound and echo timing

Answer C

Question 19

Ultrasound is used to find a crack inside a cuboid metal block. A probe is held against the top surface and sends a short pulse into the block. Reflections return from the crack and from the bottom surface.



The reflections return after $0.12\ \mathrm{ms}$ and $0.20\ \mathrm{ms}$. The speed of ultrasound in the metal is $5000\ \mathrm{m\ s^{-1}}$.

What is the ratio

$$\frac{\text{thickness of block}}{\text{depth of crack below the top surface}}?$$

A $2:1$

B $3:2$

C $5:3$

D $4:3$

E $5:4$

WORKED SOLUTION

Echo times are round-trip times, so

depth of crack:

$$d = \frac{vt}{2} = \frac{5000 \times 0.12 \times 10^{-3}}{2} = 0.30 \text{ m}.$$

Thickness of block:

$$h = \frac{5000 \times 0.20 \times 10^{-3}}{2} = 0.50 \text{ m}.$$

Therefore

$$h : d = 0.50 : 0.30 = 5 : 3.$$

P-20

Kinematics and minimum-time travel

Answer C

Question 20

A straight railway line between two cities is $60\ \mathrm{km}$ long. A train starts from rest, accelerates uniformly at $1.5\ \mathrm{m\ s^{-2}}$ to a maximum speed of $120\ \mathrm{m\ s^{-1}}$, then later brakes uniformly at $2.0\ \mathrm{m\ s^{-2}}$ to come to rest at the second city.

What is the minimum possible journey time?

- **A** $430\ \mathrm{s}$
- **B** $500\ \mathrm{s}$
- **C** $570\ \mathrm{s}$
- **D** $640\ \mathrm{s}$
- **E** $860\ \mathrm{s}$

WORKED SOLUTION

Acceleration phase:

$$t_1 = \frac{120}{1.5} = 80\ \mathrm{s}, \qquad s_1 = \frac{120^2}{2(1.5)} = 4800\ \mathrm{m}.$$

Braking phase:

$$t_2 = \frac{120}{2.0} = 60\ \mathrm{s}, \qquad s_2 = \frac{120^2}{2(2.0)} = 3600\ \mathrm{m}.$$

Total distance used in speeding up and slowing down:

$$4800 + 3600 = 8400\ \mathrm{m}.$$

Remaining cruising distance:

$$60000 - 8400 = 51600\ \mathrm{m}.$$

Cruising time:

$$t_3 = \frac{51600}{120} = 430\ \mathrm{s}.$$

Total time:

$$80 + 60 + 430 = 570\ \mathrm{s}.$$

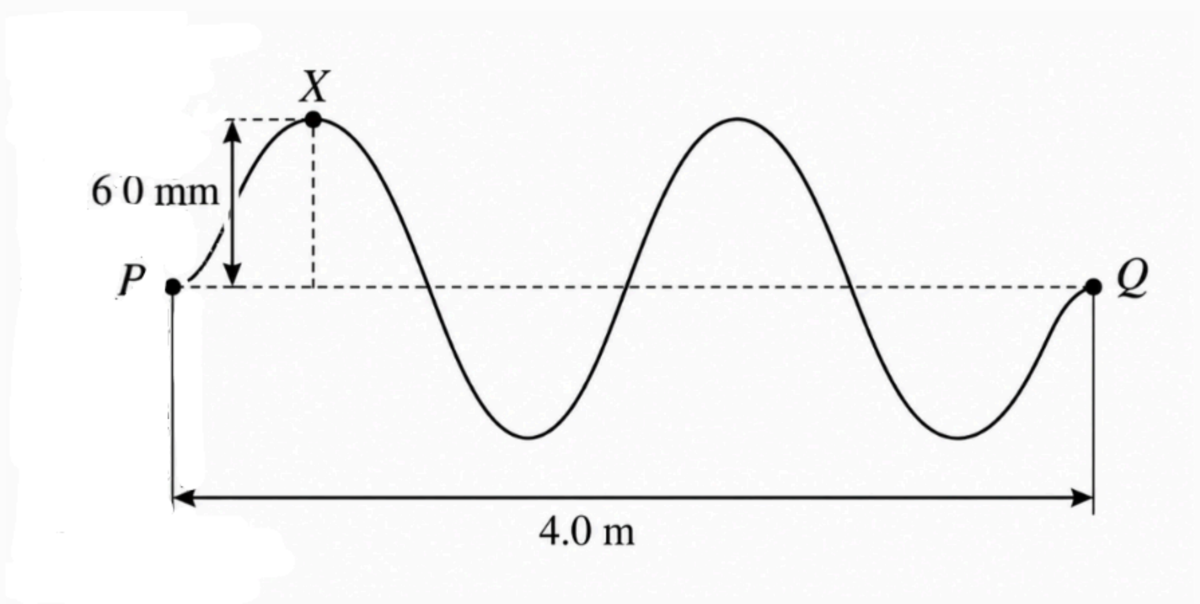
P-21

Waves, phase and SHM

Answer B

Question 21

A transverse wave on a string has speed $500 \mathrm{~m~s^{-1}}$. The horizontal distance between points P and Q shown is $4.0 \mathrm{~m}$.



At time $t = 0$, point X is at its maximum displacement of $6.0 \mathrm{~mm}$ above equilibrium.

What is the displacement of point X at $t = 7.0 \mathrm{~ms}$, and in which direction is it moving at that instant?

- **A** $6.0 \mathrm{~mm}$ above equilibrium, moving downward
- **B** $0 \mathrm{~mm}$, moving upward
- **C** $0 \mathrm{~mm}$, moving downward
- **D** $6.0 \mathrm{~mm}$ below equilibrium, moving upward
- **E** between 0 and $6.0 \mathrm{~mm}$ below equilibrium, moving downward

WORKED SOLUTION

From the diagram, the distance from P to Q spans two complete wavelengths. Therefore

$$2\lambda = 4.0 \mathrm{~m},$$

so

$$\lambda = 2.0 \mathrm{~m}.$$

The frequency of the wave is therefore

$$f = \frac{v}{\lambda} = \frac{500}{2.0} = 250 \text{ Hz}.$$

Hence the period is

$$T = \frac{1}{f} = \frac{1}{250} = 0.004 \text{ s} = 4.0 \text{ ms}.$$

Now

$$7.0 \text{ ms} = 1.75T.$$

This is equivalent to $0.75T$ after one complete cycle. Starting from maximum positive displacement, after $0.75T$ the point is at equilibrium and moving upward.

Therefore the displacement is

$$\boxed{0 \text{ mm, moving upward}}.$$

P-22

Radioactivity and half-life

Answer **B**

Question 22

A radioactive nuclide X decays in a single stage to a stable nuclide R. Another radioactive nuclide Y decays in a single stage to a stable nuclide S.

When a rock formed, it contained equal numbers of atoms of X, Y, R and S.

The half-life of X is T years and the half-life of Y is $2T$ years.

What is the value of

$$\frac{\text{number of atoms of } R}{\text{number of atoms of } S} \quad \text{at a time } 4T \text{ years after the rock formed?}$$

A $\frac{17}{20}$

B $\frac{31}{28}$

C $\frac{5}{4}$

D $\frac{6}{5}$

E 2

WORKED SOLUTION

Let each initial number be N.

For X: after $4T$, four half-lives have passed, so

$$X = \frac{N}{16}.$$

Thus $\frac{15N}{16}$ atoms have decayed into R, giving

$$R = N + \frac{15N}{16} = \frac{31N}{16}.$$

For Y: after $4T$, two half-lives have passed, so

$$Y = \frac{N}{4}.$$

Hence $\frac{3N}{4}$ atoms have decayed into S, giving

$$S = N + \frac{3N}{4} = \frac{7N}{4}.$$

Therefore

$$\frac{R}{S} = \frac{31N/16}{7N/4} = \frac{31}{28}.$$

P-23

Waves and relative motion

Answer D

Question 23

A ship travels into a wave that is moving in the opposite direction to the ship. The ship has speed $8.0\ \mathrm{m\ s^{-1}}$ and the wave has speed $3.0\ \mathrm{m\ s^{-1}}$.

The front of the ship rises and falls with a time period of $8.0\ \mathrm{s}$.

What are the wavelength of the wave and the period of the wave for a stationary buoy?

- **A** $24\ \mathrm{m},\ 8.0\ \mathrm{s}$
- **B** $40\ \mathrm{m},\ 13.3\ \mathrm{s}$
- **C** $64\ \mathrm{m},\ 21.3\ \mathrm{s}$
- **D** $88\ \mathrm{m},\ 29.3\ \mathrm{s}$
- **E** $88\ \mathrm{m},\ 8.0\ \mathrm{s}$

WORKED SOLUTION

Relative speed between ship and wave crests is

$$8.0 + 3.0 = 11.0\ \mathrm{m\ s^{-1}}.$$

So the wavelength is

$$\lambda = v_{rel} T_{ship} = 11.0 \times 8.0 = 88\ \mathrm{m}.$$

For a stationary buoy, the wave period is

$$T = \frac{\lambda}{v_{wave}} = \frac{88}{3.0} = 29.3\ \mathrm{s}.$$

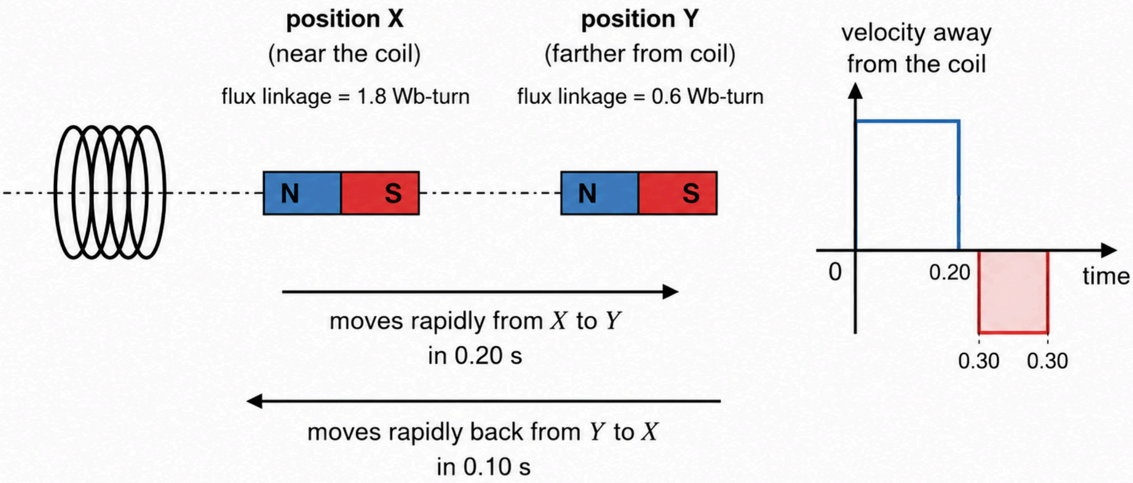
P-24

Electromagnetic induction

Answer D

Question 24

A bar magnet is placed at position X close to one end of a coil and on its axis. It is moved rapidly to a more distant position Y, and then back to X.



The flux linkage of the coil is $1.8\ \mathrm{Wb\text{-}turn}$ when the magnet is at X and $0.6\ \mathrm{Wb\text{-}turn}$ when it is at Y. The motion from X to Y takes $0.20\ \mathrm{s}$, while the return motion from Y to X takes $0.10\ \mathrm{s}$.

What is the greatest magnitude of the induced e.m.f. during the complete motion?

A $3.0\ \mathrm{V}$

B $6.0\ \mathrm{V}$

C $9.0\ \mathrm{V}$

D $12\ \mathrm{V}$

E $18\ \mathrm{V}$

WORKED SOLUTION

The change in flux linkage between X and Y is

$$\Delta(N\Phi) = 1.8 - 0.6 = 1.2\ \mathrm{Wb\text{-}turn}.$$

For the outward motion, the average induced e.m.f. magnitude is

$$\mathcal{E}_1 = \frac{1.2}{0.20} = 6.0\ \mathrm{V}.$$

For the return motion, the same flux-linkage change happens in $0.10\ \mathrm{s}$, so

$$\mathcal{E}_2 = \frac{1.2}{0.10} = 12\ \mathrm{V}.$$

The greatest magnitude is $12\ \mathrm{V}$.

P-25

Seismic waves

Answer **B**

Question 25

When an earthquake occurs, both P-waves and S-waves are produced at the same time and travel along the same path. P-waves travel at $5.0\ \mathrm{km\ s^{-1}}$ and S-waves travel at $3.0\ \mathrm{km\ s^{-1}}$.

A seismic monitoring station detects the P-waves $30\ \mathrm{s}$ before it detects the S-waves.

How long after the earthquake does the P-wave reach the station?

- **A** $30\ \mathrm{s}$
- **B** $45\ \mathrm{s}$
- **C** $60\ \mathrm{s}$
- **D** $75\ \mathrm{s}$
- **E** $90\ \mathrm{s}$

WORKED SOLUTION

Let the distance to the station be d km. Then

$$\frac{d}{3.0} - \frac{d}{5.0} = 30.$$

So

$$d\left(\frac{1}{3} - \frac{1}{5}\right) = 30 \Rightarrow d\left(\frac{2}{15}\right) = 30 \Rightarrow d = 225\ \mathrm{km}.$$

The P-wave travel time is therefore

$$t_P = \frac{225}{5.0} = 45\ \mathrm{s}.$$

P-26

Pressure, ideal gas law and geometry

Answer C

Question 26

A tall smooth cylinder contains air at atmospheric pressure 1.00×10^5 Pa. The density of the air is $1.20\ \mathrm{kg\ m^{-3}}$. A heavy piston of mass $50.0\ \mathrm{kg}$ and cross-sectional area $0.0200\ \mathrm{m^2}$ is placed on top and allowed to fall slowly until it rests in equilibrium.

The initial height of the trapped air column is $0.50\ \mathrm{m}$. Assume the air behaves as an ideal gas and its temperature remains constant. Take $g = 10\ \mathrm{N\ kg^{-1}}$.

By how far does the piston fall?

A $0.040\ \mathrm{m}$

B $0.080\ \mathrm{m}$

C $0.10\ \mathrm{m}$

D $0.125\ \mathrm{m}$

E $0.20\ \mathrm{m}$

WORKED SOLUTION

The piston adds pressure

$$\frac{mg}{A} = \frac{50.0 \times 10}{0.0200} = 2.5 \times 10^4\ \mathrm{Pa}.$$

So the final pressure is

$$1.00 \times 10^5 + 2.5 \times 10^4 = 1.25 \times 10^5\ \mathrm{Pa}.$$

At constant temperature, pV is constant, so volume decreases by a factor

$$\frac{V_f}{V_i} = \frac{p_i}{p_f} = \frac{1.00}{1.25} = 0.80.$$

The cross-sectional area is constant, so the height also decreases by a factor of 0.80:

$$h_f = 0.80 \times 0.50 = 0.40\ \mathrm{m}.$$

Therefore the piston falls by

$$0.50 - 0.40 = 0.10\ \mathrm{m}.$$

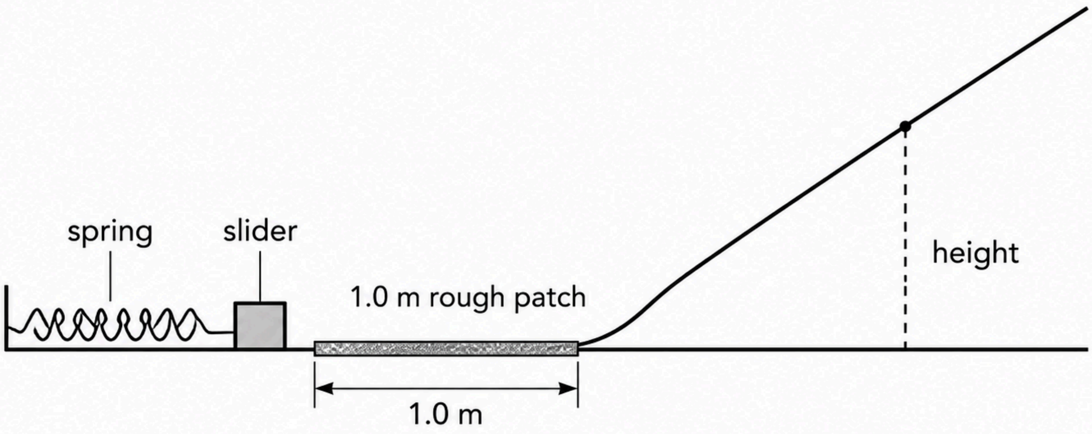
P-27

Springs, energy scaling and friction

Answer C

Question 27

A small slider of mass $30\ \mathrm{g}$ is at rest near the bottom of a frictionless slope and in contact with a light uncompressed spring as shown.



In one test, compressing the spring by $5.0\ \mathrm{cm}$ sends the $30\ \mathrm{g}$ slider up the slope to a maximum vertical height of $20\ \mathrm{cm}$.

The slider is then replaced by a $20\ \mathrm{g}$ slider. The spring is compressed by $15\ \mathrm{cm}$. During the motion, the slider must first cross a rough horizontal patch of length $1.0\ \mathrm{m}$ on which a constant resistive force of $0.20\ \mathrm{N}$ acts. Elsewhere resistance is negligible.

To what maximum vertical height does the new slider rise? Take $g = 10\ \mathrm{N\ kg^{-1}}$.

- **A** $120\ \mathrm{cm}$
- **B** $150\ \mathrm{cm}$
- **C** $170\ \mathrm{cm}$
- **D** $180\ \mathrm{cm}$
- **E** $270\ \mathrm{cm}$

WORKED SOLUTION

From the first test, spring energy equals gain in gravitational potential energy:

$$\frac{1}{2}k(0.050)^2 = 0.030 \times 10 \times 0.20.$$

So

$$0.00125k = 0.060 \Rightarrow k = 48 \text{ N m}^{-1}.$$

With the new slider and compression 0.15 m, spring energy is

$$E_s = \frac{1}{2} \times 48 \times (0.15)^2 = 0.54 \text{ J}.$$

Work done against the rough patch is

$$W_f = 0.20 \times 1.0 = 0.20 \text{ J}.$$

So energy left for gravitational potential energy is

$$0.54 - 0.20 = 0.34 \text{ J}.$$

For the $20 \text{ g} = 0.020 \text{ kg}$ slider,

$$mgh = 0.34 \Rightarrow h = \frac{0.34}{0.020 \times 10} = 1.7 \text{ m} = 170 \text{ cm}.$$

ANSWER KEY

ESAT Mock Test 5 — Mathematics 2

Twenty-seven challenging Mathematics 2 questions emphasising structural insight, efficient exact reasoning, algebra, geometry and calculus.

Mathematics 2 answer key

1 D	2 B	3 E	4 C	5 B	6 C	7 B	8 B	9 C
10 B	11 B	12 B	13 C	14 B	15 B	16 C	17 C	18 D
19 D	20 C	21 C	22 B	23 C	24 C	25 B	26 E	27 D

ESAT MOCK TEST 5

Mathematics 2

27 hard ESAT-style questions with complete worked solutions.

M2-01

Polynomials

Answer D

Question 1

The polynomial $P(x) = x^4 + ax^3 + bx^2 - 44x + 24$ has $(x - 1)$ and $(x - 2)$ as factors. What is the value of $a^2 + b$?

A 49

B 61

C 68

D 75

E 82

WORKED SOLUTION

Since $x = 1$ and $x = 2$ are roots, $P(1)=0$ and $P(2)=0$.
From $P(1)=0$: $1 + a + b - 44 + 24 = 0$, so $a + b = 19$.
From $P(2)=0$: $16 + 8a + 4b - 88 + 24 = 0$, so $2a + b = 12$.
Subtracting gives $a = -7$, and then $b = 26$.
Hence $a^2 + b = 49 + 26 = $ **75**.

M2-02

Calculus / stationary points

Answer **B**

Question 2

The curve $y = x^3 + px^2 + qx + 10$ has a local maximum at $x = -1$ and a local minimum at $x = 3$. What is the sum of the y-coordinates of these two stationary points?

- **A** -8
- **B** -2
- **C** 2
- **D** 6
- **E** 12

WORKED SOLUTION

Differentiate: $y' = 3x^2 + 2px + q$. Since the stationary points are at $x = -1$ and $x = 3$, we have $y' = 3(x+1)(x-3) = 3x^2 - 6x - 9$.

So $2p = -6$ and $q = -9$, hence $p = -3$.

The curve is $y = x^3 - 3x^2 - 9x + 10$.

At $x = -1$, $y = -1 - 3 + 9 + 10 = 15$.

At $x = 3$, $y = 27 - 27 - 27 + 10 = -17$.

The sum is $15 + (-17) =$ $\mathbf{-2}$.

M2-03

Coordinate geometry

Answer E

Question 3

The line $y = mx + 3$ is reflected in the line $y = x$. The reflected line has y-intercept $-\frac{2}{3}$. What is the value of m?

A $\frac{3}{2}$

B 2

C $\frac{5}{2}$

D 3

E $\frac{9}{2}$

WORKED SOLUTION

Reflecting in $y = x$ swaps x and y. So the reflected line is $x = my + 3$, which rearranges to $y = (x - 3)/m = (1/m)x - 3/m$.

The y-intercept of the reflected line is therefore $-3/m$.

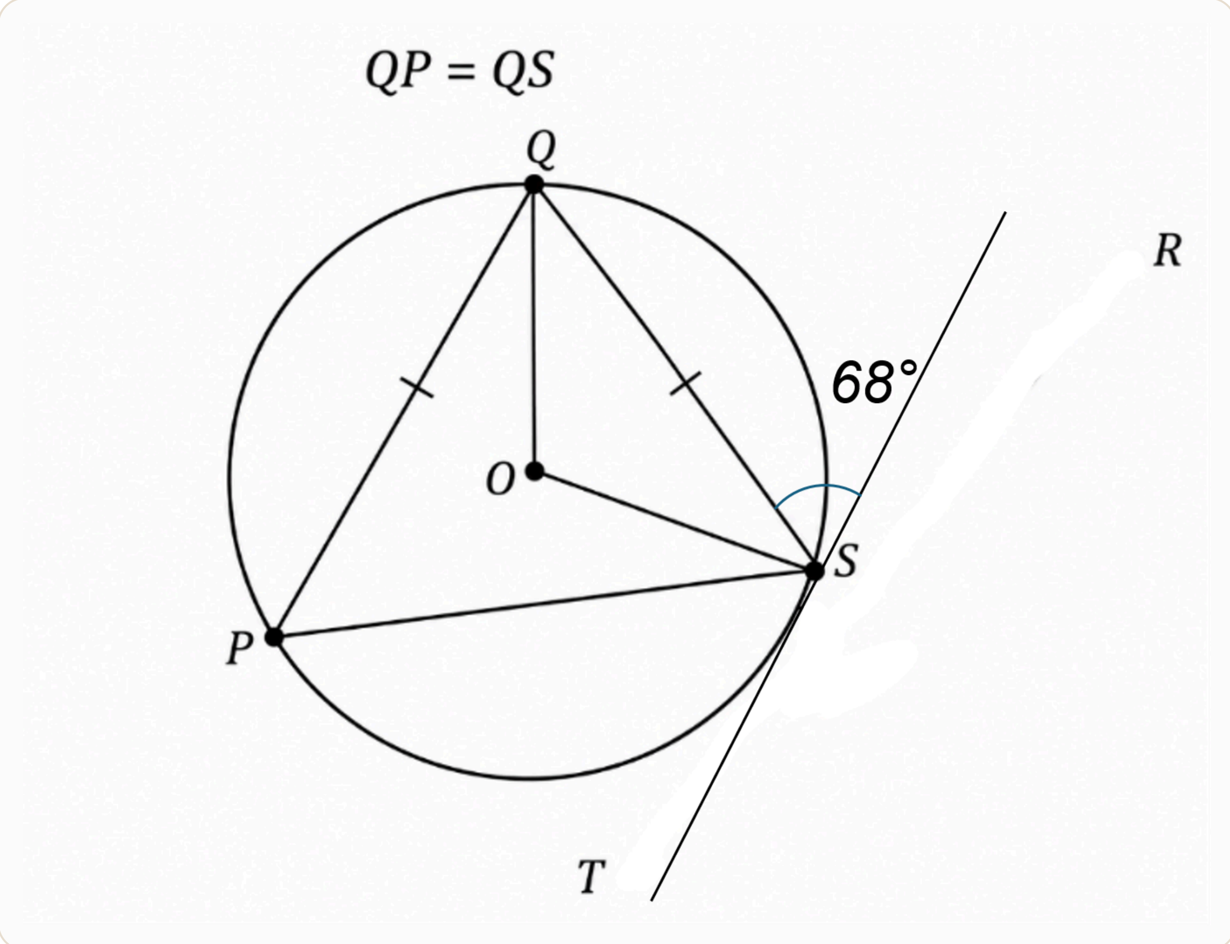
Given $-3/m = -2/3$, we get $3/m = 2/3$, so $9 = 2m$ and $m =$ **9/2**.

M2-04

Circle theorems

Answer C

Question 4



In the diagram, O is the centre of the circle and RT is a tangent at S. Points P and Q lie on the circumference, $QP = QS$, and the marked angle is $\angle QSR = 68^\circ$. What is the size of $\angle POS$?

- **A** 44°
- **B** 68°
- **C** 88°
- **D** 112°
- **E** 136°

WORKED SOLUTION

By the tangent–chord theorem applied to chord QS,

$$\angle QPS = \angle QSR = 68^\circ.$$

Since $QP = QS$, triangle PQS is isosceles with base PS. Hence

$$\angle PSQ = 68^\circ, \qquad \angle PQS = 180^\circ - 68^\circ - 68^\circ = 44^\circ.$$

Chord PS subtends angle $\angle PQS$ at the circumference and angle $\angle POS$ at the centre. Therefore

$$\angle POS = 2\angle PQS = 2(44^\circ) = \boxed{88^\circ}.$$

M2-05

Integration / area

Answer **B**

Question 5

What is the total area enclosed between the curve $y = x^2 - 4x + 3$ and the line $y = -x + 1$, for $0 \le x \le 3$?

A $\frac{7}{6}$

B $\frac{11}{6}$

C $\frac{13}{6}$

D $\frac{17}{6}$

E $\frac{23}{6}$

WORKED SOLUTION

Subtract the line from the curve: $(x^2 - 4x + 3) - (-x + 1) = x^2 - 3x + 2 = (x-1)(x-2)$.

This is positive on [0,1], negative on [1,2], and positive on [2,3].

So total area $= \int_0^1 (x^2-3x+2)dx - \int_1^2 (x^2-3x+2)dx + \int_2^3 (x^2-3x+2)dx$.

An antiderivative is $x^3/3 - 3x^2/2 + 2x$.

Evaluating gives 5/6 + 1/6 + 5/6 = **11/6**.

M2-06

Surds

Answer C

Question 6

Simplify $\sqrt{(18 + 8\sqrt{5})} - \sqrt{(18 - 8\sqrt{5})}$. Write the answer in the form $k\sqrt{2}$. What is k?

A 2

B 3

C 4

D 5

E 6

WORKED SOLUTION

Write $18 + 8\sqrt{5}$ as $a + b + 2\sqrt{ab}$ with $a + b = 18$ and $ab = 80$. The numbers are 10 and 8.

So $\sqrt{(18 + 8\sqrt{5})} = \sqrt{10} + \sqrt{8} = \sqrt{10} + 2\sqrt{2}$.

Similarly, $\sqrt{(18 - 8\sqrt{5})} = \sqrt{10} - 2\sqrt{2}$.

Subtracting gives $(\sqrt{10} + 2\sqrt{2}) - (\sqrt{10} - 2\sqrt{2}) = \mathbf{4\sqrt{2}}$.

Hence $k = \mathbf{4}$.

M2-07

Tangents

Answer B

Question 7

The line $y = 2x + k$ is a tangent to the curve $y = 3x^2 - 10x + 7$. What is the value of k?

A -7

B -5

C -3

D 1

E 5

WORKED SOLUTION

At the point of tangency the line and curve have one repeated intersection.

Set them equal: $3x^2 - 10x + 7 = 2x + k$, so $3x^2 - 12x + (7 - k) = 0$.

For tangency the discriminant is 0:

$(-12)^2 - 4\cdot3\cdot(7-k) = 0$.

$144 - 12(7-k) = 0 \Rightarrow 144 - 84 + 12k = 0 \Rightarrow 60 + 12k = 0$.

So $k = \mathbf{-5}$.

M2-08

Differentiation

Answer **B**

Question 8

Given $y = (3\sqrt{x} - 2/\sqrt{x})^2$, find dy/dx when $x = ½$.

- **A** −9
- **B** −7
- **C** −5
- **D** 7
- **E** 9

WORKED SOLUTION

First simplify: $y = 9x - 12 + 4/x$.

Differentiate: $dy/dx = 9 - 4/x^2$.

At $x = 1/2$, $x^2 = 1/4$, so $4/x^2 = 16$.

Therefore $dy/dx = 9 - 16 =$ **−7**.

M2-09

Integration

Answer **C**

Question 9

Find the value of $\int_1^4 (3x^2 + 1)/(x\sqrt{x})\,dx$.

- **A** 12
- **B** 14
- **C** 15
- **D** 16
- **E** 18

WORKED SOLUTION

Rewrite the integrand: $(3x^2 + 1)/(x\sqrt{x}) = 3x^{1/2} + x^{-3/2}$.

So $\int (3x^{1/2} + x^{-3/2})\,dx = 2x^{3/2} - 2x^{-1/2}$.

Evaluate from 1 to 4:

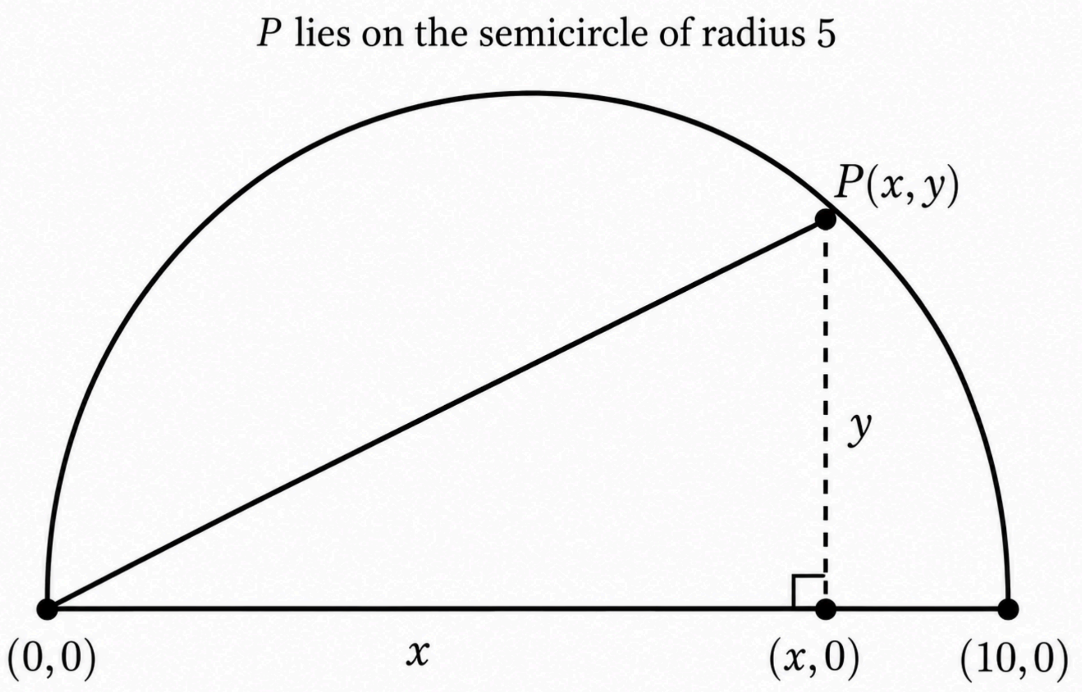
$[2x^{3/2} - 2x^{-1/2}]_1^4 = (2\cdot8 - 2\cdot1/2) - (2\cdot1 - 2\cdot1) = 15 - 0 =$ **15**.

M2-10

Coordinate geometry / optimisation

Answer **B**

Question 10



A point P(x,y) lies on the semicircle with diameter from (0,0) to (10,0). A right triangle has vertices at (0,0), (x,0), and P(x,y). What is the maximum possible area of the triangle?

A $25\sqrt{3/8}$

B $75\sqrt{3/8}$

C $25\sqrt{3/4}$

D $75\sqrt{3/4}$

E $125\sqrt{3/8}$

WORKED SOLUTION

The semicircle has centre (5,0) and radius 5, so $y = \sqrt{(25 - (x-5)^2)} = \sqrt{(10x - x^2)}$.

The triangle area is $A = 1/2\ xy = 1/2\ x\sqrt{(10x - x^2)}$.

To maximise A, maximise $A^2 = 1/4\ x^2(10x - x^2) = 1/4(10x^3 - x^4)$.

Differentiate: $d(A^2)/dx = 1/4(30x^2 - 4x^3) = x^2(15 - 2x)/2$.

So the maximum occurs at $x = 15/2$.

Then $y = \sqrt{(10 \cdot 15/2 - (15/2)^2)} = \sqrt{(75/4)} = 5\sqrt{3}/2$.

Hence $A = 1/2 \cdot 15/2 \cdot 5\sqrt{3}/2 =$ **$75\sqrt{3}/8$**.

M2-11

Inequalities

Answer **B**

Question 11

What is the complete set of real values of x for which $x^2(x^2 + 4) < 21$?

A $-2 < x < 2$

B $-\sqrt{3} < x < \sqrt{3}$

C $x < -\sqrt{3}$ or $x > \sqrt{3}$

D $-\sqrt{7} < x < \sqrt{7}$

E $-1 < x < 1$

WORKED SOLUTION

Let $u = x^2$, so $u \geq 0$. Then the inequality becomes $u(u+4) < 21$, i.e. $u^2 + 4u - 21 < 0$.

Factor: $(u+7)(u-3) < 0$, so $-7 < u < 3$.

Since $u = x^2 \geq 0$, we get $0 \leq u < 3$.

Thus $x^2 < 3$, so $\mathbf{-\sqrt{3} < x < \sqrt{3}}$.

M2-12

Similar solids

Answer **B**

Question 12

Four mathematically similar solids W, X, Y and Z satisfy the following:

• the ratio of lengths W:X is 1:2

• the ratio of surface areas X:Y is 4:9

• the ratio of volumes Y:Z is 8:27

What is the ratio of the volumes of W:Z?

A 1:81

B 8:729

C 4:243

D 8:243

E 16:729

WORKED SOLUTION

Length ratio W:X = 1:2.

Surface area ratio X:Y = 4:9, so length ratio X:Y = 2:3.

Volume ratio Y:Z = 8:27, so length ratio Y:Z = 2:3.

Hence the overall length ratio W:Z = 1:2 × 2:3 × 2:3 = 2:9.

Volumes scale as the cube of length, so W:Z volume ratio = $2^3:9^3$ = **8:729**.

M2-13

Logarithms

Answer C

Question 13

What is the mean of $\log_2 64$, $\log_4 64$, and $\log_8 64$?

A 3

B $10/3$

C $11/3$

D 4

E $13/3$

WORKED SOLUTION

$\log_2 64 = 6$, $\log_4 64 = 3$, and $\log_8 64 = 2$.

So the mean is $(6 + 3 + 2)/3 =$ **11/3**.

M2-14

Trigonometric comparison

Answer B

Question 14

Which of the following is largest in value? (All angles are in radians.)

A sin 1

B cos 0.5

C tan 0.5

D sin 0.5

E cos 1

WORKED SOLUTION

On $0 < \theta < \pi/2$, $\sin\theta$ and $\tan\theta$ increase, while $\cos\theta$ decreases.

Now $\sin 1 = \cos(\pi/2 - 1) = \cos(0.5708\ldots)$. Since $0.5 < 0.5708\ldots$, and cosine decreases on this interval, $\cos 0.5 > \cos 0.5708\ldots = \sin 1$.

Also $\tan 0.5 < 1$ and in fact $\tan 0.5 < \cos 0.5$ numerically, while $\sin 0.5$ and $\cos 1$ are smaller still.

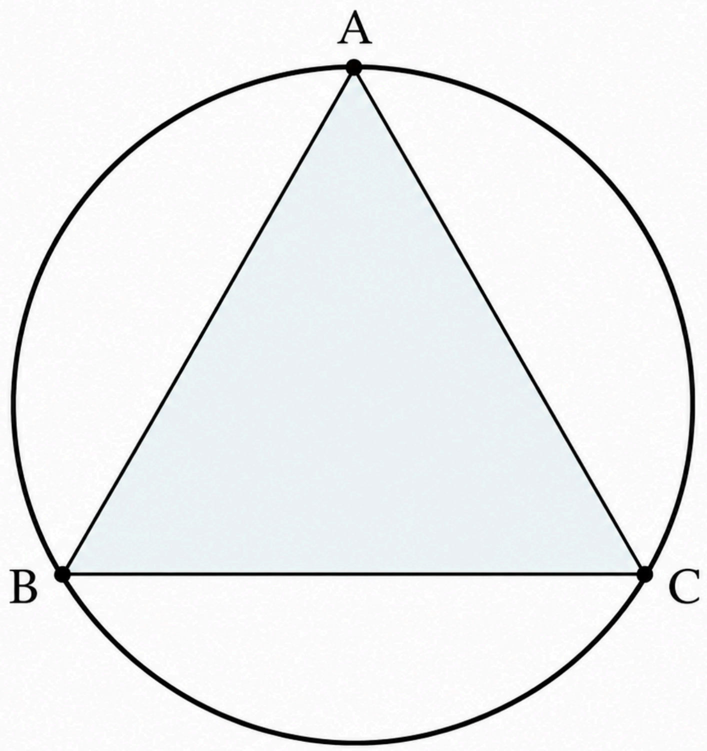
So the largest value is **cos 0.5**.

M2-15

Circle geometry / area

Answer **B**

Question 15



An equilateral triangle of side 6 cm is inscribed in a circle. What is the area inside the circle but outside the triangle?

- **A** $12\pi - 6\sqrt{3}$
- **B** $12\pi - 9\sqrt{3}$
- **C** $6\pi - 9\sqrt{3}$
- **D** $4\pi - 9\sqrt{3}$
- **E** $12\pi - 18\sqrt{3}$

WORKED SOLUTION

For an equilateral triangle of side s, the circumradius is $R = s/\sqrt{3}$.

So here $R = 6/\sqrt{3} = 2\sqrt{3}$.

Circle area $= \pi R^2 = \pi(2\sqrt{3})^2 = 12\pi$.

Triangle area $= (\sqrt{3}/4)s^2 = (\sqrt{3}/4)\cdot 36 = 9\sqrt{3}$.

Required area $= 12\pi - 9\sqrt{3}$, so the answer is $\mathbf{12\pi - 9\sqrt{3}}$.

M2-16

Sequences and series

Answer C

Question 16

An arithmetic sequence has first term a and common difference a. If S_n is the sum of the first n terms, which of the following is equal to $S_{2n} - 3S_n$?

- **A** $an(n+1)/2$
- **B** $an(2n-1)/2$
- **C** $an(n-1)/2$
- **D** $a(n-1)(n+1)$
- **E** an^2

WORKED SOLUTION

The terms are a, 2a, 3a, …, na, so $S_n = a(1 + 2 + \ldots + n) = an(n+1)/2$.

Then $S_{2n} = a\cdot 2n(2n+1)/2 = an(2n+1)$.

So $S_{2n} - 3S_n = an(2n+1) - 3[an(n+1)/2]$

$= [2an(2n+1) - 3an(n+1)]/2 = an(n-1)/2$.

Thus the correct expression is **$an(n-1)/2$**.

M2-17

Exponential equations

Answer C

Question 17

The equation $2 \cdot 5^{2x} - 7 \cdot 5^{x} + 3 = 0$ has two real solutions. What is the sum of these two solutions?

A $\log_5 3$

B $\log_5(1/2)$

C $\log_5(3/2)$

D $\log_5 6$

E $\log_5(2/3)$

WORKED SOLUTION

Let $u = 5^x$. Then $2u^2 - 7u + 3 = 0$, which factors as $(2u-1)(u-3)=0$.

So $u = 1/2$ or $u = 3$.

Hence $x = \log_5(1/2)$ or $x = \log_5 3$.

The sum is $\log_5(1/2) + \log_5 3 = \log_5(3/2)$.

Therefore the answer is **$\log_5(3/2)$**.

M2-18

Coordinate geometry

Answer **D**

Question 18

The line L with equation $y = mx + c$, where $m > 0$ and $c \geq 0$, passes through the point (2,4). A line is drawn through (2,4) perpendicular to L. The triangle enclosed between the two lines and the y-axis has area 5 square units. What is the larger possible value of m?

- **A** ½
- **B** 1
- **C** 3⁄2
- **D** 2
- **E** 5⁄2

WORKED SOLUTION

Since L passes through (2,4), its equation is $y = mx + (4 - 2m)$. So its y-intercept is $4 - 2m$.
The perpendicular line through (2,4) has slope $-1/m$, so its equation is $y - 4 = -(1/m)(x - 2)$.
Its y-intercept is $4 + 2/m$.
The base of the triangle on the y-axis has length $(4 + 2/m) - (4 - 2m) = 2m + 2/m$.
The horizontal distance from (2,4) to the y-axis is 2, so
Area $= 1/2 \times (2m + 2/m) \times 2 = 2(m + 1/m)$.
Set this equal to 5:
$2(m + 1/m) = 5 \Rightarrow 2m^2 - 5m + 2 = 0 \Rightarrow (2m-1)(m-2)=0$.
Thus $m = 1/2$ or 2, and the larger value is **2**.

M2-19

Algebraic structure

Answer **D**

Question 19

How many distinct real solutions does the equation $(x^2 - 2x - 3)^2 = x^2 - 2x - 3$ have?

A 1

B 2

C 3

D 4

E 5

WORKED SOLUTION

Let $u = x^2 - 2x - 3$. Then the equation becomes $u^2 = u$, so $u(u-1)=0$.

Hence $u = 0$ or $u = 1$.

If $u = 0$, then $x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1)=0 \Rightarrow x = 3$ or -1.

If $u = 1$, then $x^2 - 2x - 4 = 0 \Rightarrow x = 1 \pm \sqrt{5}$.

These are four distinct real solutions, so the answer is **4**.

M2-20

Quadratics

Answer C

Question 20

The curve $y = x^2 - x - 6$ intersects the x-axis at the points A and B, and has a minimum at the point C. A rectangle ABDE has two of its vertices at A and B, and C lies on the edge DE, between D and E. What is the area of the rectangle?

A 25

B $\frac{125}{8}$

C $\frac{125}{4}$

D 35

E 45

WORKED SOLUTION

The roots of $x^2 - x - 6 = 0$ are $x = 3$ and $x = -2$, so $AB = 5$.
The x-coordinate of the minimum is $x = -b/(2a) = 1/2$.
Then y at C is $(1/2)^2 - 1/2 - 6 = 1/4 - 1/2 - 6 = -25/4$.
Since C lies on the top edge DE of the rectangle while A and B lie on the x-axis, the height of the rectangle is $25/4$.
Area = base × height = $5 \times 25/4$ = **125/4**.

M2-21

Trigonometric equations

Answer C

Question 21

It is given that $9\cos x + \tan x \cdot \sin x = 6$, where $0° < x < 90°$. What are the possible values of $\tan x$?

A $\frac{1}{2}$ or $\frac{1}{4}$

B $1/\sqrt{3}$ or $1/\sqrt{15}$

C $\sqrt{3}$ or $\sqrt{15}$

D 2 or 4

E $\sqrt{3}$ or $2\sqrt{2}$

WORKED SOLUTION

Let $c = \cos x$. Then $\tan x \cdot \sin x = \sin^2 x / \cos x = (1-c^2)/c$.

So the equation becomes $9c + (1-c^2)/c = 6$.

Multiply by c: $9c^2 + 1 - c^2 = 6c$, so $8c^2 - 6c + 1 = 0$.

Factor: $(4c-1)(2c-1)=0$, giving $c = 1/4$ or $c = 1/2$.

If $\cos x = 1/2$, then $\tan x = \sqrt{3}$.

If $\cos x = 1/4$, then $\sin x = \sqrt{(1-1/16)} = \sqrt{15}/4$, so $\tan x = (\sqrt{15}/4)/(1/4) = \sqrt{15}$.

Therefore the possible values are **$\sqrt{3}$ or $\sqrt{15}$**.

M2-22

Distance from point to line

Answer B

Question 22

A straight line passes through the points $(0, 3a)$ and $(2a, 0)$, where a is a positive constant. What is the perpendicular distance of the point $P(a, 4a)$ from this line?

A $5a/\sqrt{13}$

B $5\sqrt{13}\,a/13$

C $2\sqrt{13}\,a/13$

D $4\sqrt{13}\,a/13$

E $3\sqrt{13}\,a/13$

WORKED SOLUTION

The line through $(0,3a)$ and $(2a,0)$ has intercept form $x/(2a) + y/(3a) = 1$, so $3x + 2y - 6a = 0$.

The perpendicular distance from $(a,4a)$ to this line is

$|3a + 2(4a) - 6a| / \sqrt{(3^2 + 2^2)} = |5a|/\sqrt{13} = 5a/\sqrt{13} =$ **$5\sqrt{13}\,a/13$**.

M2-23

Intersections of curves

Answer C

Question 23

At how many distinct points do the curves $y = (x - 3)(x^2 - 4x - 5)$ and $y = -x^2 + 6x - 9$ meet?

A 1

B 2

C 3

D 4

E 5

WORKED SOLUTION

Set the equations equal:

$(x - 3)(x^2 - 4x - 5) = -x^2 + 6x - 9.$

Bring everything to one side:

$(x - 3)(x^2 - 4x - 5) + x^2 - 6x + 9 = 0.$

Since $x^2 - 6x + 9 = (x-3)^2$, factor out $(x-3)$:

$(x-3)[x^2 - 4x - 5 + (x-3)] = (x-3)(x^2 - 3x - 8)=0.$

So $x = 3$, or $x = (3 \pm \sqrt{41})/2$.

These are three distinct x-values, so the curves meet at **3** distinct points.

M2-24

Geometric progressions

Answer C

Question 24

A geometric progression has first term $u_1 = a$ and common ratio r. The sum to infinity is $\frac{8}{5}$, and the sum to infinity of the even-numbered terms ($u_2 + u_4 + u_6 + \cdots$) is $\frac{3}{5}$. What is the value of a + r?

A $\frac{21}{25}$

B $\frac{6}{5}$

C $\frac{31}{25}$

D $\frac{8}{5}$

E $\frac{41}{25}$

WORKED SOLUTION

For the whole GP, $a/(1-r) = 8/5$.

The even-numbered terms form a GP with first term ar and ratio r^2, so $ar/(1-r^2) = 3/5$.

But $a = (8/5)(1-r)$. Substitute into the second equation:

$[(8/5)(1-r)r] / [(1-r)(1+r)] = 3/5$.

This simplifies to $(8r/5)/(1+r) = 3/5$.

So $8r = 3 + 3r$, hence $5r = 3$ and $r = 3/5$.

Then $a = (8/5)(1-3/5) = (8/5)(2/5) = 16/25$.

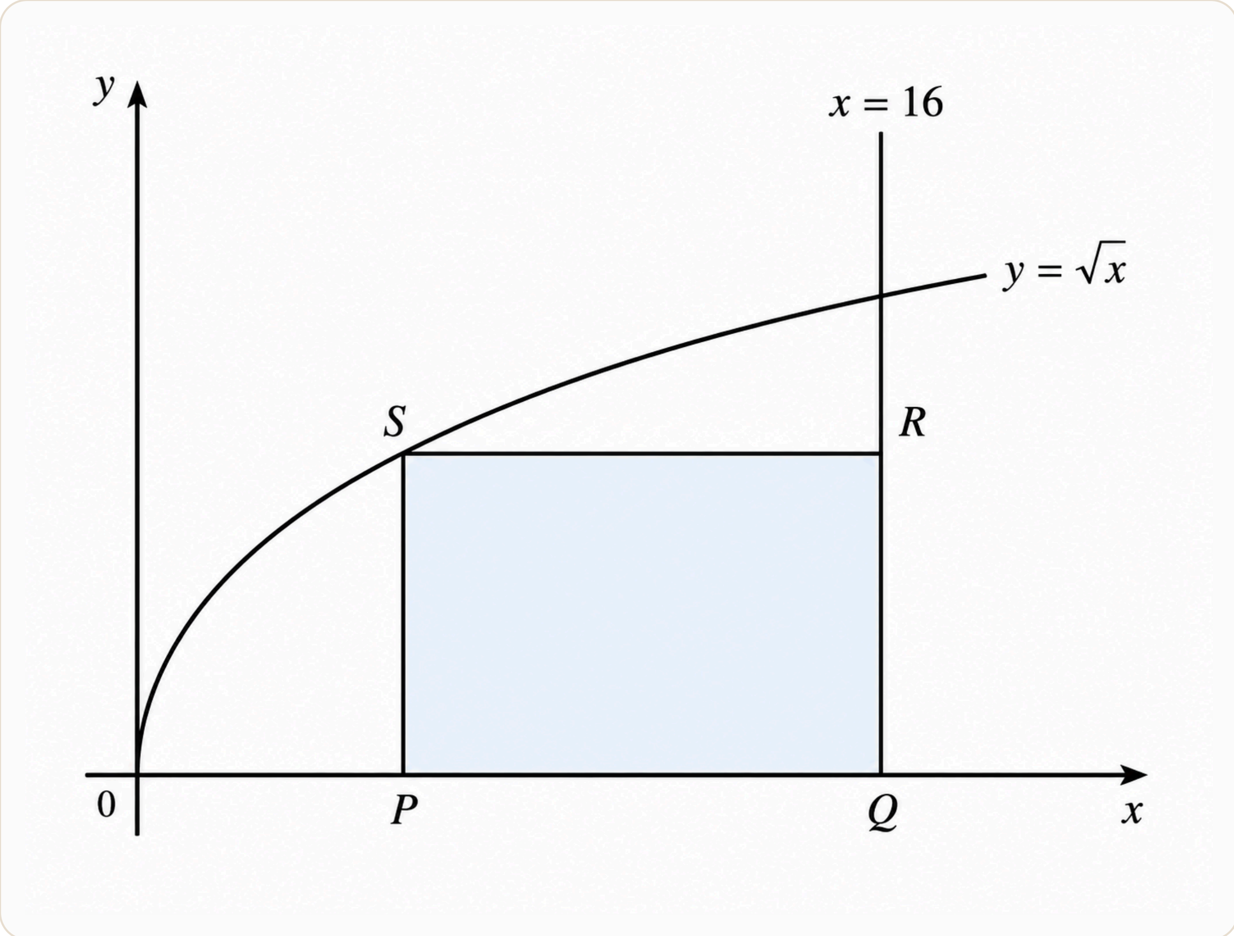
Therefore $a + r = 16/25 + 15/25 =$ **31/25**.

M2-25

Optimisation with curves

Answer B

Question 25



PQRS is a rectangle. P and Q lie on the x-axis, Q and R lie on the line $x = 16$, and S lies on the curve $y = \sqrt{x}$. What is the maximum possible area of the rectangle?

- **A** $64\sqrt{3}/9$
- **B** $128\sqrt{3}/9$
- **C** $64\sqrt{3}/3$
- **D** $128\sqrt{3}/3$
- **E** $32\sqrt{3}$

WORKED SOLUTION

Let $S = (x, \sqrt{x})$. Then $P = (x,0)$, $Q = (16,0)$, and $R = (16,\sqrt{x})$.

The width of the rectangle is $16-x$ and its height is $\sqrt{x}$, so

$A = (16-x)\sqrt{x}$.

Let $t = \sqrt{x}$, so $x = t^2$. Then $A = (16-t^2)t = 16t - t^3$.

Differentiate: $dA/dt = 16 - 3t^2$.

Set to zero: $16 - 3t^2 = 0 \Rightarrow t = 4/\sqrt{3}$.

Then $A_{max} = 16\cdot(4/\sqrt{3}) - (4/\sqrt{3})^3 = 64/\sqrt{3} - 64/(3\sqrt{3}) = 128/(3\sqrt{3}) = \mathbf{128\sqrt{3}/9}$.

M2-26

Maximum gradient

Answer **E**

Question 26

Find the maximum value of the gradient of the curve $y = 1 - 2x + 4x^{3/2} - x^2$, where $x > 0$.

A ½

B 1

C 3⁄2

D 2

E 5⁄2

WORKED SOLUTION

The gradient is $dy/dx = -2 + 6\sqrt{x} - 2x$.

Let $t = \sqrt{x}$, where $t > 0$. Then the gradient becomes

$g(t) = -2 + 6t - 2t^2 = -2(t^2 - 3t + 1)$.

This is a downward-opening quadratic, so its maximum occurs at $t = -b/(2a) = 3/2$.

Then $g_{max} = -2 + 6\cdot(3/2) - 2\cdot(9/4) = -2 + 9 - 9/2 =$ **5/2**.

M2-27

Trigonometry in triangles

Answer **D**

Question 27

The internal angles of a triangle are α, β and θ, and
$3 \tan \alpha - 2 \sin \beta = 2$,
$5 \tan \alpha + 6 \sin \beta = 8$.
What is the value of θ in degrees?

- **A** 15
- **B** 45
- **C** 75
- **D** 105
- **E** 135

WORKED SOLUTION

Let $u = \tan \alpha$ and $v = \sin \beta$. Then we have
$3u - 2v = 2$ and $5u + 6v = 8$.
Multiply the first by 3: $9u - 6v = 6$.
Add to the second: $14u = 14$, so $u = 1$.
Then $3(1) - 2v = 2$, so $v = 1/2$.
Hence $\tan \alpha = 1$. Since α is an internal angle of a triangle, $\alpha = 45°$.
Also $\sin \beta = 1/2$, so an internal angle β could initially be 30° or 150°. But $\beta = 150°$ is impossible because $\alpha + \beta$ would then be 195°, greater than 180°. Thus $\beta = 30°$.
Therefore $\theta = 180° - 45° - 30° =$ **105°**.