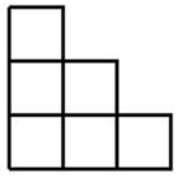
AMC 10
MOCK TEST 7
Solution Book

Combinatorics and Probability

ThrivingScholars

1. It is required to shade at least one of the six small squares in the diagram on the right so that the resulting figure has exactly one axis of symmetry. In how many different ways can this be done?



A 6 B 9 C 10 D 12 E 15

SOLUTION **E**

E

6		
4	5	
1	2	3

We number the squares to identify them. The only line of symmetry possible is the diagonal through 1 and 5. For a symmetric shading, if 4 is shaded, then so too must be 2; so either both are shaded or neither. Likewise 3 and 6 go together and provide 2 more choices. Whether 1 is shaded or not will not affect a symmetry, and this gives a further 2 choices; and the same applies to 5. Overall, therefore, there are $2^4 = 16$ choices. However, one of these is the choice to shade no squares, which is excluded by the question.

2. A bag contains hundreds of glass marbles, each one coloured either red, orange, green or blue. There are more than 2 marbles of each colour.

Marbles are drawn randomly from the bag, one at a time, and not replaced.

How many marbles must be drawn from the bag in order to ensure at least three marbles of the same colour are drawn?

A 4 B 7 C 9 D 12 E 13

SOLUTION **C**

C If at most two marbles of each colour are chosen, the maximum number we can choose is 8, corresponding to 2 of each. Therefore, if 9 are chosen, we must have at least 3 of one colour, but this statement is not true if 9 is replaced by any number less than 9.

3. There are 120 different arrangements of the five letters in the word ANGLE. If all 120 are listed in alphabetical order starting with AEGLN and finishing with NLGEA, which position in the list does ANGLE occupy?

A 18th B 20th C 22nd D 24th E 26th

SOLUTION **C**

C There are 24 arrangements of the letters in the word ANGLE with A as the first letter. In alphabetical order AEGLN is first and ANLGE is last ie 24th. ANLEG is the 23rd and hence ANGLE is the 22nd.

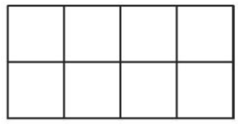
4. As a special treat, Sammy is allowed to eat five sweets from his very large jar which contains many sweets of each of three flavours – Lemon, Orange and Strawberry. He wants to eat his five sweets in such a way that no two consecutive sweets have the same flavour. In how many ways can he do this?

A 32 B 48 C 72 D 108 E 162

SOLUTION **B**

B There are three options for Sammy's first choice and then two options for each subsequent choice. Therefore the number of possible ways is $3 \times 2 \times 2 \times 2 \times 2 = 48$.

5. Diane has five identical blue disks, two identical red disks and one yellow disk. She wants to place them on the grid opposite so that each cell contains exactly one disk. The two red disks are not to be placed in cells that share a common edge.



How many different-looking completed grids can she produce?

A 96 B 108 C 144 D 180 E 216

SOLUTION **B**

Note: The key to a problem of this kind is deciding the order in which to consider the placing of the differently coloured disks. In this case it is best to consider first the number of different ways the two red disks may be placed, because they are subject to the condition that they should not be put in cells that share an edge.

In the diagram we have labelled the cells so that we can refer to them.

P	Q	R	S
T	U	V	W

If the first red disk is placed the cell P, then the second red disk may be placed in any one of the 5 cells R, S, U, V and W.

Likewise, if it placed in the any of the cells S, T and Q, there are 5 possible cells for the second red disk.

If the first red disk is placed any one of the cells Q, R, U and V, there are 4 choices for the second red disk.

This gives $5 \times 4 + 4 \times 4 = 36$ ways to place the two red disks, but each possible pair of cells has been counted twice.

Therefore there are $36 \div 2 = 18$ different ways to place the two red disks.

Once the red disks have been placed, there remain 6 cells in which the yellow disk may be placed.

Once the red and yellow disks have been placed, the 5 blue disks must be placed in the remaining 5 empty cells. This may be done in just 1 way.

This gives $18 \times 6 \times 1 = 108$ different-looking ways of filling the grid.

FOR INVESTIGATION

14.1 How many different-looking grids can Diane produce if she has one yellow disk, three identical red disks and four identical blue disks, and there are no restrictions other than that each cell should contain exactly one disk?

14.2 How many different-looking grids can Diane produce if she has one yellow disk, two identical red disks, two identical blue disks and three identical green disks, and there are no restrictions other than that each cell should contain exactly one disk?

14.3 How many different-looking grids can Diane produce if she has one yellow disk, two identical red disks, two identical blue disks and three identical green disks, with each cell containing exactly one disk and the two red disks not in cells that share a common edge?

6. Six students who share a house all speak exactly two languages. Helga speaks only English and German; Ina speaks only German and Spanish; Jean-Pierre speaks only French and Spanish; Karim speaks only German and French; Lionel speaks only French and English whilst Mary speaks only Spanish and English. If two of the students are chosen at random, what is the probability that they speak a common language?

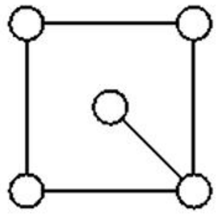
A $\frac{1}{2}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D $\frac{4}{5}$ E $\frac{5}{6}$

SOLUTION **D**

D Note that each student has a language in common with exactly four of the other five students. For instance, Jean-Pierre has a language in common with each of Ina, Karim, Lionel and Mary. Only Helga does not have a language in common with Jean-Pierre. So whichever two students are chosen, the probability that they have a language in common is 4/5.

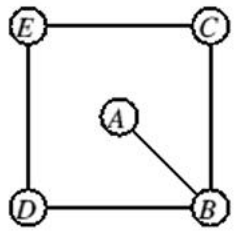
7. The diagram shows five discs connected by five line segments.
Three colours are available to colour these discs.
In how many different ways is it possible to colour all five discs if discs which are connected by a line segment are to have different colours?

A 6 B 12 C 30 D 36 E 48



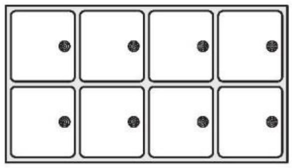
SOLUTION **D**

D Disc A may have any one of three colours and, for each of these, disc B may have two colours. So these two discs may be coloured in six different ways.



If discs C and D have the same colour, then they may be coloured in two different ways and, for each of these, disc E may have two colours. So the discs may be coloured in 24 different ways if C and D are the same colour. However, if discs C and D are different colours, then C may have one of two colours, but the colours of discs D and E are then determined. So the discs may be coloured in 12 different ways if C and D are different colours. In total, therefore, the discs may be coloured in 36 different ways.

8. A set of cupboards containing eight identical blue doors is arranged in a 2 by 4 grid as shown. A fussy decorator wishes to paint three of the doors red such that at least one door in each row is painted red and at least two of the four corners are painted red.



How many ways are there to do this?

A 12 B 24 C 36 D 40 E 56

Solution **B**

To solve this problem we need to count the number of ways of painting the doors systematically so that each possible arrangement is counted once, and once only.

A	B	C	D
E	F	G	H

We label the doors with the letters A, B, C, D, E, F, G and H, as shown in the diagram on the right, so that we can refer to them.

We know that at least two of the doors A, D, E and H must be painted red. We list the possible cases according to which is the first of these doors that is painted red.

The first row of the diagrams below covers the cases where door A is painted red. Diagram 1 deals with the cases where door D is also red. The third red door must be in the second row. Hence there are four doors which could be the third red door. These are numbered 1, 2, 3 and 4 in the diagram. Diagram 2 deals with the cases where door A is red, door D is not red, but door E is red. In this case there is already a red door in each row. Hence there are five possibilities for the third red door, numbered 1, 2, 3, 4 and 5 in the diagram.

We leave it to the reader to check that all the remaining possibilities are covered in a similar way in diagrams 3 to 6, and that no case is counted twice.

R	×	×	R
1	2	3	4

diagram 1

R	1	2	×
R	3	4	5

diagram 2

R	1	2	×
×	3	4	R

diagram 3

×	1	2	R
R	3	4	5

diagram 4

×	1	2	R
×	3	4	R

diagram 5

×	1	2	×
R	×	×	R

diagram 6

We can calculate the number of possible cases by adding up the numbers corresponding to the six diagrams. Hence there are $4 + 5 + 4 + 5 + 4 + 2 = 24$ ways to paint the doors so as to meet the required conditions.

For investigation

16.1 Check that diagrams 1 to 6 cover all of the possible cases once each.

16.2 How many ways are there to paint four of the doors red, so that there is at least one red door in each row, and at least two of the four corner doors are red?

9. Three dice, each showing numbers 1 to 6 are coloured red, blue and yellow respectively. Each of the dice is rolled once. The total of the numbers rolled is 10. In how many different ways can this happen?

A 36 B 30 C 27 D 24 E 21

SOLUTION **C**

We note first that, because the dice are coloured, two outcomes with total 10, but with different numbers rolled on particular dice, count as being different. For example

red : 6 blue : 3 yellow : 1

counts as being different from the outcome

red : 6 blue : 1 yellow : 3.

With three different numbers there are three choices for the dice which rolls the first number, then two choices for the dice which rolls the second number, leaving just one choice for the dice which rolls the third number. This gives a total of $3 \times 2 \times 1 = 6$ arrangements for the three numbers.

It can be checked that when two of the numbers are the same these can occur in 3 different ways.

It is not possible to have three equal scores with total 10.

To solve this problem we now list in the following table all possible ways a total of 10 may be obtained by throwing three dice. In each row of the table we also give the number of different ways the three numbers in the row may be arranged between the three dice.

scores	no. of ways
6, 3, 1	6
6, 2, 2	3
5, 4, 1	6
5, 3, 2	6
4, 4, 2	3
4, 3, 3	3

Therefore the total number of different ways of achieving a total of 10 is $6+3+6+6+3+3 = 27$.

FOR INVESTIGATION

15.1 Check that when two of the numbers are the same they can occur in 3 different ways on the three dice.

15.2 In how many different ways can the total of the numbers rolled be 12?

15.3 For $3 \leq T \leq 18$, calculate the number of different ways in which a total of T can be rolled using the three dice.

What do you notice about the answers?

10. A hockey team consists of 1 goalkeeper, 4 defenders, 4 midfielders and 2 forwards. There are 4 substitutes: 1 goalkeeper, 1 defender, 1 midfielder and 1 forward. A substitute may only replace a player of the same category eg: midfielder for midfielder. Given that a maximum of 3 substitutes may be used and that there are still 11 players on the pitch at the end, how many different teams could finish the game?

A 110 B 118 C 121 D 125 E 132

SOLUTION **B**

B Firstly, we note that of the players on the pitch at the end of the game, the goalkeeper is one of two players; the four defenders form one of five different possible combinations, as do the four midfielders, and the two forwards form one of three different possible combinations. So, if up to four substitutes were allowed, the number of different teams which could finish the game would be $2 \times 5 \times 5 \times 3$, that is 150.

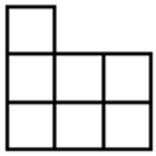
From this number we must subtract the number of these teams which require four substitutions to be made. This is $1 \times 4 \times 4 \times 2$, that is 32, so the required number of teams is 118.

11. The figure shown alongside is made from seven small squares. Some of these squares are to be shaded so that:

(i) at least two squares are shaded;

(ii) two squares meeting along an edge or at a corner are not both shaded.

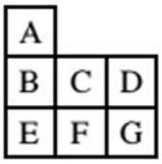
How many ways are there to do this?



A 4 B 8 C 10 D 14 E 18

SOLUTION **C**

C



Label the squares as shown. Possible pairs to be shaded which include A are AD, AE, AF and AG. Pairs excluding A are BD, BG, DE, EG. Triples must include A and there are two possibilities, ADE and AEG. This gives 10 ways of shading the grid,

12. Fnargs are either red or blue and have 2, 3 or 4 heads. A group of six Fnargs consisting of one of each possible form is made to line up such that no immediate neighbours are the same colour nor have the same number of heads. How many ways are there of lining them up from left to right?

A 12 B 24 C 60 D 120 E 720

SOLUTION **A**

A Let the six Fnargs in their final positions be denoted by $F_1F_2F_3F_4F_5F_6$. There are six choices for F_1. Once this Fnarg is chosen, the colours of the Fnargs must alternate all along the line and so we need only consider the number of heads. There are $3 - 1 = 2$ choices for F_2 as the number of heads for $F_2 \neq$ the number of heads for F_1. There is only one choice for F_3 as F_3 cannot have the same number of heads as F_2 or F_1 (F_3 and F_1 are the same colour and so have different numbers of heads). There is only one choice for F_4 as it is completely determined by F_3 and F_2, just as F_3 was completely determined by F_2 and F_1. There is only one choice for each of F_5 and F_6 as they are the last of each colour of Fnargs. The total number of ways of lining up the Fnargs is $6 \times 2 \times 1 \times 1 \times 1 \times 1$ which is 12.

13. Aaron has to choose a three-digit code for his bike lock. The digits can be chosen from 1 to 9. To help him remember them, Aaron chooses three different digits in increasing order, for example 278. How many such codes can be chosen?

A 779　　B 504　　C 168　　D 84　　E 9

SOLUTION **D**

D One way to count the possible codes is in descending numerical order of the three-digit codes. The list begins: 789; 689, 679, 678; 589, 579, 578, 569, 568, 567; ... . Each initial digit n produces part of the list with the $(8 - n)$th triangular number of possible codes, where $n \leqslant 7$. The total number of possible codes is then the sum of these triangular numbers $1 + 3 + 6 + 10 + 15 + 21 + 28$ including 1 code starting with the digit 7, all the way to 28 codes starting with the digit 1. The total number of codes that Aaron can choose is 84.

14. Aaron has to choose a three-digit code for his bike lock. The digits can be chosen from 1 to 9. To help him remember them, Aaron chooses three different digits in increasing order, for example 278.

How many such codes can be chosen?

A 779 B 504 C 168 D 84 E 9

Solution **D**

Method 1

We count the number of ways in which Aaron can choose three digits in increasing order.

We first consider the case when the first digit Aaron chooses is 1.

If the second digit he chooses is 2, he has 7 remaining choices for the third digit, namely any of the digits from 3 to 9, inclusive. If the second digit Aaron chooses is 3, he has 6 remaining choices for the third digit, namely any of the digits from 4 to 9, inclusive, and so on. Finally, we see that if the second digit Aaron chooses is 8, he has just one choice, 9, for the third digit. He cannot choose 9 as the second digit, as that would leave no choice for the third digit.

Therefore, the number of different codes with first digit 1 that Aaron can choose is $7 + 6 + 5 + 4 + 3 + 2 + 1$, that is, 28.

Similarly, the number of different codes with first digit 2, that Aaron can choose is $6+5+4+3+2+1$, that is, 21.

And so on, until finally, if the first digit Aaron chooses is 7, he has just one choice of code, namely 789.

Therefore the total number of codes that Aaron can choose is $28 + 21 + 15 + 10 + 6 + 3 + 1$, that is, 84.

Method 2

This method assumes some previous knowledge of *binomial coefficients* (see the note below).

Given any three different non-zero digits, there is only one way in which Aaron can use them to make a code, as he wishes to arrange them in increasing order.

Therefore the number of different codes that Aaron can choose is the number of different ways in which Aaron can choose 3 digits from the 9 non-zero digits. This number, '9 choose 3 ', is written $\binom{9}{3}$ or sometimes ${}_9C_3$ or ${}^9C_3$. The number $\binom{9}{3}$ is the coefficient of x^3 in the expansion of $(1 + x)^9$. It is what is called a *binomial coefficient*.

The general formula for the binomial coefficients is given by $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. Therefore the number of different codes that Aaron can choose is

$$\binom{9}{3} = \frac{9!}{3!6!} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84.$$

15. A bag contains four balls each of which is coloured either red or white. If one ball is drawn at random from the bag but not replaced and then a second ball is drawn at random, the probability that both balls are red is $\frac{1}{2}$.

What is the probability that both balls are white?

A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{6}$ E 0

SOLUTION **E**

Suppose that r of the four balls are red. There are r ways to choose a red ball first, and this leaves $r - 1$ ways to choose a second red ball. So there are are $r \times (r - 1)$ ways in which both balls are red.

With 4 balls in the bag there are $4 \times 3 = 12$ ways to choose first one of the four balls and then one of the three of the remaining balls.

Since the probability that both balls are red is $\frac{1}{2}$,

$$\frac{r(r-1)}{12} = \frac{1}{2}.$$

Hence

$$r(r-1) = 6.$$

Therefore

$$r^2 - r - 6 = 0.$$

That is,

$$(r-3)(r+2) = 0.$$

Hence $r = 3$ or $r = -2$. Since $r \geq 0$, we conclude that $r = 3$.

It follows that there is just one white ball in the bag. Hence the probability that both balls are white is 0.

FOR INVESTIGATION

17.1 The following argument is taken from Lewis Carroll's *Pillow Problems* (1893).

Problem: Suppose a bag contains two counters each of which is red or white with equal probability. Ascertain their colours without taking them out of the bag.

Answer: One counter is white and the other is red.

Full solution: The chances that the counters in the given bag are (a) two white, (b) one white and one red and (c) two red are, respectively, $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$.

Add a red counter. Then the chances that the counters in the bag are now (a) two white and one red, (b) one white and two red and (c) three red are $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$, as before.

Hence the chance of now drawing a red counter from the bag is $\frac{1}{4} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} + \frac{1}{4} \times 1 = \frac{1}{12} + \frac{1}{3} + \frac{1}{4} = \frac{2}{3}$. Hence the bag must now contain one white and two red counters since with any other combination the chance of drawing a red counter would not be $\frac{2}{3}$.

Hence before the red counter was added the bag contained one white counter and one red counter.

Is this argument correct?

16. How many pairs of numbers (m, n) are there such that the following statement is true?

'A regular m-sided polygon has an exterior angle of size $n°$ and a regular n-sided polygon has an exterior angle of size $m°$.'

A 24 B 22 C 20 D 18 E 16

SOLUTION **C**

We first note that a polygon has at least 3 sides, so we need consider only cases where $m \geq 3$ and $n \geq 3$.

The exterior angle of a regular m-sided polygon is $\left(\frac{360}{m}\right)^{\circ}$. Hence, the condition for a regular m-sided polygon to have an exterior angle of $n°$ is that $n = \frac{360}{m}$. This condition may be rewritten as $mn = 360$.

Similarly, this is the condition that a regular n-sided polygon has an exterior angle of $m°$.

Therefore the pair of numbers (m, n) satisfies the condition of the question if, and only if m and n are both at least 3 and $mn = 360$.

There are 10 ways of expressing 360 as the product of positive integers which are both at least 3. These are

$$3 \times 120,\ 4 \times 90,\ 5 \times 72,\ 6 \times 60,\ 8 \times 45,\ 9 \times 40,\ 10 \times 36,\ 12 \times 30,\ 15 \times 24 \text{ and } 18 \times 20.$$

Each of these 10 factorizations can be taken in either order to give two pairs (m, n) which meet the required condition. For example, corresponding to the factorization 3×120 we see that (m, n) may be either $(3, 120)$ or $(120, 3)$.

It follows that there are 10×2, that is, 20 pairs of numbers (m, n) meeting the required condition.

FOR INVESTIGATION

19.1 Prove that the exterior angle of a regular polygon with n sides is $\left(\frac{360}{n}\right)^{\circ}$.

19.2 For how many different values of n is there a regular polygon with n sides whose interior angles are each an integer number of degrees?

17. There are 10 girls in a mixed class. If two pupils from the class are selected at random to represent the class on the School Council, then the probability that both are girls is 0.15. How many boys are in the class?

A 10 B 12 C 15 D 18 E 20

SOLUTION **C**

C Let the number of boys in the class be x. Hence $\frac{10}{10 + x} \times \frac{9}{9 + x} = \frac{3}{20}$.
Simplifying gives $1800 = 3(10 + x)(9 + x)$ and then $x^2 + 19x - 510 = 0$.
Factorising gives $(x + 34)(x - 15) = 0$ and, since $x \neq -34$, $x = 15$.

18. A bracelet is to be made by threading four identical red beads and four identical yellow beads onto a hoop. How many different bracelets can be made?

A 4 B 8 C 12 D 18 E 24

SOLUTION **B**

B In this solution, the notation $p / q / r / s / \ldots$ represents p beads of one colour, followed by q beads of the other colour, followed by r beads of the first colour, followed by s beads of the second colour etc.

Since the colours alternate, there must be an even number of these sections of beads. If there are just two sections, then the necklace is 4/4 and there is only one such necklace. If there are four, then each colour is split either 2, 2 or 3, 1. So the possibilities are 2/3/2/1 (which can occur in two ways, with the 3 being one colour or the other) or 2/2/2/2 (which can occur in one way) or 3/3/1/1(also one way). Note that 3/2/1/2 appears to be another possibility, but is the same as 2/3/2/1 rotated.

If there are six sections, then each colour must be split into 2, 1, 1 and the possibilities are 2/2/1/1/1/1 (one way)or 2/1/1/2/1/1 (one way). Finally, if there are eight, then the only possible necklace is 1/1/1/1/1/1/1/1. In total that gives 8 necklaces.

19. A bag contains m blue and n yellow marbles. One marble is selected at random from the bag and its colour is noted. It is then returned to the bag along with k other marbles of the same colour. A second marble is now selected at random from the bag. What is the probability that the second marble is blue?

A $\frac{m}{m+n}$ B $\frac{n}{m+n}$ C $\frac{m}{m+n+k}$ D $\frac{m+k}{m+n+k}$ E $\frac{m+n}{m+n+k}$

SOLUTION **A**

A The probability that the second marble is blue equals P(2nd marble is blue given that the 1st marble is blue) + P(2nd marble is blue given that the 1st marble is yellow), which is

$$\frac{m}{m+n} \times \frac{m+k}{m+n+k} + \frac{n}{m+n} \times \frac{m}{m+n+k} = \frac{m^2+mk+mn}{(m+n)(m+n+k)} = \frac{m(m+k+n)}{(m+n)(m+n+k)} = \frac{m}{m+n}.$$

Note: this expression is independent of k.

20. The numbers $1, 2, \ldots, 9$ are randomly placed into the 9 squares of a 3×3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

$\textbf{(A)}\ \frac{1}{21} \qquad \textbf{(B)}\ \frac{1}{14} \qquad \textbf{(C)}\ \frac{5}{63} \qquad \textbf{(D)}\ \frac{2}{21} \qquad \textbf{(E)}\ \frac{1}{7}$

Solution **B**

Solution 1

Note that odd sums can only be formed by (e,e,o) or (o,o,o), so we focus on placing the evens: we need to have each even be with another even in each row/column. Because there are only 5 odd numbers overall, we can only have 1 (o,o,o) row and 1 (o,o,o) column. It can be seen that there are $3 \cdot 3 = 9$ ways to do this. There are then $5!$ ways to permute the odd numbers, and $4!$ ways to permute the even numbers, thus giving the answer as $\frac{9 \cdot 5! \cdot 4!}{9!} = \boxed{\textbf{(B)}\ \frac{1}{14}}$.

~Petallstorm (Minor edits by JeffersonJ and monkey_land)

Solution 2

By the Pigeonhole Principle, there must be at least one row with 2 or more odd numbers in it. Therefore, that row must contain 3 odd numbers in order to have an odd sum. The same thing can be done with the columns. Thus we simply have to choose one row and one column to be filled with odd numbers, so the number of valid odd/even configurations (without regard to which particular odd and even numbers are placed where) is $3 \cdot 3 = 9$. The denominator will be $\binom{9}{4}$, the total number of ways we could choose which 4 of the 9 squares will contain an even number. Hence the answer is

$$\frac{9}{\binom{9}{4}} = \boxed{\textbf{(B)}\ \frac{1}{14}}$$

21. A child builds towers using identically shaped cubes of different colors. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

$\textbf{(A)}\ 24 \qquad \textbf{(B)}\ 288 \qquad \textbf{(C)}\ 312 \qquad \textbf{(D)}\ 1,260 \qquad \textbf{(E)}\ 40,320$

Solution **D**

Arranging eight cubes is the same as arranging the nine cubes first, and then removing the last cube. In other words, there is a one-to-one correspondence between every arrangement of nine cubes, and every actual valid arrangement. Thus, we initially get $9!$. However, we have overcounted, because the red cubes can be permuted to have the same overall arrangement, and the same applies with the blue and green cubes. Thus, we have to divide by the $2!$ ways to arrange the red cubes, the $3!$ ways to arrange the blue cubes, and the $4!$ ways to arrange the green cubes. Thus we have $\frac{9!}{2!\cdot 3!\cdot 4!} = \boxed{\textbf{(D)}\ 1,260}$ different possible towers.

Note: this can be written more compactly as

$$\binom{9}{2,3,4} = \binom{9}{2}\binom{9-2}{3}\binom{9-(2+3)}{4} = \boxed{1,260}$$

22. Real numbers between 0 and 1, inclusive, are chosen in the following manner. A fair coin is flipped. If it lands heads, then it is flipped again and the chosen number is 0 if the second flip is heads, and 1 if the second flip is tails. On the other hand, if the first coin flip is tails, then the number is chosen uniformly at random from the closed interval $[0,1]$. Two random numbers x and y are chosen independently in this manner. What is the probability that $|x-y|>\tfrac{1}{2}$?

(A) $\frac{1}{3}$ **(B)** $\frac{7}{16}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{2}{3}$

Solution **B**

There are several cases depending on what the first coin flip is when determining x and what the first coin flip is when determining y.

The four cases are:

Case 1: x is either 0 or 1, and y is either 0 or 1.

Case 2: x is either 0 or 1, and y is chosen from the interval $[0,1]$.

Case 3: x is is chosen from the interval $[0,1]$, and y is either 0 or 1.

Case 4: x is is chosen from the interval $[0,1]$, and y is also chosen from the interval $[0,1]$.

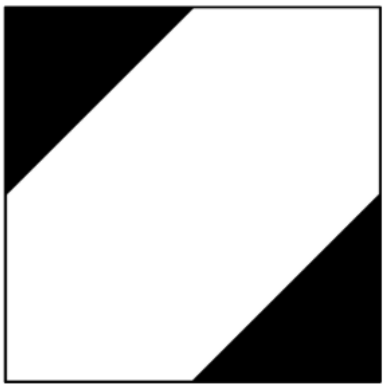
Each case has a $\frac{1}{4}$ chance of occurring (as it requires two coin flips).

For Case 1, we need x and y to be different. Therefore, the probability for success in Case 1 is $\frac{1}{2}$.

For Case 2, if x is 0, we need y to be in the interval $\left(\frac{1}{2},1\right]$. If x is 1, we need y to be in the interval $\left[0,\frac{1}{2}\right)$. Regardless of what x is, the probability for success for Case 2 is $\frac{1}{2}$.

By symmetry, Case 3 has the same success rate as Case 2.

For Case 4, we must use geometric probability because there are an infinite number of pairs (x,y) that can be selected, whether they satisfy the inequality or not. Graphing $|x-y|>\frac{1}{2}$ gives us the following picture where the shaded area is the set of all the points that fulfill the inequality:



The shaded area is $\frac{1}{4}$, which means the probability for success for case 4 is $\frac{1}{4}$ (since the total area of the bounding square, containing all possible pairs, is 1).

Adding up the success rates from each case, we get:

$$\left(\frac{1}{4}\right)\cdot\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}\right)=\boxed{\textbf{(B)}\ \frac{7}{16}}.$$

23. Tom and Geri have a competition. Initially, each player has one attempt at hitting a target. If one player hits the target and the other does not then the successful player wins. If both players hit the target, or if both players miss the target, then each has another attempt, with the same rules applying. If the probability of Tom hitting the target is always $\frac{4}{5}$ and the probability of Geri hitting the target is always $\frac{2}{3}$, what is the probability that Tom wins the competition?

A $\frac{4}{15}$ B $\frac{8}{15}$ C $\frac{2}{3}$ D $\frac{4}{5}$ E $\frac{13}{15}$

SOLUTION **C**

C Tom wins after one attempt each if he hits the target and Geri misses. The probability of this happening is $\frac{4}{5} \times \frac{1}{3} = \frac{4}{15}$. Similarly the probability that Geri wins after one attempt is $\frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$. So the probability that both competitors will have at least one more attempt is $1 - \frac{4}{15} - \frac{2}{15} = \frac{3}{5}$.
Therefore the probability that Tom wins after two attempts each is $\frac{3}{5} \times \frac{4}{15}$. The probability that neither Tom nor Geri wins after two attempts each is $\frac{3}{5} \times \frac{3}{5}$. So the probability that Tom wins after three attempts each is $\left(\frac{3}{5}\right)^2 \times \frac{4}{15}$ and, more generally, the probability that he wins after n attempts each is $\left(\frac{3}{5}\right)^{n-1} \times \frac{4}{15}$.
Therefore the probability that Tom wins is $\frac{4}{15} + \left(\frac{3}{5}\right) \times \frac{4}{15} + \left(\frac{3}{5}\right)^2 \times \frac{4}{15} + \left(\frac{3}{5}\right)^3 \times \frac{4}{15} + \ \ldots.$
This is the sum to infinity of a geometric series with first term $\frac{4}{15}$ and common ratio $\frac{3}{5}$. Its value is $\frac{4}{15} \div \left(1 - \frac{3}{5}\right) = \frac{2}{3}$.

24. Peter has 25 cards, each printed with a different integer from 1 to 25. He wishes to place N cards in a single row so that the numbers on every adjacent pair of cards have a prime factor in common.

What is the largest value of N for which this is possible?

A 16 B 18 C 20 D 22 E 24

SOLUTION **C**

C There are five cards in Peter's set that are printed with an integer that has no prime factors in common with any other number from 1 to 25. The five numbers are 1 (which has no prime factors) and the primes 13, 17, 19 and 23. These cards cannot be placed anywhere in the row of N cards. One possible row is: 11, 22, 18, 16, 12, 10, 8, 6, 4, 2, 24, 3, 9, 21, 7, 14, 20, 25, 15, 5. So the longest row is of 20 cards.

25. All the digits of a number are different, the first digit is not zero, and the sum of the digits is 36. There are $N \times 7!$ such numbers. What is the value of N?

A 72 B 97 C 104 D 107 E 128

SOLUTION **D**

D The sum of 10 different digits is 45. As the sum of the digits in the question is 36 then digits adding to 9 are omitted.

The combinations of digits satisfying this are:

$$9;\ 1+8;\ 2+7;\ 3+6;\ 4+5;\ 1+2+6;\ 1+3+5;\ 2+3+4.$$

When '0' is not involved there are $(8! + 4 \times 7! + 3 \times 6!)$ numbers, whereas when '0' is used there are $(8 \times 8! + 4 \times 7 \times 7! + 3 \times 6 \times 6!)$.

This gives a total of $9 \times 8! + (4+28) \times 7! + (3+18) \times 6! = (72+32+3) \times 7! = 107 \times 7!$

Hence $N = 107$.