TMUA MOCK FULL TEST 1
Solution Book

- All Topics

ThrivingScholars

Paper 1

Question 1

A square has centre $(3, 4)$ and one corner at $(1, 5)$. Another corner is at

A $(1, 3)$

B $(5, 5)$

C $(2, 2)$

D $(5, 2)$

E $(4, 2)$

Answer C

1. The centre is at $(3, 4)$ and one corner is at $(1, 5)$, which is a displacement of $(-2, 1)$. There's another corner opposite, at $(3, 4) - (-2, 1) = (5, 3)$, but this is not one of the options. To find the other corners, we need to find a vector with equal magnitude, but at right-angles to $(-2, 1)$. Rotating by $90°$ gives $(1, 2)$. The other corners are at $(3, 4) \pm (1, 2)$, which includes $(2, 2)$. The answer is (2,2).

Question 2

What is the value of $\int_0^1 (e^x - x)(e^x + x)\, \mathrm{d}x$?

A $\dfrac{3e^2 - 5}{6}$

B $\dfrac{e^2 + 3}{6}$

C $\dfrac{3e^2 - 2}{6}$

D $\dfrac{2e^2 - 3}{6}$

E $\dfrac{3e^2 + 2}{6}$

Answer A

Using the difference of two squares, this is $\int_0^1 e^{2x} - x^2\, \mathrm{d}x$. Notice that, since the derivative of e^{2x} is $2e^{2x}$, this integral is

$$\left[\frac{1}{2}e^{2x} - \frac{x^3}{3}\right]_0^1 = \frac{e^2}{2} - \frac{1}{3} - \frac{1}{2} = \frac{3e^2 - 5}{6}$$

The answer is $\dfrac{3e^2 - 5}{6}$.

Question 3

A regular dodecagon is a 12-sided polygon with all sides the same length and all internal angles equal. If I construct a regular dodecagon by connecting 12 equally-spaced points on a circle of radius 1, then the area of this polygon is

A 3

B $3\sqrt{3}$

C $3\sqrt{2}$

D $2\sqrt{2}$

E $6+3\sqrt{3}$

Answer A

Connect each point to the centre of the circle to split the shape into 12 isosceles triangles, each with angle 30° at the centre. Then the area of each triangle is $\frac{1}{2} \times 1 \times 1 \times \sin 30^\circ = \frac{1}{4}$, and there are 12 triangles, for a total area of 3.
The answer is 3.

Question 4

The positive number a satisfies

$$\int_0^a \left(\sqrt{x} + x^2\right)\, \mathrm{d}x = 5$$

if

A $a = \sqrt{3}$

B $a = \left(\sqrt{21} - 1\right)^{1/3}$

C $a = 3^{2/3}$

D $a = 5^{2/3}$

E $a = (\sqrt{6} - 1)^{2/3}$

Answer C

The integral is

$$\int_0^a \sqrt{x} + x^2\, \mathrm{d}x = \int_0^a x^{1/2} + x^2\, \mathrm{d}x = \left[\frac{2}{3}x^{3/2} + \frac{1}{3}x^3\right]_0^a = \frac{2}{3}a^{3/2} + \frac{1}{3}a^3.$$

So we have $2a^{3/2} + a^3 = 15$. This factorises as $(a^{3/2} - 3)(a^{3/2} + 5) = 0$. Since $a > 0$ we want $a^{3/2} > 0$, so it's 3, so $a = 3^{2/3}$.

The answer is $a = 3^{2/3}$.

Question 5

Tangents to $y = e^x$ are drawn at (p, e^p) and (q, e^q). These tangents cross the x-axis at a and b respectively. It follows that, for all p and q,

A $pa = qb$

B $p - a = q - b$

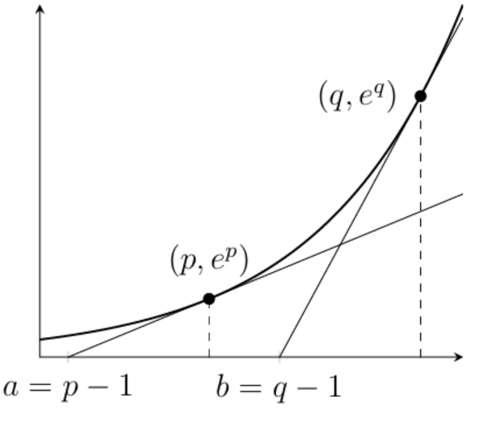
C $p - a < q - b$

D $p + q = a + b$

E $p - a > q - b$

Answer B

The gradient at p is e^p and so the tangent is $y = e^p(x - p) + e^p$. This crosses the x-axis when $e^p(a-p)+e^p = 0$ which happens if $a = p-1$. Similarly $q = 1-b$ so $p-a = q-b$ (they're both 1).



The answer is $p - a = q - b$.

Question 6

How many real solutions x are there to the equation $x|x| + 1 = 3|x|$?

[Note that $|x|$ is equal to x if $x \geq 0$, and equal to $-x$ otherwise.]

A 1

B 3

C 4

D 0

E 2

Answer B

Either $x \geq 0$ and $x^2 + 1 = 3x$ or $x < 0$ and $-x^2 + 1 = -3x$. The first equation has solutions $\frac{1}{2}\left(3 \pm \sqrt{5}\right)$ which are both positive. The second equation has solutions $\frac{1}{2}\left(3 \pm \sqrt{13}\right)$ and only one of these is actually negative. So there are three solutions in total.
The answer is 3.

Question 7

If the expression $\left(x+1+\frac{1}{x}\right)^4$ is fully expanded term-by-term and like terms are collected together, there is one term which is independent of x. The value of this term is

A 51

B 81

C 10

D 19

E 14

Answer D

We'll calculate the square of $(x+1+x^{-1})$ first;

$$x^2+2x+3+2x^{-1}+x^{-2}$$

(I drew a square grid to keep track of all the cross-terms, and there's a nice pattern which helps). Now if we were to square this expression, the constant term independent of x would be

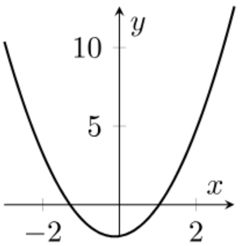
$$2(x^2)(x^{-2})+2(2x)(2x^{-1})+3^2$$

Most of the terms have a factor of 2 because they occur in either order. This sum is 2+8+9 = 19. The answer is 19.

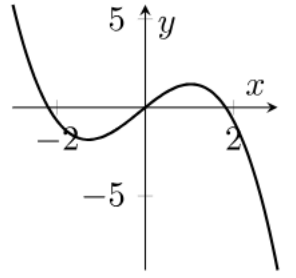
Question 8

The following five graphs are, in some order, plots of $y = f(x)$, $y = g(x)$, $y = h(x)$, $y = \frac{\mathrm{d}f}{\mathrm{d}x}$ and $y = \frac{\mathrm{d}g}{\mathrm{d}x}$; that is, three unknown functions and the derivatives of the first two of those functions. Which graph is a plot of $h(x)$?

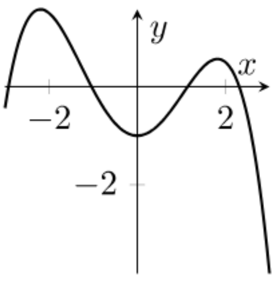
A



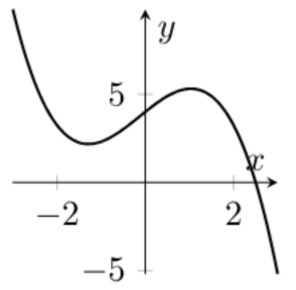
D



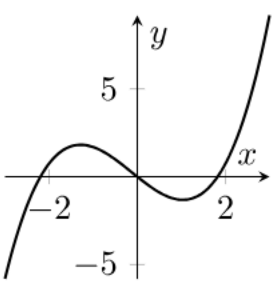
B



E

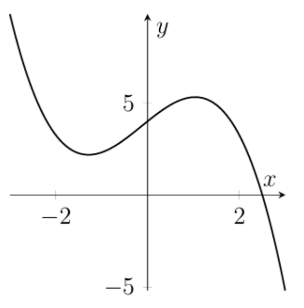


C



One of the options looks like a quadratic, three of the options look like cubics, and one of the options looks like a quartic (a polynomial of degree 4), so we might expect that one of the cubics is the derivative of the quartic, and the quadratic is the derivative of another of the cubics, leaving one remaining cubic that could be $h(x)$. Checking the sign and the locations of the zeros of the possible derivatives leaves one function over as a possible candidate for $h(x)$;

Answer E



We should check that it's definitely not the derivative of any of the other options, and that its derivative is not any of the other options. Note that it's only got one zero, and it's negative for large x. This is enough to see that it's not the derivative of any of the other functions. It's also not the case that the derivative of this graph is any of the other graphs; such a graph would have two zeros for the two turning points, but only of the functions has two zeros, and the sign of that graph is wrong (positive where the gradient of this graph is negative). So we conclude that the graph above is the "odd one out".
The answer is the graph above.

Question 9

Let s_k denote the sum of the kth powers of the roots of the polynomial $x^3 - 5x^2 + 8x - 13$. In particular, $s_0 = 3$, $s_1 = 5$, and $s_2 = 9$. Let a, b, and c be real numbers such that $s_{k+1} = a\, s_k + b\, s_{k-1} + c\, s_{k-2}$ for $k = 2$, 3, ... What is $a + b + c$?

$\textbf{(A)}\ -6 \qquad \textbf{(B)}\ 0 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 10 \qquad \textbf{(E)}\ 26$

Answer D

Let p, q, and r be the roots of the polynomial. Then,

$p^3 - 5p^2 + 8p - 13 = 0$

$q^3 - 5q^2 + 8q - 13 = 0$

$r^3 - 5r^2 + 8r - 13 = 0$

Adding these three equations, we get

$(p^3 + q^3 + r^3) - 5(p^2 + q^2 + r^2) + 8(p + q + r) - 39 = 0$

$s_3 - 5s_2 + 8s_1 = 39$

39 can be written as $13s_0$, giving

$s_3 = 5s_2 - 8s_1 + 13s_0$

We are given that $s_{k+1} = a\, s_k + b\, s_{k-1} + c\, s_{k-2}$ is satisfied for $k = 2$, 3, ..., meaning it must be satisfied when $k = 2$, giving us $s_3 = a\, s_2 + b\, s_1 + c\, s_0$.

Therefore, $a = 5, b = -8$, and $c = 13$ by matching coefficients.

$5 - 8 + 13 = \boxed{\textbf{(D)}\ 10}$.

Question 10

In $\triangle ABC$ with integer side lengths, $\cos A = \frac{11}{16}$, $\cos B = \frac{7}{8}$, and $\cos C = -\frac{1}{4}$. What is the least possible perimeter for $\triangle ABC$?

$\textbf{(A) } 9 \qquad \textbf{(B) } 12 \qquad \textbf{(C) } 23 \qquad \textbf{(D) } 27 \qquad \textbf{(E) } 44$

Answer A

Solution 1

Notice that by the Law of Sines, $a : b : c = \sin A : \sin B : \sin C$, so let's flip all the cosines using $\sin^2 x + \cos^2 x = 1$ ($\sin x$ is positive for $0^\circ < x < 180^\circ$, so we're good there).

$$\sin A = \frac{3\sqrt{15}}{16}, \qquad \sin B = \frac{\sqrt{15}}{8}, \qquad \text{and} \qquad \sin C = \frac{\sqrt{15}}{4}$$

These are in the ratio $3 : 2 : 4$, so our minimal triangle has side lengths 2, 3, and 4. $\boxed{\textbf{(A) } 9}$ is our answer.

Solution 2

$\angle ACB$ is obtuse since its cosine is negative, so we let the foot of the altitude from C to AB be H. Let $AH = 11x$, $AC = 16x$, $BH = 7y$, and $BC = 8y$. By the Pythagorean Theorem, $CH = \sqrt{256x^2 - 121x^2} = 3x\sqrt{15}$ and $CH = \sqrt{64y^2 - 49y^2} = y\sqrt{15}$. Thus, $y = 3x$. The sides of the triangle are then $16x$, $11x + 7(3x) = 32x$, and $24x$, so for some integers a, b, $16x = a$ and $24x = b$, where a and b are minimal. Hence, $\frac{a}{16} = \frac{b}{24}$, or $3a = 2b$. Thus the smallest possible positive integers a and b that satisfy this are $a = 2$ and $b = 3$, so $x = \frac{1}{8}$. The sides of the triangle are 2, 3, and 4, so $\boxed{\textbf{(A) } 9}$ is our answer.

Question 11

Define binary operations $\diamondsuit$ and $\heartsuit$ by

$$a \diamondsuit b = a^{\log_7(b)} \qquad \text{and} \qquad a \heartsuit b = a^{\frac{1}{\log_7(b)}}$$

for all real numbers a and b for which these expressions are defined. The sequence (a_n) is defined recursively by $a_3 = 3 \heartsuit 2$ and

$$a_n = (n \heartsuit (n-1)) \diamondsuit a_{n-1}$$

for all integers $n \geq 4$. To the nearest integer, what is $\log_7(a_{2019})$?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

Answer D

First note that by log properties $a \diamondsuit b = 7^{(\log_7 a)(\log_7 b)}$ and $a \heartsuit b = 7^{\frac{\log_7 a}{\log_7 b}} = 7^{\log_b a}$.

Now, define $b_n = \log_7(a_n)$. Thus $b_3 = \log_7(3 \heartsuit 2) = \log_7(7^{\log_2 3}) = \log_2 3$.

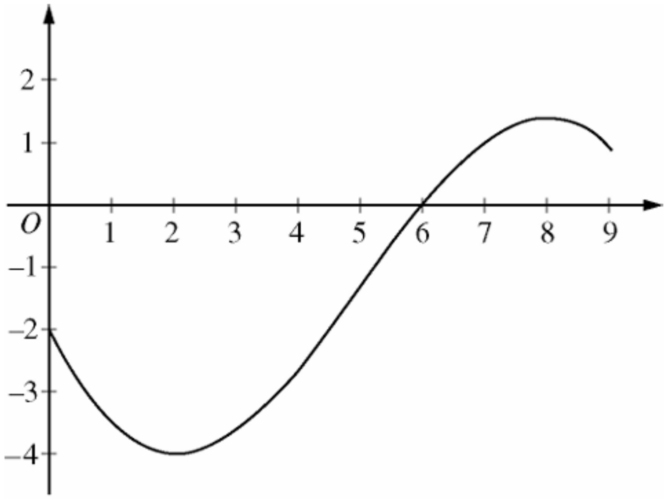
Taking logs of both sides of the recursion and using the definition of $\diamondsuit$ gives $\log_7(a_n) = \log_7(7^{\log_7 n \heartsuit (n-1) \log_7 a_{n-1}})$.

The logs and the exponent cancel to $\log_7((n \heartsuit (n-1))^{\log_7(a_{n-1})})$, and by the definition of $\heartsuit$, ths is $\log_7(7^{(\log_{n-1} n) \log_7(a_{n-1})})$, which quickly simplifies to $\log_7(a_{n-1}) \log_{n-1} n = b_{n-1} \log_{n-1} n$.

Thus $b_n = b_{n-1} \log_{n-1} n$. From this, we have $b_4 = b_3 \log_3 4 = \log_2 3 \log_3 4 = \log_2 4$, $b_5 = \log_4 5 \log_2 4 = \log_2 5$, and in general, $b_n = \log_2 n$.

Finally, $\log_7(a_{2019}) = b_{2019} = \log_2 2019$. Since $2^{11} = 2048$ and 2019 is slightly less than 2048, $\log_2 2019 \approx \boxed{\text{(D) } 11}$.

Question 12



Graph of f

The graph of a differentiable function f is shown above. If $h(x) = \int_0^x f(t)\,dt$, which of the following is true?

(A) $h(6) < h'(6) < h''(6)$

(B) $h(6) < h''(6) < h'(6)$

(C) $h'(6) < h(6) < h''(6)$

(D) $h''(6) < h(6) < h'(6)$

(E) $h''(6) < h'(6) < h(6)$

Answer A

Let $h(x) = \int_0^x f(t)\,dt$. Then by FTC:

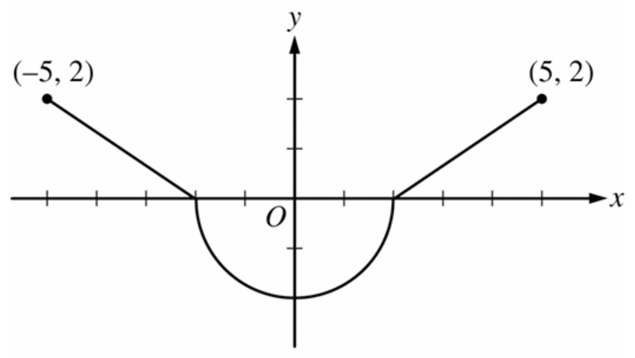
$$h'(x) = f(x), \qquad h''(x) = f'(x).$$

From the graph:

- For $0 < x < 6$, $f(x) < 0$. Hence the net area from 0 to 6 is negative, so $h(6) < 0$.
- The curve crosses the x-axis at $x = 6$, so $f(6) = 0 \Rightarrow h'(6) = 0$.
- At $x = 6$ the graph is increasing (positive slope), so $f'(6) > 0 \Rightarrow h''(6) > 0$.

Thus $h(6) < h'(6) < h''(6)$, which is choice **A**.

Question 13



Graph of f'

The graph of f', the derivative of a function f, consists of two line segments and a semicircle, as shown in the figure above. If $f(2) = 1$, then $f(-5) =$

(A) $2\pi - 2$

(B) $2\pi - 3$

(C) $2\pi - 5$

(D) $6 - 2\pi$

(E) $4 - 2\pi$

Answer A

By FTC,

$$f(2) - f(-5) = \int_{-5}^{2} f'(x)\,dx$$

which is the **signed area** under the graph of f' from -5 to 2.

From the figure:

- On $[-5, -2]$, f' is a line segment above the x-axis, forming a triangle with base 3 and height 2.
  Area $= \frac{1}{2} \cdot 3 \cdot 2 = 3$ (positive).
- On $[-2, 2]$, f' is a semicircle of radius 2 **below** the axis.
  Area magnitude $= \frac{1}{2}\pi(2)^2 = 2\pi$, so signed area $= -2\pi$.

Thus,

$$\int_{-5}^{2} f'(x)\,dx = 3 - 2\pi.$$

Given $f(2) = 1$,

$$f(-5) = f(2) - \big(3 - 2\pi\big) = 1 - 3 + 2\pi = 2\pi - 2.$$

So the correct choice is **(A)** $2\pi - 2$.

Question 14

The function f is defined by $f(x) = \frac{x}{x+2}$. What points (x, y) on the graph of f have the property that the line tangent to f at (x, y) has slope $\frac{1}{2}$?

(A) $(0,0)$ only

(B) $\left(\frac{1}{2}, \frac{1}{5}\right)$ only

(C) $(0,0)$ and $(-4,2)$

(D) $(0,0)$ and $\left(4, \frac{2}{3}\right)$

(E) There are no such points.

Answer C $(0,0)$ and $(-4,2)$

$$f(x) = \frac{x}{x+2} \Rightarrow f'(x) = \frac{(x+2)\cdot 1 - x\cdot 1}{(x+2)^2} = \frac{2}{(x+2)^2}.$$

Set the slope equal to $\frac{1}{2}$:

$$\frac{2}{(x+2)^2} = \frac{1}{2} \Rightarrow (x+2)^2 = 4 \Rightarrow x = 0 \text{ or } x = -4.$$

Points on the graph:

$$f(0) = 0 \Rightarrow (0,0), \qquad f(-4) = \frac{-4}{-2} = 2 \Rightarrow (-4,2).$$

So the tangent has slope $\frac{1}{2}$ at $(0,0)$ and $(-4,2)$.

Question 15

A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$ Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

$\textbf{(A)}\ 2020 \qquad \textbf{(B)}\ 4039 \qquad \textbf{(C)}\ 6057 \qquad \textbf{(D)}\ 6061 \qquad \textbf{(E)}\ 8078$

Answer E

Using the recursive formula, we find $a_3 = \frac{3}{11}$, $a_4 = \frac{3}{15}$, and so on. It appears that $a_n = \frac{3}{4n-1}$, for all n. Setting $n = 2019$, we find $a_{2019} = \frac{3}{8075}$, so the answer is $\boxed{\textbf{(E)}\ 8078}$.

Question 16

Let g be a function with first derivative given by $g'(x) = \int_0^x e^{-t^2}\,dt$. Which of the following must be true on the interval $0 < x < 2$?

(A) g is increasing, and the graph of g is concave up.

(B) g is increasing, and the graph of g is concave down.

(C) g is decreasing, and the graph of g is concave up.

(D) g is decreasing, and the graph of g is concave down.

(E) g is decreasing, and the graph of g has a point of inflection on $0 < x < 2$.

Answer A

- For $x > 0$, the integrand $e^{-t^2} > 0$. Hence

$$g'(x) = \int_0^x e^{-t^2}\,dt > 0$$

  so g is increasing on $(0, 2)$.
- By the FTC,

$$g''(x) = \frac{d}{dx} g'(x) = e^{-x^2} > 0$$

  for all x. Therefore the graph of g is concave up on $(0, 2)$.

So g is increasing and concave up ⇒ choice (A).

Question 17

Positive real numbers $x \neq 1$ and $y \neq 1$ satisfy $\log_2 x = \log_y 16$ and $xy = 64$. What is $(\log_2 \tfrac{x}{y})^2$?

(A) $\frac{25}{2}$ **(B)** 20 **(C)** $\frac{45}{2}$ **(D)** 25 **(E)** 32

Answer B

Let $\log_2 x = \log_y 16 = k$, so that $2^k = x$ and $y^k = 16 \implies y = 2^{\frac{4}{k}}$. Then we have $(2^k)(2^{\frac{4}{k}}) = 2^{k+\frac{4}{k}} = 2^6$.

We therefore have $k + \dfrac{4}{k} = 6$, and deduce $k^2 - 6k + 4 = 0$. The solutions to this are $k = 3 \pm \sqrt{5}$.

To solve the problem, we now find

$$\begin{aligned}(\log_2 \tfrac{x}{y})^2 &= (\log_2 x - \log_2 y)^2 \\ &= (k - \tfrac{4}{k})^2 = (3 \pm \sqrt{5} - \tfrac{4}{3 \pm \sqrt{5}})^2 \\ &= (3 \pm \sqrt{5} - [3 \mp \sqrt{5}])^2 \\ &= (3 \pm \sqrt{5} - 3 \pm \sqrt{5})^2 \\ &= (\pm 2\sqrt{5})^2 \\ &= \boxed{\textbf{(B) } 20}.\end{aligned}$$

Question 18

How many solutions does the equation $\tan(2x) = \cos(\frac{x}{2})$ have on the interval $[0, 2\pi]$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer E

We count the intersections of the graphs of $y = \tan(2x)$ and $y = \cos\left(\frac{x}{2}\right)$:

1. The graph of $y = \tan(2x)$ has a period of $\frac{\pi}{2}$, asymptotes at $x = \frac{\pi}{4} + \frac{k\pi}{2}$, and zeros at $x = \frac{k\pi}{2}$ for some integer k.

   On the interval $[0, 2\pi]$, the graph has five branches:

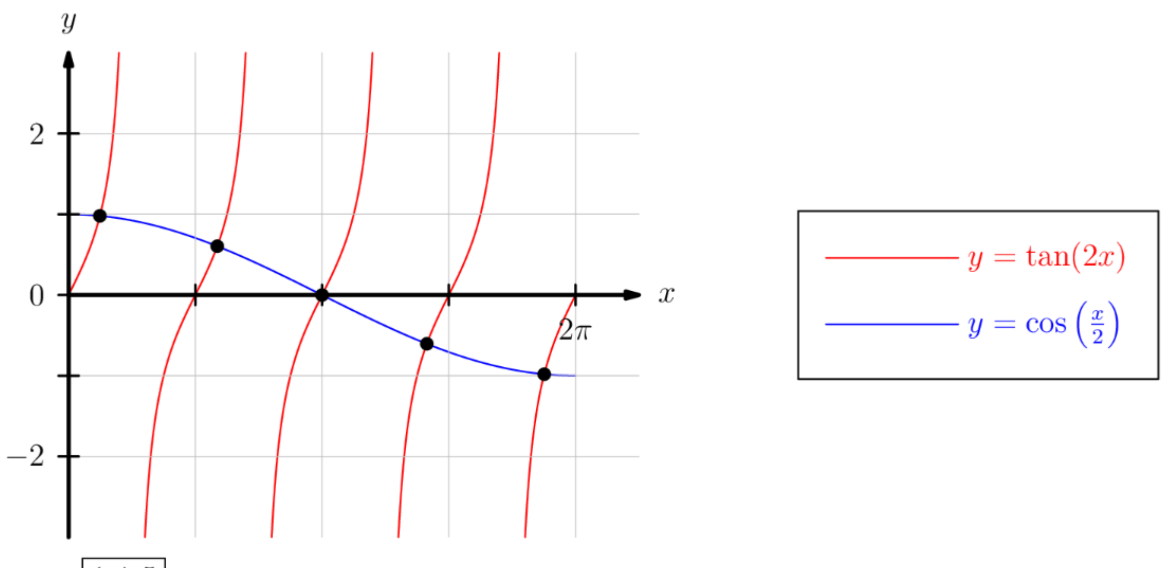
$$\left[0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right), \left(\frac{7\pi}{4}, 2\pi\right].$$

   Note that $\tan(2x) \in [0, \infty)$ for the first branch, $\tan(2x) \in (-\infty, \infty)$ for the three middle branches, and $\tan(2x) \in (-\infty, 0]$ for last branch. Moreover, all branches are strictly increasing.

2. The graph of $y = \cos\left(\frac{x}{2}\right)$ has a period of 4π and zeros at $x = \pi + 2k\pi$ for some integer k.

   On the interval $[0, 2\pi]$, note that $\cos\left(\frac{x}{2}\right) \in [-1, 1]$. Moreover, the graph is strictly decreasing.

The graphs of $y = \tan(2x)$ and $y = \cos\left(\frac{x}{2}\right)$ intersect once on each of the five branches of $y = \tan(2x)$, as shown below:



Therefore, the answer is **(E)** 5.

Question 19

What is the sum of all possible values of k for which the polynomials $x^2 - 3x + 2$ and $x^2 - 5x + k$ have a root in common?

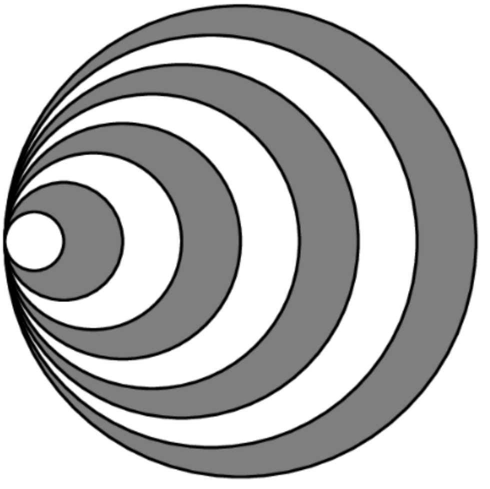
$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 5 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 10$

Answer E

We factor $x^2 - 3x + 2$ into $(x-1)(x-2)$. Thus, either 1 or 2 is a root of $x^2 - 5x + k$. If 1 is a root, then $1^2 - 5 \cdot 1 + k = 0$, so $k = 4$. If 2 is a root, then $2^2 - 5 \cdot 2 + k = 0$, so $k = 6$. The sum of all possible values of k is $\boxed{\textbf{(E)}\ 10}$.

Question 20

An even number of circles are nested, starting with a radius of 1 and increasing by 1 each time, all sharing a common point. The region between every other circle is shaded, starting with the region inside the circle of radius 2 but outside the circle of radius 1. An example showing 8 circles is displayed below. What is the least number of circles needed to make the total shaded area at least 2023π?



$\textbf{(A)}\ 46$ $\textbf{(B)}\ 48$ $\textbf{(C)}\ 56$ $\textbf{(D)}\ 60$ $\textbf{(E)}\ 64$

Answer E

Notice that the area of the shaded region is $(2^2\pi - 1^2\pi) + (4^2\pi - 3^2\pi) + (6^2\pi - 5^2\pi) + \cdots + (n^2\pi - (n-1)^2\pi)$ for any even number n. Using the difference of squares, this simplifies to $(1 + 2 + 3 + 4 + \cdots + n)\pi$. So, we are basically finding the smallest n such that $\frac{n(n+1)}{2} > 2023 \Rightarrow n(n+1) > 4046$. Since $60(61) > 60^2 = 3600$, the only option higher than 60 is $\boxed{\textbf{(E)}\ 64}$.

Paper 2

Question 1

A number is chosen at random from among the first 100 positive integers, and a positive integer divisor of that number is then chosen at random. What is the probability that the chosen divisor is divisible by 11?

$\textbf{(A)}\ \frac{4}{100} \qquad \textbf{(B)}\ \frac{9}{200} \qquad \textbf{(C)}\ \frac{1}{20} \qquad \textbf{(D)}\ \frac{11}{200} \qquad \textbf{(E)}\ \frac{3}{50}$

Answer B

In order for the divisor chosen to be a multiple of 11, the original number chosen must also be a multiple of 11. Among the first 100 positive integers, there are 9 multiples of 11; 11, 22, 33, 44, 55, 66, 77, 88, 99. We can now perform a little casework on the probability of choosing a divisor which is a multiple of 11 for each of these 9, and see that the probability is 1/2 for each. The probability of choosing these 9 multiples in the first place is $\frac{9}{100}$, so the final probability is $\frac{9}{100} \cdot \frac{1}{2} = \frac{9}{200}$, so the answer is $\boxed{\textbf{(B)}\ \frac{9}{200}}$.

$11 = 1, 11 \Rightarrow \frac{1}{2}$

$22 = 2 \times 11 : 1, 2, 11, 22 \Rightarrow \frac{1}{2}$

$33 = 3 \times 11 : 1, 3, 11, 33 \Rightarrow \frac{1}{2}$

$44 = 2^2 \times 11 : 1, 2, 4, 11, 22, 44 \Rightarrow \frac{1}{2}$

$55 = 5 \times 11 : 1, 5, 11, 55 \Rightarrow \frac{1}{2}$

$66 = 2 \times 3 \times 11 : 1, 2, 3, 6, 11, 22, 33, 66 \Rightarrow \frac{1}{2}$

$77 = 7 \times 11 : 1, 7, 11, 77 \Rightarrow \frac{1}{2}$

$88 = 2^3 \times 11 : 1, 2, 4, 8, 11, 22, 44, 88 \Rightarrow \frac{1}{2}$

$99 = 3^2 \times 11 : 1, 3, 9, 11, 33, 99 \Rightarrow \frac{1}{2}$

Question 2

Let $P(x)$ be the unique polynomial of minimal degree with the following properties:

- $P(x)$ has a leading coefficient 1,
- 1 is a root of $P(x)-1$,
- 2 is a root of $P(x-2)$,
- 3 is a root of $P(3x)$, and
- 4 is a root of $4P(x)$.

The roots of $P(x)$ are integers, with one exception. The root that is not an integer can be written as $\frac{m}{n}$, where m and n are relatively prime integers. What is $m+n$?

$\textbf{(A) } 41 \qquad \textbf{(B) } 43 \qquad \textbf{(C) } 45 \qquad \textbf{(D) } 47 \qquad \textbf{(E) } 49$

Answer D

From the problem statement, we find $P(2-2)=0$, $P(9)=0$ and $4P(4)=0$. Therefore, we know that 0, 9, and 4 are roots. So, we can factor $P(x)$ as $x(x-9)(x-4)(x-a)$, where a is the unknown root. Since $P(x)-1=0$, we plug in $x=1$ which gives $1(-8)(-3)(1-a)=1$, so $24(1-a)=1 \implies 1-a=1/24 \implies a=23/24$. Therefore, our answer is $23+24=\boxed{\textbf{(D) } 47}$

Question 3

How many three-digit positive integers N satisfy the following properties?

- The number N is divisible by 7.
- The number formed by reversing the digits of N is divisible by 5.

$\textbf{(A)}\ 13 \qquad \textbf{(B)}\ 14 \qquad \textbf{(C)}\ 15 \qquad \textbf{(D)}\ 16 \qquad \textbf{(E)}\ 17$

Answer B

Multiples of 5 will always end in 0 or 5, and since the numbers have to be a three-digit numbers, it cannot start with 0 (otherwise it would be a two-digit number), narrowing our choices to 3-digit numbers starting with 5. Since the numbers must be divisible by 7, all possibilities have to be in the range from $7 \cdot 72$ to $7 \cdot 85$ inclusive(504 to 595).

(Add 1 to include 72)

$85 - 72 + 1 = 14$. $\boxed{\textbf{(B)}\ 14}$.

You can also take 497 away from each of the numbers(removing the hundreds digit and adding three to each of the numbers), resulting in the numbers {7, 14, 21..., 84, 91, 98}. Dividing each of them by 7, you get the numbers {1, 2, 3..., 12, 13, 14}. Therefore, the answer is $\boxed{\textbf{(B)}\ \textbf{14}}$

Question 4

Let $f(x) = 3 - 2x$. A differentiable function g satisfies $g'(x) = f(x)f'(x)(x-3)$.
For this question, take "g is increasing at x" to mean $g'(x) \geq 0$.
Decide which of the following statements are true.

1. $x \leq \frac{3}{2}$ is a **sufficient** condition for g to be increasing at x.
2. $x \geq 3$ is a **necessary** condition for g to be increasing at x.
3. $x \leq \frac{3}{2}$ **or** $x \geq 3$ is a **necessary and sufficient** condition for g to be increasing at x.

How many of the statements 1–3 are true?

(A) 0 (B) 1 (C) 2 (D) 3

Answer C

$f'(x) = -2$, so
$g'(x) = f(x)f'(x)(x-3) = (3-2x)(-2)(x-3) = 2(3-2x)(3-x)$.
Hence $g'(x) \geq 0$ exactly when $x \leq \frac{3}{2}$ or $x \geq 3$.

- (1) True (sufficient).
- (2) False (not necessary; $x \leq \frac{3}{2}$ also works).
- (3) True (precisely characterizes where $g'(x) \geq 0$).

So exactly **2** statements are true → choice (C).

Question 5

Maureen is keeping track of the mean of her quiz scores this semester. If Maureen scores an 11 on the next quiz, her mean will increase by 1. If she scores an 11 on each of the next three quizzes, her mean will increase by 2. What is the mean of her quiz scores currently?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 7 \qquad \textbf{(E)}\ 8$

Answer D

Let a represent the amount of tests taken previously and x the mean of the scores taken previously.

We can write the following equations:

$$\frac{ax+11}{a+1} = x+1 \qquad (1)$$

$$\frac{ax+33}{a+3} = x+2 \qquad (2)$$

Multiplying equation (1) by $(a+1)$ and solving, we get:

$$ax+11 = ax+a+x+1$$

$$11 = a+x+1$$

$$a+x = 10 \qquad (3)$$

Multiplying equation (2) by $(a+3)$ and solving, we get:

$$ax+33 = ax+2a+3x+6$$

$$33 = 2a+3x+6$$

$$2a+3x = 27 \qquad (4)$$

Solving the system of equations for (3) and (4), we find that $a = 3$ and $x = \boxed{\textbf{(D)}\ 7}$.

Question 6

What is the least value of n such that $n!$ is a multiple of 2024?

$\textbf{(A)}\ 11 \qquad \textbf{(B)}\ 21 \qquad \textbf{(C)}\ 22 \qquad \textbf{(D)}\ 23 \qquad \textbf{(E)}\ 253$

Answer D

Note that $2024 = 2^3 \cdot 11 \cdot 23$ in the prime factorization. Since $23!$ is a multiple of $2^3, 11,$ and 23, we conclude that $23!$ is a multiple of 2024. Therefore, we have $n = \boxed{\textbf{(D)}\ 23}$.

Question 7

How many angles θ with $0 \leq \theta \leq 2\pi$ satisfy $\log(\sin(3\theta)) + \log(\cos(2\theta)) = 0$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer A

$$\log(\sin(3\theta)) + \log(\cos(2\theta)) = 0$$

$$\log(\sin(3\theta)\cos(2\theta)) = 0$$

$$\sin(3\theta)\cos(2\theta) = 1$$

$$\text{Since } -1 \leq \sin(x), \cos(x) \leq 1 \Rightarrow \sin(3\theta) = \cos(2\theta) = \pm 1$$

BUT note that $\log(-1)$ is not real

$$\Rightarrow \sin(3\theta) = \cos(2\theta) = 1$$

$$3\theta = \frac{\pi}{2} + 2\pi n; 2\theta = 2\pi m (m, n \in \mathbb{Z})$$

$$\theta = \frac{\pi}{6} + \frac{2\pi n}{3}; \theta = \pi m$$

$$\Rightarrow \theta \text{ has no solution}$$

Giving us $\boxed{(\text{A})\ 0}$.

Question 8

On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

"Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.

"Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

"Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

$\textbf{(A) } 7 \qquad \textbf{(B) } 12 \qquad \textbf{(C) } 21 \qquad \textbf{(D) } 27 \qquad \textbf{(E) } 31$

Answer A

Note that:

- Truth-tellers would answer yes-no-no to the three questions in this order.
- Liars would answer yes-yes-no to the three questions in this order.
- Alternaters who responded truth-lie-truth would answer no-no-no to the three questions in this order.
- Alternaters who responded lie-truth-lie would answer yes-yes-yes to the three questions in this order.

Suppose that there are T truth-tellers, L liars, and A alternaters who responded lie-truth-lie.

The conditions of the first two questions imply that

$$\begin{aligned} T + L + A &= 22, \\ L + A &= 15. \end{aligned}$$

Subtracting the second equation from the first, we have $T = 22 - 15 = \boxed{\textbf{(A) } 7}$.

Question 9

Which expression is equal to

$$\left|a-2-\sqrt{(a-1)^2}\right|$$

for $a<0$?

$\textbf{(A)}\ 3-2a \qquad \textbf{(B)}\ 1-a \qquad \textbf{(C)}\ 1 \qquad \textbf{(D)}\ a+1 \qquad \textbf{(E)}\ 3$

Answer A

We have

$$\begin{aligned}\left|a-2-\sqrt{(a-1)^2}\right| &= |a-2-|a-1|| \\ &= |a-2-(1-a)| \\ &= |2a-3| \\ &= \boxed{\textbf{(A)}\ 3-2a}.\end{aligned}$$

Question 10

What is the least possible value of $(xy-1)^2+(x+y)^2$ for real numbers x and y?

$\textbf{(A) } 0 \qquad \textbf{(B) } \frac{1}{4} \qquad \textbf{(C) } \frac{1}{2} \qquad \textbf{(D) } 1 \qquad \textbf{(E) } 2$

Answer D

We expand the original expression, then factor the result by grouping:

$$\begin{aligned}(xy-1)^2+(x+y)^2 &= \left(x^2y^2-2xy+1\right)+\left(x^2+2xy+y^2\right) \\ &= x^2y^2+x^2+y^2+1 \\ &= x^2\left(y^2+1\right)+\left(y^2+1\right) \\ &= \left(x^2+1\right)\left(y^2+1\right).\end{aligned}$$

Clearly, both factors are positive. By the Trivial Inequality, we have

$$\left(x^2+1\right)\left(y^2+1\right)\geq(0+1)(0+1)=\boxed{\textbf{(D) } 1}.$$

Note that the least possible value of $(xy-1)^2+(x+y)^2$ occurs at $x=y=0$.

Question 11

In the following list of numbers, the integer n appears n times in the list for $1 \leq n \leq 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, 200, \ldots, 200$$

What is the median of the numbers in this list?

$\textbf{(A)}\ 100.5 \qquad \textbf{(B)}\ 134 \qquad \textbf{(C)}\ 142 \qquad \textbf{(D)}\ 150.5 \qquad \textbf{(E)}\ 167$

Answer C

There are $1+2+..+199+200 = \frac{(200)(201)}{2} = 20100$ numbers in total. Let the median be k. We want to find the median k such that

$$\frac{k(k+1)}{2} = 20100/2,$$

or

$$k(k+1) = 20100.$$

Note that $\sqrt{20100} \approx 142$. Plugging this value in as k gives

$$\frac{1}{2}(142)(143) = 10153.$$

$10153 - 142 < 10050$, so 142 is the 152nd and 153rd numbers, and hence, our desired answer. $\boxed{\text{(C) } 142}$.

Note that we can derive $\sqrt{20100} \approx 142$

Question 12

What is the sum of all real numbers x for which $|x^2 - 12x + 34| = 2$?

$\textbf{(A)}\ 12 \qquad \textbf{(B)}\ 15 \qquad \textbf{(C)}\ 18 \qquad \textbf{(D)}\ 21 \qquad \textbf{(E)}\ 25$

Answer C

Split the equation into two cases, where the value inside the absolute value is positive and nonpositive.

Case 1:

The equation yields $x^2 - 12x + 34 = 2$, which is equal to $(x-4)(x-8) = 0$. Therefore, the two values for the positive case is 4 and 8.

Case 2:

Similarly, taking the nonpositive case for the value inside the absolute value notation yields $x^2 - 12x + 34 = -2$. Factoring and simplifying gives $(x-6)^2 = 0$, so the only value for this case is 6.

Summing all the values results in $4 + 8 + 6 = \boxed{\textbf{(C)}\ 18}$.

Question 13

What is the value of

$$1 + 2 + 3 - 4 + 5 + 6 + 7 - 8 + \cdots + 197 + 198 + 199 - 200?$$

$\textbf{(A)}\ 9,800 \qquad \textbf{(B)}\ 9,900 \qquad \textbf{(C)}\ 10,000 \qquad \textbf{(D)}\ 10,100 \qquad \textbf{(E)}\ 10,200$

Answer B

Looking at the numbers, you see that every set of 4 has 3 positive numbers and 1 negative number. Calculating the sum of the first couple sets gives us $2 + 10 + 18... + 394$. Clearly, this pattern is an arithmetic sequence. By using the formula we get $\frac{2+394}{2} \cdot 50 = \boxed{\textbf{(B)}\ 9900}$.

Question 14

Four distinct real numbers a, b, c, and d are used to define four points

$$A = (a, b), \quad B = (b, c), \quad C = (c, d), \quad D = (d, a).$$

The quadrilateral $ABCD$ has all four sides the same length

A if and only if $(a - b)^2 = (c - d)^2$

B for no values of a, b, c, d

C if and only if $(a - d)^2 = (b - c)^2$

D if and only if $a - b + c - d = 0$

E if and only if $(a - c)^2 = (b - d)^2$

Answer D

We must have $|AB| = |BC|$ so $\sqrt{(b-a)^2 + (c-b)^2} = \sqrt{(c-b)^2 + (d-c)^2}$.
We must also have $|BC| = |CD|$ so $\sqrt{(c-b)^2 + (d-c)^2} = \sqrt{(d-c)^2 + (a-d)^2}$.
These conditions are equivalent to $(b-a)^2 = (d-c)^2$ and $(c-b)^2 = (a-d)^2$ respectively.
Using the difference of two squares, the first is equivalent to $(a-b+c-d)(a-b-c+d) = 0$ and the second is equivalent to $(a-b+c-d)(a+b-c-d) = 0$.

In each case, we can't have both brackets equal to zero because $c \neq d$ and $b \neq c$ because the numbers are distinct. So either $a-b+c-d = 0$ or both of $a-b-c+d = 0$ and $a+b-c-d = 0$. That second case would imply that $a-c = 0$, but the numbers are distinct so that's impossible. So we're left with just the case that $a-b+c-d = 0$. We can also check that $CD = DA$ in this case, because $\sqrt{(d-c)^2 + (a-d)^2} = \sqrt{(a-d)^2 + (b-a)^2}$ rearranges to $(d-c)^2 = (b-a)^2$ which is one of the equations we already had.
The answer is "if and only if $a - b + c - d = 0$".

Question 15

Tom runs a small reptile education booth at the weekend market. He keeps notes on 13 snakes: 4 are purple and 5 are happy. From observing their "math tricks," he knows that

- all of his happy snakes can add,
- none of his purple snakes can subtract, and
- any snake that can't subtract also can't add.

Which of these conclusions must be true?

A) Being purple is a necessary condition for being happy.

B) Being able to subtract is a sufficient condition for being happy.

C) Being able to add is a sufficient condition for being happy.

D) Not being purple is a necessary condition for being happy.

E) Not being able to subtract is a necessary condition for being happy.

Answer D

Why D? From the premises, purple ⇒ can't subtract ⇒ can't add ⇒ not happy, so happy ⇒ not purple (i.e., "not purple" is **necessary** for "happy").

Question 16

Two transformations are said to commute if applying the first followed by the second gives the same result as applying the second followed by the first. Consider these four transformations of the coordinate plane:

- a translation 2 units to the right,
- a 90°-rotation counterclockwise about the origin,
- a reflection across the x-axis, and

Of the 3 pairs of distinct transformations from this list, how many commute?

(commute" means the order of the two transformations doesn't matter)

(A) 1 **(B)** 2 **(C)** 3 **(D)** None

Answer A

Label the given transformations $T_1, T_2, T_3,$ and T_4, respectively. The rules of transformations are:

- $T_1: \ (x, y) \rightarrow (x+2, y)$
- $T_2: \ (x, y) \rightarrow (-y, x)$
- $T_3: \ (x, y) \rightarrow (x, -y)$

Note that:

- Applying T_1 and then T_2 gives $(x, y) \rightarrow (x+2, y) \rightarrow (-y, x+2)$.
  Applying T_2 and then T_1 gives $(x, y) \rightarrow (-y, x) \rightarrow (-y+2, x)$.
  Therefore, T_1 and T_2 do not commute. One counterexample is the preimage $(0, 0)$.
- Applying T_1 and then T_3 gives $(x, y) \rightarrow (x+2, y) \rightarrow (x+2, -y)$.
  Applying T_3 and then T_1 gives $(x, y) \rightarrow (x, -y) \rightarrow (x+2, -y)$.
  Therefore, T_1 and T_3 commute. They form a **glide reflection**.
- Applying T_2 and then T_3 gives $(x, y) \rightarrow (-y, x) \rightarrow (-y, -x)$.
  Applying T_3 and then T_2 gives $(x, y) \rightarrow (x, -y) \rightarrow (y, x)$.
  Therefore, T_2 and T_3 do not commute. One counterexample is the preimage $(1, 0)$.

Question 17

Positive real numbers a and b have the property that

$$\sqrt{\log a} + \sqrt{\log b} + \log \sqrt{a} + \log \sqrt{b} = 100$$

and all four terms on the left are positive integers, where log denotes the base 10 logarithm. What is ab?

(A) 10^{52} **(B)** 10^{100} **(C)** 10^{144} **(D)** 10^{164} **(E)** 10^{200}

Answer D

Since $\sqrt{\log a}$ is a positive integer, we get $\log a = x^2$ for some integer x; since $\log \sqrt{a} = \frac{1}{2} \log a$ is a positive integer, we get $x = 2m$. Thus $a = 10^{4m^2}$; similarly $b = 10^{4n^2}$. Substituting, we get $2(m + n + m^2 + n^2) = 100$, i.e. $m(m+1) + n(n+1) = 50$. It follows that $m, n \leq 6$. The values of $m(m+1)$ for $m = 1, \ldots, 6$ are

m	1	2	3	4	5	6
$m(m+1)$	2	6	12	20	30	42

Two of those values must add up to 50 and we see that $20 + 30 = 50$, so $m = 4, n = 5$ and $ab = 10^{4(m^2+n^2)} = 10^{4(4^2+5^2)}$, and our answer is $\boxed{\textbf{(D)}\ 10^{164}}$.

Question 18

The equation $x^2 - 4kx + y^2 - 4y + 8 = k^3 - k$ is the equation of a circle

A if and only if either $-4 < k < -1$ or $k > 1$

B if and only if $k < -1$

C for all real values of k

D if and only if $k > 1$

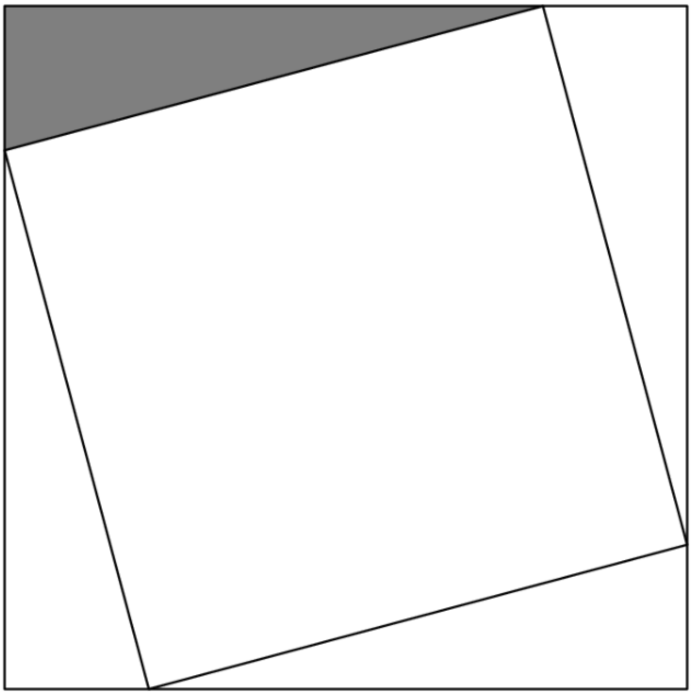
E if and only if either $-1 < k < 0$ or $k > 1$

Answer A

Complete the square for x and for y $(x - 2k)^2 - 4k^2 + (y - 2)^2 - 4 + 8 = k^3 - k$ This is the equation of a circle with centre $(2k, 2)$ and $r^2 = k^3 + 4k^2 - k - 4$ provided that expression is positive. Factorising the cubic as $(k + 4)(k - 1)(k + 1)$ reveals that this happens if and only if either $-4 < k < -1$ or if $k > 1$.
The answer is "if and only if either $-4 < k < -1$ or $k > 1$".

Question 19

A square of area 2 is inscribed in a square of area 3, creating four congruent triangles, as shown
What is the ratio of the shorter leg to the longer leg in the shaded right triangle?



(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $2-\sqrt{3}$ **(D)** $\sqrt{3}-\sqrt{2}$ **(E)** $\sqrt{2}-1$

Answer C

The side lengths of the inner square and outer square are $\sqrt{2}$ and $\sqrt{3}$ respectively. Let the shorter side of our triangle be x, thus the longer is $\sqrt{3}-x$. Hence, by the Pythagorean Theorem, we have

$$(\sqrt{3}-x)^2+x^2=(\sqrt{2})^2$$

$$3-2\sqrt{3}x+x^2+x^2=2$$

$$2x^2-2\sqrt{3}x+1=0$$

By the quadratic formula, we find that $x=\frac{\sqrt{3}\pm 1}{2}$, so the answer is $\frac{\sqrt{3}-1}{\sqrt{3}+1}=\boxed{\textbf{(C)}\ 2-\sqrt{3}}$.

Question 20

Define

$$P(x) = (x-1^2)(x-2^2)\cdots(x-100^2).$$

How many integers n are there such that $P(n) \le 0$?

(A) 4900 **(B)** 4950 **(C)** 5000 **(D)** 5050 **(E)** 5100

Answer E

We perform casework on $P(n) \le 0$:

1. $P(n) = 0$

   In this case, there are 100 such integers n :

$$1^2, 2^2, 3^2, \ldots, 100^2.$$

2. $P(n) < 0$

   There are 100 factors in $P(x)$, and we need an odd number of them to be negative. We construct the table below:

Interval of x	# of Negative Factors	Valid?
$(-\infty, 1^2)$	100	
$(1^2, 2^2)$	99	✓
$(2^2, 3^2)$	98	
$(3^2, 4^2)$	97	✓
$(4^2, 5^2)$	96	
$(5^2, 6^2)$	95	✓
$(6^2, 7^2)$	94	
$\vdots$	$\vdots$	$\vdots$
$(99^2, 100^2)$	1	✓
$(100^2, \infty)$	0	

Note that there are 50 valid intervals of x. We count the integers in these intervals:

$$\begin{aligned}(2^2-1^2-1)+(4^2-3^2-1)+(6^2-5^2-1)+\cdots+(100^2-99^2-1) &= \underbrace{(2^2-1^2)}_{(2+1)(2-1)}+\underbrace{(4^2-3^2)}_{(4+3)(4-3)}+\underbrace{(6^2-5^2)}_{(6+5)(6-5)}+\cdots+\underbrace{(100^2-99^2)}_{(100+99)(100-99)}-50 \\ &= \underbrace{(2+1)+(4+3)+(6+5)+\cdots+(100+99)}_{1+2+3+4+5+6+\cdots+99+100}-50 \\ &= \frac{101(100)}{2}-50 \\ &= 5000.\end{aligned}$$

In this case, there are 5000 such integers n.

Together, the answer is $100 + 5000 = \boxed{\textbf{(E) } 5100}$.