ESAT COMBINED SOLUTION BOOK

ESAT Mock Test 4

Mathematics 1 · Physics · Mathematics 2

- 27 multiple-choice questions per module
- Recommended time: 40 minutes per module
- Non-calculator
- Hard ESAT-style, specification-aligned revision set

SOLUTION BOOK

ThrivingScholars

ANSWER KEY

ESAT Mock Test — Mathematics 1

Hard ESAT-style Mathematics 1 questions with carefully selected syllabus coverage and fully worked solutions.

Mathematics 1 answer key

1	C	2	E	3	B	4	D	5	A	6	C	7	E	8	B	9	D
10	A	11	B	12	C	13	E	14	D	15	A	16	B	17	C	18	E
19	D	20	A	21	C	22	E	23	B	24	D	25	A	26	C	27	E

ESAT MOCK TEST

Mathematics 1

27 ESAT-style questions with diagnostic options, compact multi-step reasoning and complete worked solutions.

M1-01

Conditional probability and symmetry

Answer C

Question 1

A bag contains 5 red, 4 blue and 3 green counters. Three counters are drawn one at a time without replacement.

Given that exactly two of the counters are the same colour and the third is a different colour, what is the probability that the first and third counters are the same colour?

A $\frac{1}{6}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D $\frac{2}{5}$

E $\frac{1}{2}$

WORKED SOLUTION

Once the three counters are known to consist of a matching pair and one counter of a different colour, the different-coloured counter is equally likely to occupy any of the three positions.

The first and third counters match exactly when the different-coloured counter is in the middle position. Hence the probability is

$$\boxed{\frac{1}{3}}.$$

M1-02

Algebraic fractions and domain

Answer E

Question 2

For real x, define

$$E = \frac{\dfrac{1}{x-2} - \dfrac{1}{x+2}}{\dfrac{1}{x-2} + \dfrac{1}{x+2}}.$$

Given that $x > 2$ and

$$E + \frac{1}{E} = \frac{17}{4},$$

what is the value of x?

- A $\frac{1}{2}$
- B 2
- C 4
- D 16
- E 8

WORKED SOLUTION

Simplifying the compound fraction gives

$$E = \frac{4/(x^2-4)}{2x/(x^2-4)} = \frac{2}{x}.$$

Let $t = E$. Then

$$t + \frac{1}{t} = \frac{17}{4} \quad \Longrightarrow \quad 4t^2 - 17t + 4 = 0,$$

so $t = 4$ or $t = \frac{1}{4}$. Since $x > 2$, $E = 2/x$ lies between 0 and 1, so $E = \frac{1}{4}$. Therefore

$$\frac{2}{x} = \frac{1}{4} \quad \Longrightarrow \quad \boxed{x = 8}.$$

M1-03

Quadratic roots and reciprocals

Answer B

Question 3

The roots of

$$3x^2 + bx - 8 = 0$$

are α and β, where $b > 0$. The reciprocals of the roots satisfy

$$\left|\frac{1}{\alpha} - \frac{1}{\beta}\right| = \frac{7}{4}.$$

What is the value of b?

- **A** 8
- **B** 10
- **C** 12
- **D** 14
- **E** 16

WORKED SOLUTION

Let $u = 1/\alpha$ and $v = 1/\beta$. From the quadratic,

$$\alpha + \beta = -\frac{b}{3}, \qquad \alpha\beta = -\frac{8}{3}.$$

Hence

$$u + v = \frac{\alpha + \beta}{\alpha\beta} = \frac{b}{8}, \qquad uv = \frac{1}{\alpha\beta} = -\frac{3}{8}.$$

Therefore

$$(u - v)^2 = (u + v)^2 - 4uv = \frac{b^2}{64} + \frac{3}{2}.$$

Using $|u - v| = 7/4$,

$$\frac{49}{16} = \frac{b^2}{64} + \frac{24}{16},$$

so $b^2 = 100$. Since $b > 0$, $\boxed{b = 10}$.

M1-04

Algebraic fractions and inequalities

Answer D

Question 4

For real x, define

$$F = \frac{\dfrac{1}{x-1} + \dfrac{1}{x+1}}{\dfrac{1}{x-1} - \dfrac{1}{x+1}}.$$

How many integer values of x satisfy all three conditions?

- $-5 \le x \le 5$;
- F is defined;
- $F^2 - 3F - 4 \le 0$.

A 2

B 3

C 5

D 4

E 6

WORKED SOLUTION

The original expression excludes $x = \pm 1$. For all other x,

$$F = \frac{2x/(x^2-1)}{2/(x^2-1)} = x.$$

The inequality becomes

$$x^2 - 3x - 4 \le 0 \quad \iff \quad (x-4)(x+1) \le 0,$$

so $-1 \le x \le 4$. The integers in this interval are $-1, 0, 1, 2, 3, 4$, but $x = -1$ and $x = 1$ are excluded by the original expression.

Thus the admissible values are $0, 2, 3, 4$, giving $\boxed{4}$ values.

M1-05

Similarity and compound proportion

Answer A

Question 5

For a family of mathematically similar open containers, the time T taken to cool through a fixed temperature range is directly proportional to the container volume and inversely proportional to the exposed surface area.

Every linear dimension of container B is $\frac{3}{2}$ times the corresponding dimension of container A. All of A's surface area is exposed, but only $\frac{4}{5}$ of the corresponding surface area of B is exposed.

What is T_B/T_A?

- **A** $\frac{15}{8}$
- **B** $\frac{27}{16}$
- **C** $\frac{9}{4}$
- **D** $\frac{27}{8}$
- **E** $\frac{5}{2}$

WORKED SOLUTION

The volume scale factor is $(3/2)^3 = 27/8$. The full surface-area scale factor is $(3/2)^2 = 9/4$, so the exposed-area scale factor is

$$\frac{4}{5} \cdot \frac{9}{4} = \frac{9}{5}.$$

Since $T \propto V/A$,

$$\frac{T_B}{T_A} = \frac{27/8}{9/5} = \frac{27}{8} \cdot \frac{5}{9} = \boxed{\frac{15}{8}}.$$

M1-06

Prime exponents and divisors

Answer C

Question 6

Let n be the smallest positive integer such that

- n is a perfect square, and
- $180n$ is a perfect cube.

How many positive divisors does n have?

A 45

B 60

C 75

D 90

E 125

WORKED SOLUTION

Since $180 = 2^2 \cdot 3^2 \cdot 5$, the exponent added by n must be even for each prime, while the resulting exponent in $180n$ must be a multiple of 3.

The smallest possible added exponents are therefore $4, 4, 2$, giving

$$n = 2^4 \cdot 3^4 \cdot 5^2.$$

Hence the number of positive divisors is

$$(4+1)(4+1)(2+1) = \boxed{75}.$$

M1-07

Standard form and algebraic structure

Answer E

Question 7

Calculate

$$\frac{(5.01 \times 10^4)^2 - (4.99 \times 10^4)^2}{(2.5 \times 10^2)^2}.$$

A 32

B 80

C 160

D 800

E 320

WORKED SOLUTION

Use the difference of two squares. The numerator is

$$\big((5.01 - 4.99) \times 10^4\big)\big((5.01 + 4.99) \times 10^4\big) = 200 \times 100000.$$

The denominator is $250^2 = 62500$. Therefore

$$\frac{20000000}{62500} = \boxed{320}.$$

M1-08

Completing the square and root structure

Answer B

Question 8

For constants a, h, c, where $c > 0$,

$$x^2 + ax + 20 = (x - h)^2 - c.$$

The two roots of $x^2 + ax + 20 = 0$ are positive, and the larger root is five times the smaller root.

What is $a + h + c$?

- **A** -2
- **B** 10
- **C** 16
- **D** 22
- **E** 4

WORKED SOLUTION

Let the roots be r and $5r$. Their product is 20, so

$$5r^2 = 20 \quad \Longrightarrow \quad r = 2.$$

The roots are therefore 2 and 10. Their midpoint is $h = 6$, and the squared half-separation is $c = 4^2 = 16$.

Also, the sum of the roots is $-a = 12$, so $a = -12$. Hence

$$a + h + c = -12 + 6 + 16 = \boxed{10}.$$

M1-09

Algebraic ratios and admissible roots

Answer D

Question 9

The positive numbers m and n satisfy $m \neq 3n$ and

$$\frac{m-3n}{m+n} + \frac{m+n}{m-3n} = \frac{10}{3}.$$

What is $m : n$?

- **A** $1:5$
- **B** $3:1$
- **C** $4:1$
- **D** $5:1$
- **E** $6:1$

WORKED SOLUTION

Let

$$t = \frac{m-3n}{m+n}.$$

Then $t + 1/t = 10/3$, so

$$3t^2 - 10t + 3 = 0 = (3t-1)(t-3).$$

If $t = 3$, then $m - 3n = 3m + 3n$, which would give $m = -3n$, impossible for positive m, n.

Thus $t = 1/3$. Hence

$$3(m-3n) = m + n \implies 2m = 10n,$$

so $\boxed{m : n = 5 : 1}$.

M1-10

Compound proportional change

Answer A

Question 10

The positive quantity P is directly proportional to $\sqrt{Q}$ and inversely proportional to R^2.

The value of Q is increased by 44%, while the value of R is decreased by 20%.

By what percentage does P increase?

A 87.5%

B 44%

C 56.25%

D 80%

E 120%

WORKED SOLUTION

The factor from Q is $\sqrt{1.44} = 1.2 = 6/5$. The factor from decreasing R to $0.8R$ is

$$\frac{1}{0.8^2} = \frac{25}{16}.$$

Therefore the total factor is

$$\frac{6}{5} \cdot \frac{25}{16} = \frac{15}{8} = 1.875.$$

Thus P increases by $0.875 = \boxed{87.5\%}$.

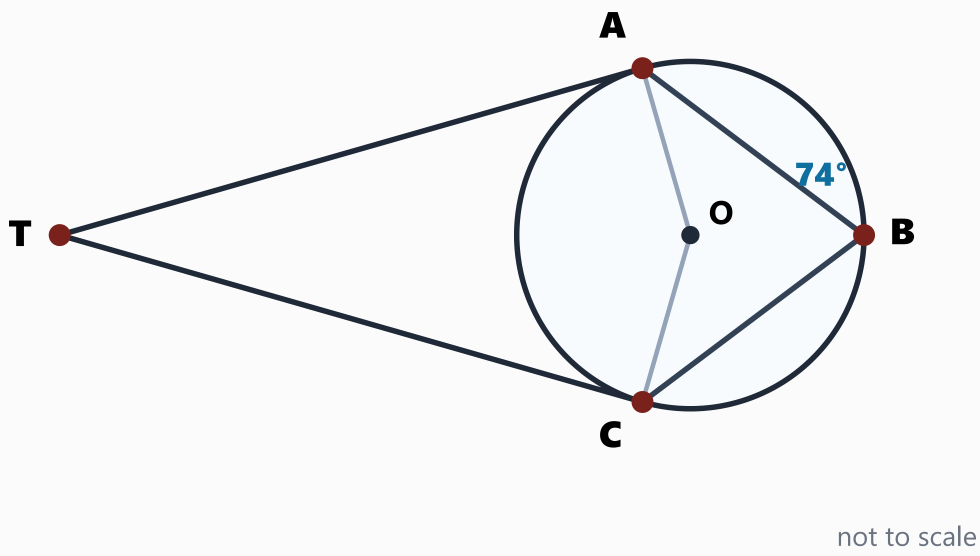
M1-11

Circle theorems

Answer **B**

Question 11

Points A, B, C lie on a circle. The tangents to the circle at A and C meet at T. The diagram is not to scale.



Given that $\angle ABC = 74°$, what is $\angle ATC$?

- **A** $28°$
- **B** $32°$
- **C** $36°$
- **D** $42°$
- **E** $74°$

WORKED SOLUTION

The angle at the centre subtending the same arc AC is twice the angle at the circumference:

$$\angle AOC = 2\angle ABC = 148°.$$

The radii OA and OC are perpendicular to the two tangents. Hence quadrilateral $AOTC$ has two right angles, so

$$\angle ATC + \angle AOC = 180°.$$

Therefore

$$\angle ATC = 180° - 148° = \boxed{32°}.$$

M1-12

Similarity, area and volume scaling

Answer C

Question 12

Two open cylinders A and B are mathematically similar. Each has one circular base and no top.

Cylinder B has radius $5\ \mathrm{cm}$ and height $10\ \mathrm{cm}$. The total surface area of cylinder A is $45\pi\ \mathrm{cm}^2$.

Cylinder A is filled to $\frac{2}{3}$ of its capacity and the liquid is poured into empty cylinder B. What fraction of B's capacity is filled?

A $\frac{9}{125}$

B $\frac{27}{125}$

C $\frac{18}{125}$

D $\frac{2}{5}$

E $\frac{3}{5}$

WORKED SOLUTION

The surface area of B is

$$\pi(5)^2 + 2\pi(5)(10) = 125\pi.$$

Thus the area scale factor $A : B$ is $45/125 = 9/25$, so the linear scale factor is $3/5$. The volume scale factor is therefore

$$\left(\frac{3}{5}\right)^3 = \frac{27}{125}.$$

Only $2/3$ of A is filled, so the fraction of B's capacity occupied is

$$\frac{2}{3} \cdot \frac{27}{125} = \boxed{\frac{18}{125}}.$$

M1-13

Coordinate geometry without solving fully

Answer **E**

Question 13

The line $y = 3x - 1$ intersects the curve $y = x^2 - 3x + 5$ at points P and Q.

What is the distance PQ?

- **A** $2\sqrt{3}$
- **B** $4\sqrt{3}$
- **C** $2\sqrt{10}$
- **D** $6\sqrt{10}$
- **E** $2\sqrt{30}$

WORKED SOLUTION

At an intersection,

$$x^2 - 3x + 5 = 3x - 1,$$

so the two x-coordinates are roots of $x^2 - 6x + 6 = 0$. Their separation is

$$\sqrt{(-6)^2 - 4(1)(6)} = \sqrt{12} = 2\sqrt{3}.$$

Because both points lie on a line of gradient 3, the corresponding change in y is three times the change in x. Hence

$$PQ = (2\sqrt{3})\sqrt{1 + 3^2} = 2\sqrt{3}\sqrt{10} = \boxed{2\sqrt{30}}.$$

M1-14

Grouped data and paired conclusions

Answer D

Question 14

A grouped distribution is shown. Its estimated mean is 25.

Interval	Frequency density
$0 \le x < 10$	1
$10 \le x < 20$	k
$20 \le x < 40$	3
$40 \le x < 50$	1

Which option gives both the frequency in $10 \le x < 20$ and the interval containing the median?

A $20;\ 10 \le x < 20$

B $30;\ 10 \le x < 20$

C $40;\ 20 \le x < 40$

D $30;\ 20 \le x < 40$

E $60;\ 20 \le x < 40$

WORKED SOLUTION

The frequencies are $10, 10k, 60, 10$, with class midpoints $5, 15, 30, 45$. Therefore

$$\frac{5(10) + 15(10k) + 30(60) + 45(10)}{80 + 10k} = 25.$$

This gives $2300 + 150k = 2000 + 250k$, so $k = 3$. The required frequency is $10k = 30$.

The total frequency is 110. The cumulative frequency after the second class is 40, and after the third class it is 100. Hence the median lies in $20 \le x < 40$.

The correct pair is $\boxed{30;\ 20 \le x < 40}$.

M1-15

Conditional probability with two restrictions

Answer A

Question 15

Two cards are drawn at random without replacement from a standard deck of 52 cards.

Given that at least one card is an ace and the two cards are of different suits, what is the probability that both cards are aces?

A $\frac{1}{25}$

B $\frac{1}{33}$

C $\frac{1}{30}$

D $\frac{1}{24}$

E $\frac{2}{25}$

WORKED SOLUTION

There are $4 \times 3/2 = 6$ unordered pairs containing two aces, and every such pair has different suits.

For a pair containing exactly one ace, choose the ace in 4 ways. The other card must be a non-ace from one of the other three suits, giving $3 \times 12 = 36$ choices. Thus there are $4 \times 36 = 144$ such pairs.

The required conditional probability is therefore

$$\frac{6}{6+144} = \boxed{\frac{1}{25}}.$$

M1-16

Loaded dice and conditional probability

Answer B

Question 16

Two independent identical unfair dice are rolled. On each die, the probability of rolling a 6 is twice the probability of rolling any one of the other five numbers, which are equally likely.

Given that the total shown is at least 10, what is the probability that at least one die shows 6?

A $\frac{11}{12}$

B $\frac{12}{13}$

C $\frac{8}{9}$

D $\frac{10}{11}$

E $\frac{13}{14}$

WORKED SOLUTION

Let the probability of each of $1, 2, 3, 4, 5$ be a. Then $P(6) = 2a$.

The ordered outcomes with total at least 10 are

$$(4,6), (5,5), (5,6), (6,4), (6,5), (6,6).$$

Their total probability, in units of a^2, is

$$2 + 1 + 2 + 2 + 2 + 4 = 13.$$

Only $(5,5)$, of weight 1, contains no 6. Hence the required probability is

$$\frac{13-1}{13} = \boxed{\frac{12}{13}}.$$

M1-17

Venn data and hidden intersection

Answer C

Question 17

Among 120 students:

- 72 study Physics, 65 study Mathematics and 50 study Chemistry;
- 38 study both Physics and Mathematics, 30 study both Mathematics and Chemistry, and 28 study both Chemistry and Physics;
- 9 study none of the three subjects.

The pairwise figures include students who study all three subjects.

A student is selected at random from those who study at least two subjects. What is the probability that the student studies exactly two?

A $\frac{5}{14}$

B $\frac{9}{20}$

C $\frac{9}{14}$

D $\frac{5}{9}$

E $\frac{11}{14}$

WORKED SOLUTION

The number studying at least one subject is $120 - 9 = 111$. Let t study all three. Inclusion–exclusion gives

$$72 + 65 + 50 - 38 - 30 - 28 + t = 111,$$

so $t = 20$.

The number studying exactly two is

$$(38 - 20) + (30 - 20) + (28 - 20) = 36.$$

Thus $36 + 20 = 56$ students study at least two subjects, and the required probability is

$$\frac{36}{56} = \boxed{\frac{9}{14}}.$$

M1-18

Counting constrained arrangements

Answer E

Question 18

A fair coin is tossed 7 times.

Given that exactly 3 heads are obtained, what is the probability that no two heads are consecutive and at least one of the first and last tosses is a head?

A $\frac{1}{5}$

B $\frac{8}{35}$

C $\frac{2}{7}$

D $\frac{3}{10}$

E $\frac{9}{35}$

WORKED SOLUTION

Given exactly three heads, every choice of three head positions is equally likely. There are

$$\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

possible triples of head positions.

The non-consecutive triples are

$$(1,3,5), (1,3,6), (1,3,7), (1,4,6), (1,4,7), (1,5,7), (2,4,6), (2,4,7), (2,5,7), (3,5,7).$$

There are therefore 10 non-consecutive arrangements. Exactly one, $(2,4,6)$, uses neither endpoint, so 9 arrangements satisfy both conditions.

The required probability is $\boxed{9/35}$.

M1-19

Conditional enumeration with dice

Answer D

Question 19

Two fair six-sided dice are rolled.

Given that the product of the two numbers is divisible by 6, what is the probability that their total is prime?

A $\frac{2}{5}$

B $\frac{7}{15}$

C $\frac{3}{5}$

D $\frac{8}{15}$

E $\frac{2}{3}$

WORKED SOLUTION

The ordered outcomes whose product is divisible by 6 are

$$(1,6),(2,3),(2,6),(3,2),(3,4),(3,6),(4,3),(4,6),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6),$$

so there are 15 admissible outcomes.

Those with prime total are

$$(1,6),(2,3),(3,2),(3,4),(4,3),(5,6),(6,1),(6,5),$$

giving 8 outcomes. Hence the probability is $\boxed{8/15}$.

M1-20

Algebraic probability without replacement

Answer **A**

Question 20

A bag contains x red balls, $x+1$ blue balls and $x+3$ green balls, where x is a positive integer.

Two balls are drawn without replacement. The probability that they are the same colour is $\frac{4}{13}$.

How many balls are in the bag?

- **A** 13
- **B** 10
- **C** 12
- **D** 14
- **E** 16

WORKED SOLUTION

The number of same-colour pairs is

$$\binom{x}{2}+\binom{x+1}{2}+\binom{x+3}{2}=\frac{3x^2+5x+6}{2}.$$

The total number of pairs is $\binom{3x+4}{2}$. Therefore

$$\frac{3x^2+5x+6}{(3x+4)(3x+3)}=\frac{4}{13}.$$

This simplifies to

$$3x^2-19x+30=0=(3x-10)(x-3).$$

Since x is an integer, $x=3$. The total number of balls is $3x+4=\boxed{13}$.

M1-21

Mixture modelling

Answer C

Question 21

A container holds a well-mixed acid solution in which the ratio of acid to water is $2:5$.

Five litres of the mixture are removed and replaced by five litres of pure water. The new ratio of acid to water is $1:3$.

What was the original volume of solution?

- A $25\ \mathrm{L}$
- B $30\ \mathrm{L}$
- C $40\ \mathrm{L}$
- D $35\ \mathrm{L}$
- E $45\ \mathrm{L}$

WORKED SOLUTION

Let the original volume be V litres. Initially the acid and water volumes are $2V/7$ and $5V/7$.

Removing 5 litres removes $10/7$ litres of acid and $25/7$ litres of water. After adding 5 litres of water, the volumes are

$$\text{acid} = \frac{2V-10}{7}, \qquad \text{water} = \frac{5V+10}{7}.$$

The new ratio is $1:3$, so

$$3(2V-10) = 5V+10.$$

Thus $V = \boxed{40\ \mathrm{L}}$.

M1-22

Extended-game probability

Answer E

Question 22

In a tennis game, a player wins each point independently with probability p, and the opponent wins with probability $q = 1 - p$. The first player to win at least four points wins the game, provided that player leads by at least two points.

What is the probability that the game ends after exactly 10 points?

- A $20p^3q^3(p^2 + q^2)$
- B $40p^3q^3(p^2 + q^2)$
- C $20p^4q^4(p^2 + q^2)$
- D $80p^4q^4(p^2 + q^2)$
- E $40p^4q^4(p^2 + q^2)$

WORKED SOLUTION

For the game to last beyond six points, the score after six points must be 3-3.

The player can win any three of the first six points. The number of such patterns is

$$\frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20,$$

so the probability of reaching 3-3 is $20p^3q^3$.

For the game still to be level after eight points, points seven and eight must be split, with probability $2pq$.

The game then ends on point ten exactly when the same player wins points nine and ten, with probability $p^2 + q^2$.

Multiplying gives

$$20p^3q^3 \cdot 2pq \cdot (p^2 + q^2) = \boxed{40p^4q^4(p^2 + q^2)}.$$

M1-23

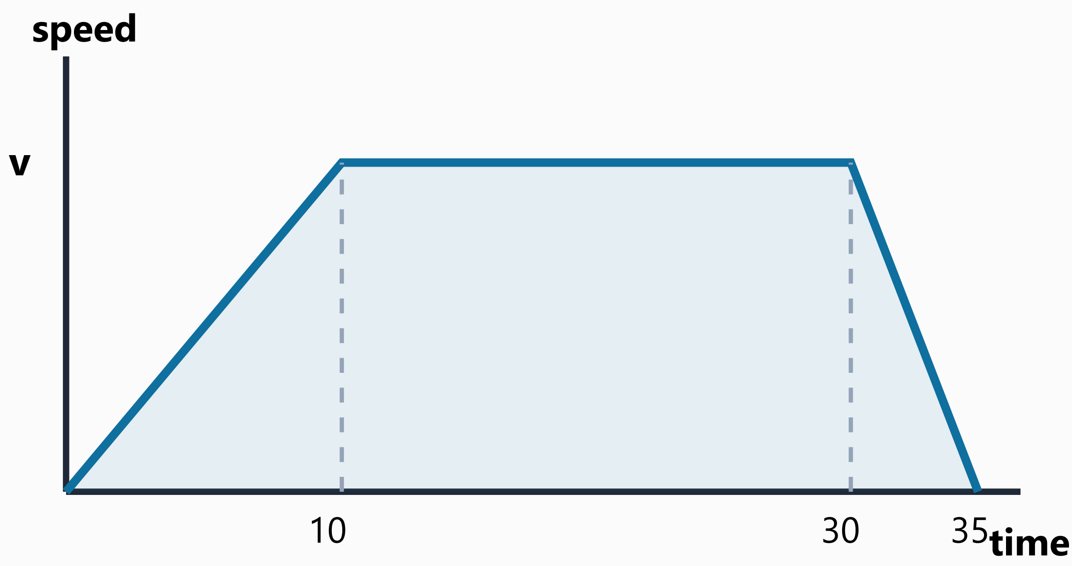
Speed–time graph

Answer B

Question 23

The speed–time graph of a vehicle consists of three straight-line sections:

- its speed rises uniformly from 0 to v during the first 10 seconds;
- it then travels at speed v for 20 seconds;
- its speed then falls uniformly to 0 during the next 5 seconds.



The total distance travelled is $660\ \mathrm{m}$. What is the average speed during the first 25 seconds?

A $18\ \mathrm{m\,s^{-1}}$

B $\frac{96}{5}\ \mathrm{m\,s^{-1}}$

C $20\ \mathrm{m\,s^{-1}}$

D $\frac{108}{5}\ \mathrm{m\,s^{-1}}$

E $24\ \mathrm{m\,s^{-1}}$

WORKED SOLUTION

The total distance is the area under the graph:

$$\frac{1}{2}(10)v + 20v + \frac{1}{2}(5)v = \frac{55}{2}v.$$

Thus

$$\frac{55}{2}v = 660,$$

so $v = 24\ \mathrm{m\,s^{-1}}$.

During the first 25 seconds, the distance is

$$\frac{1}{2}(10)(24) + 15(24) = 120 + 360 = 480 \text{ m}.$$

Therefore the average speed is

$$\frac{480}{25} = \boxed{\frac{96}{5} \text{ m s}^{-1}}.$$

M1-24

Volume conservation and surface scaling

Answer D

Question 24

A solid sphere is melted and recast into n identical smaller spheres, with no material lost.

The total surface area of the smaller spheres is three times the surface area of the original sphere.

Which option gives both n and the ratio

$$\frac{\text{radius of one smaller sphere}}{\text{radius of the original sphere}}?$$

A $n = 9;\ \frac{1}{3}$

B $n = 27;\ \frac{1}{9}$

C $n = 81;\ \frac{1}{3}$

D $n = 27;\ \frac{1}{3}$

E $n = 81;\ \frac{1}{9}$

WORKED SOLUTION

Let the original radius be R and each smaller radius be r. Volume conservation gives

$$nr^3 = R^3,$$

so $r/R = n^{-1/3}$. The total surface-area ratio is therefore

$$n\left(\frac{r}{R}\right)^2 = n \cdot n^{-2/3} = n^{1/3}.$$

This ratio is 3, so $n = 27$, and then $r/R = 1/3$. The correct pair is $\boxed{n = 27;\ 1/3}$.

M1-25

Composite solid with an internal surface

Answer A

Question 25

A right circular cylinder has radius r and height $3r$. A hemispherical cavity of radius r is hollowed out from one circular end, so the cavity opening occupies the whole end face.

The numerical value of the remaining solid's volume in cm^3 is equal to the numerical value of its total exposed surface area in cm^2.

You may use: $V_{\text{sphere}} = \frac{4}{3}\pi r^3$ and $A_{\text{sphere}} = 4\pi r^2$.

What is r, in centimetres?

A. $\frac{27}{7}$

B. $\frac{18}{7}$

C. 3

D. $\frac{24}{7}$

E. $\frac{30}{7}$

WORKED SOLUTION

The remaining volume is

$$3\pi r^3 - \frac{2}{3}\pi r^3 = \frac{7}{3}\pi r^3.$$

The exposed area consists of the curved cylinder, the intact circular base and the curved hemispherical cavity:

$$6\pi r^2 + \pi r^2 + 2\pi r^2 = 9\pi r^2.$$

Equating numerical values,

$$\frac{7}{3}\pi r^3 = 9\pi r^2.$$

Since $r > 0$, $\boxed{r = 27/7}$.

M1-26

Symmetric algebraic expressions

Answer C

Question 26

The real numbers x and y satisfy

$$x + y = 7, \qquad x^2 + y^2 = 29.$$

What is $x^4 + y^4$?

- **A** 561
- **B** 601
- **C** 641
- **D** 681
- **E** 721

WORKED SOLUTION

From

$$(x+y)^2 = x^2 + y^2 + 2xy,$$

we get $49 = 29 + 2xy$, so $xy = 10$. Therefore

$$x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 = 29^2 - 2(10^2) = 841 - 200 = \boxed{641}.$$

M1-27

Exponential symmetry

Answer E

Question 27

The equation

$$2^x + 2^{-x} = \frac{5}{2}$$

has two real solutions x_1 and x_2.

What is $4^{x_1} + 4^{x_2}$?

- A $\frac{9}{4}$
- B $\frac{13}{4}$
- C 5
- D $\frac{21}{4}$
- E $\frac{17}{4}$

WORKED SOLUTION

Let $u = 2^x > 0$. Then

$$u + \frac{1}{u} = \frac{5}{2} \quad \Longrightarrow \quad 2u^2 - 5u + 2 = 0.$$

Thus $u = 2$ or $u = 1/2$, giving $x_1 = 1$ and $x_2 = -1$. Therefore

$$4^{x_1} + 4^{x_2} = 4 + \frac{1}{4} = \boxed{\frac{17}{4}}.$$

ANSWER KEY

ESAT Mock Test — Physics

Hard ESAT-style Physics questions using coherent physical models, mathematical reasoning and fully worked solutions.

Physics answer key

1 A	2 D	3 E	4 C	5 B	6 A	7 E	8 D	9 B
10 C	11 C	12 A	13 D	14 B	15 E	16 A	17 D	18 C
19 E	20 B	21 A	22 D	23 C	24 E	25 B	26 B	27 D

ESAT MOCK TEST

Physics

27 ESAT-style questions using coherent physical models, mathematical bridges and diagnostic distractors.

P-01

Electrostatic induction and earthing

Answer A

Question 1

Two identical neutral metal spheres A and B touch each other on insulating stands. A negatively charged rod is brought close to sphere A, without touching it.

While the rod remains in place, the spheres are separated. The rod is then removed. Sphere B is moved far from A, briefly connected to Earth, and then disconnected.

Which statement describes the final state?

- **A** Electrons flow from B to Earth; A is positive and B is neutral.
- **B** Electrons flow from Earth to B; A is positive and B is negative.
- **C** Electrons flow from B to Earth; A is neutral and B is positive.
- **D** No charge flows to Earth; A is positive and B is negative.
- **E** Electrons flow from A to Earth; both spheres finish neutral.

WORKED SOLUTION

The negative rod repels electrons from A into B. Separating the spheres while the rod is present leaves A positively charged and B negatively charged.

Removing the rod does not change the net charge on either isolated sphere. When B is earthed, its excess electrons flow to Earth until B is neutral. Sphere A remains positively charged.

The answer is $\boxed{\text{A}}$.

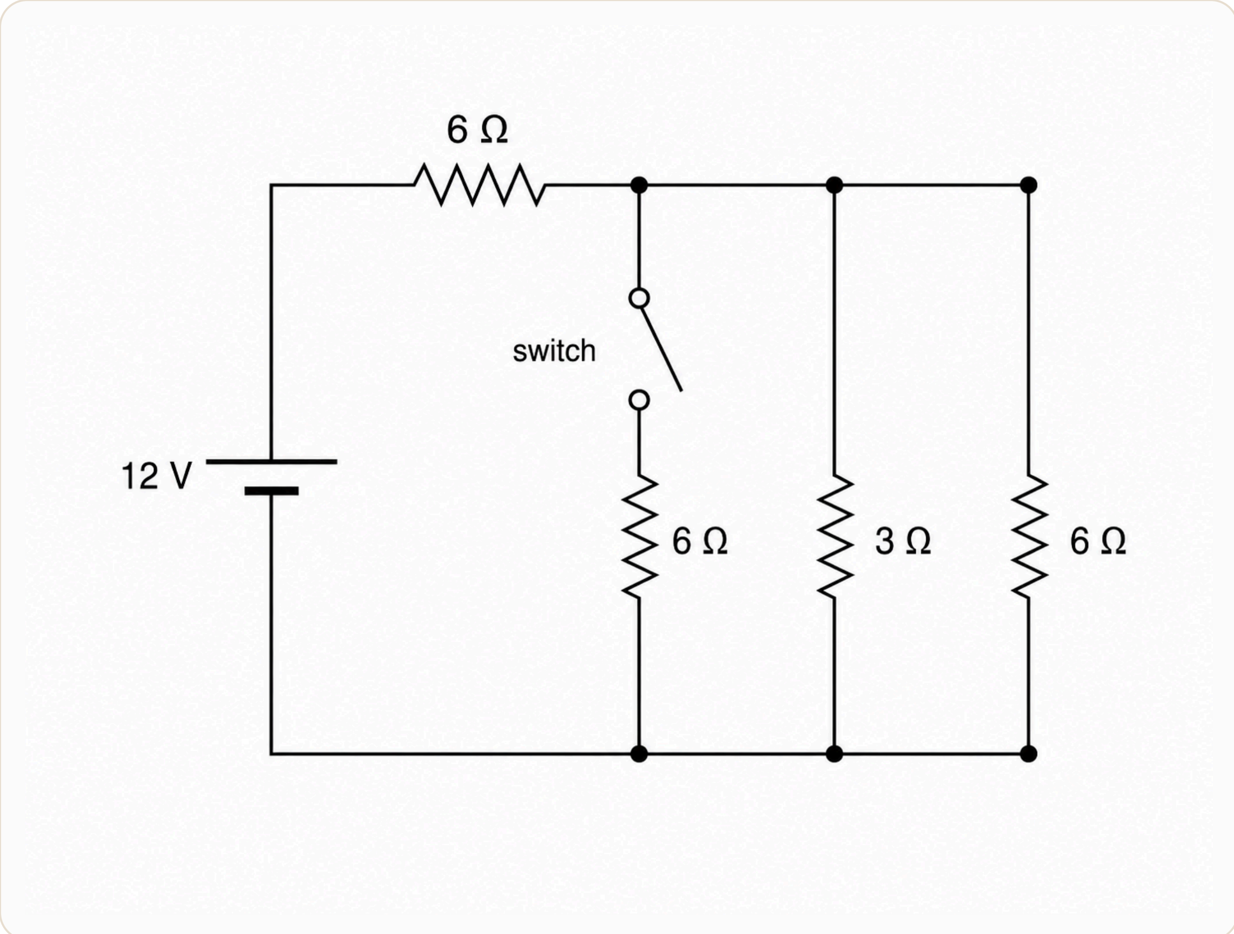
P-02

Series–parallel circuits

Answer **D**

Question 2

The circuit contains an ideal $12\ \mathrm{V}$ battery. A $6\,\Omega$ resistor is in series with a parallel network. Initially the switch is open, so the parallel network contains only the $3\,\Omega$ and $6\,\Omega$ branches. Closing the switch adds a second $6\,\Omega$ branch in parallel.



What is the ratio

$$\frac{\text{power in the series } 6\,\Omega \text{ resistor after closing the switch}}{\text{power in that resistor before closing the switch}}\,?$$

A $\frac{225}{256}$

B $\frac{15}{14}$

C $\frac{16}{15}$

D $\frac{256}{225}$

E $\frac{6}{5}$

WORKED SOLUTION

With the switch open, the parallel resistance is

$$R_{\parallel} = \left(\frac{1}{3} + \frac{1}{6}\right)^{-1} = 2\,\Omega.$$

The total resistance is $8\,\Omega$, so the current through the series resistor is

$$I_{\text{open}} = \frac{12}{8} = \frac{3}{2}\ \text{A}.$$

When the switch is closed,

$$R_{\parallel} = \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{6}\right)^{-1} = \frac{3}{2}\ \Omega.$$

The total resistance is $6 + \frac{3}{2} = \frac{15}{2}\ \Omega$, so

$$I_{\text{closed}} = \frac{12}{15/2} = \frac{8}{5}\ \text{A}.$$

For the same resistor, $P = I^2R$, so the required ratio is

$$\left(\frac{I_{\text{closed}}}{I_{\text{open}}}\right)^2 = \left(\frac{8/5}{3/2}\right)^2 = \left(\frac{16}{15}\right)^2 = \boxed{\frac{256}{225}}.$$

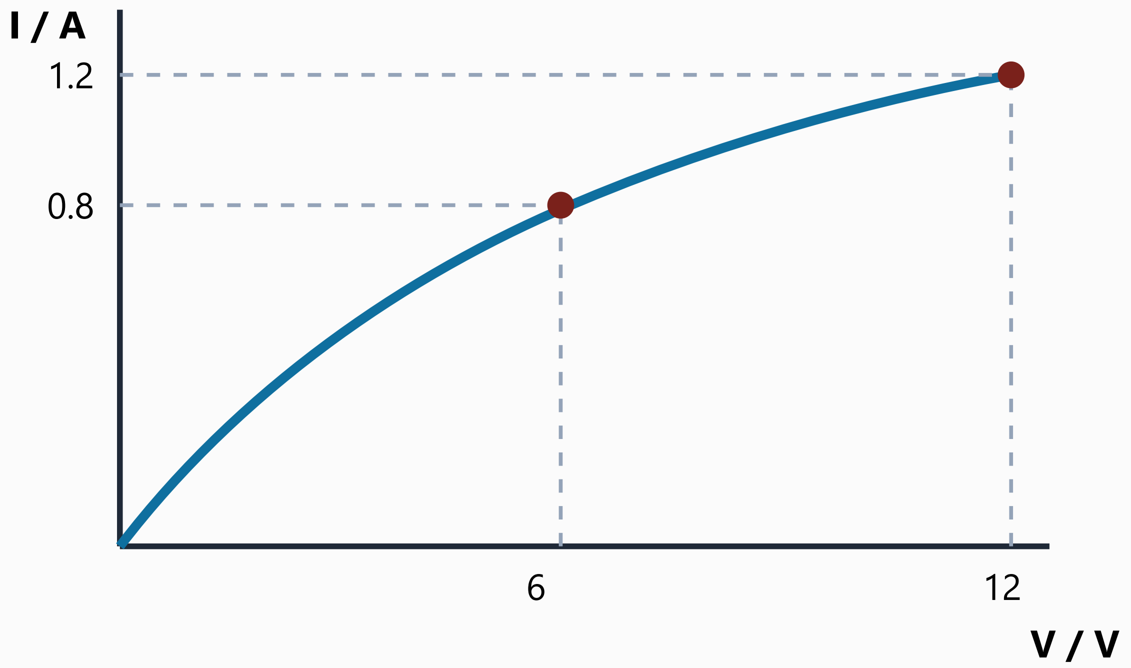
P-03

Filament-lamp characteristics

Answer E

Question 3

The graph shows the current–voltage characteristic of a filament lamp. The marked points are exact.



Two identical lamps are connected first in series across a $12\ \mathrm{V}$ supply, and then in parallel across the same supply.

What is the ratio

$$\frac{\text{total current with the lamps in parallel}}{\text{total current with the lamps in series}}\ ?$$

A $\frac{3}{2}$

B 2

C $\frac{5}{2}$

D $\frac{8}{3}$

E 3

WORKED SOLUTION

In series, the identical lamps carry the same current and therefore have equal potential differences. Each lamp has $6\ \mathrm{V}$ across it. From the graph, the series current is $0.8\ \mathrm{A}$.

In parallel, each lamp has $12\ \mathrm{V}$ across it and carries $1.2\ \mathrm{A}$. The total current is therefore $2.4\ \mathrm{A}$.

The required ratio is

$$\frac{2.4}{0.8} = \boxed{3}.$$

P-04

Thermistor circuit and paired ratios

Answer C

Question 4

An NTC thermistor is connected in series with a fixed $3.0\ \mathrm{k\Omega}$ resistor across an ideal $6.0\ \mathrm{V}$ supply.

The thermistor resistance is $6.0\ \mathrm{k\Omega}$ at $20^\circ\mathrm{C}$ and $1.0\ \mathrm{k\Omega}$ at $60^\circ\mathrm{C}$.

Which option gives both

$$\frac{P_{\text{thermistor},60}}{P_{\text{thermistor},20}} \quad \text{and} \quad \frac{P_{\text{fixed},60}}{P_{\text{fixed},20}}\ ?$$

- **A** $\frac{32}{27}$; $\frac{81}{16}$
- **B** $\frac{27}{32}$; $\frac{16}{81}$
- **C** $\frac{27}{32}$; $\frac{81}{16}$
- **D** $\frac{3}{8}$; $\frac{81}{16}$
- **E** $\frac{27}{32}$; $\frac{9}{4}$

WORKED SOLUTION

The circuit currents are

$$I_{20} = \frac{6}{9.0\ \mathrm{k\Omega}}, \qquad I_{60} = \frac{6}{4.0\ \mathrm{k\Omega}},$$

so $I_{60}/I_{20} = 9/4$.

For the fixed resistor, $P = I^2R$, so

$$\frac{P_{\text{fixed},60}}{P_{\text{fixed},20}} = \left(\frac{9}{4}\right)^2 = \frac{81}{16}.$$

For the thermistor, the resistance also changes:

$$\frac{P_{\text{thermistor},60}}{P_{\text{thermistor},20}} = \left(\frac{9}{4}\right)^2 \frac{1}{6} = \frac{27}{32}.$$

The correct pair is $\boxed{27/32;\ 81/16}$.

P-05

Transmission loss and threshold voltage

Answer B

Question 5

A transformer receives electrical power at $100\ \mathrm{kW}$ and is 80% efficient. Its output is transmitted through two cables whose combined resistance is $5.0\,\Omega$.

The power loss in the cables must not exceed 2.0% of the transformer's output power.

What is the minimum transmission voltage?

- A $2.0\ \mathrm{kV}$
- B $2\sqrt{5}\ \mathrm{kV}$
- C $4.0\ \mathrm{kV}$
- D $5.0\ \mathrm{kV}$
- E $5\sqrt{2}\ \mathrm{kV}$

WORKED SOLUTION

The transformer output is $80\ \mathrm{kW}$. The largest permitted cable loss is

$$0.02(80000) = 1600\ \mathrm{W}.$$

Since $P_{\text{loss}} = I^2R$,

$$I^2(5) = 1600 \quad\Longrightarrow\quad I = 8\sqrt{5}\ \mathrm{A}.$$

The minimum voltage occurs at this maximum current:

$$V = \frac{P}{I} = \frac{80000}{8\sqrt{5}} = 2000\sqrt{5}\ \mathrm{V} = \boxed{2\sqrt{5}\ \mathrm{kV}}.$$

P-06

Motor effect, force and torque

Answer A

Question 6

A rectangular coil has 50 turns. On each turn, each active side has length $0.20\ \mathrm{m}$, and the two active sides are $0.10\ \mathrm{m}$ apart. The sides are at right angles to a uniform magnetic field of flux density $0.40\ \mathrm{T}$. A current of $3.0\ \mathrm{A}$ flows.

You may use: **moment of a couple = one force × perpendicular separation of the two forces.**

Which statement is correct when the plane of the coil is parallel to the magnetic field?

- **A** The torque is $1.2\ \mathrm{N\ m}$, and reversing both current and field leaves its direction unchanged.
- **B** The torque is $1.2\ \mathrm{N\ m}$, and reversing both current and field reverses its direction.
- **C** The torque is $2.4\ \mathrm{N\ m}$, and reversing only the current leaves its direction unchanged.
- **D** The torque is $12\ \mathrm{N\ m}$, and reversing both current and field leaves its direction unchanged.
- **E** The forces cancel, so the torque is zero.

WORKED SOLUTION

The total force on each active side is

$$F = NBIL = 50(0.40)(3.0)(0.20) = 12\ \mathrm{N}.$$

The two forces form a couple with separation $0.10\ \mathrm{m}$, so

$$\tau = 12(0.10) = 1.2\ \mathrm{N\ m}.$$

Reversing either current or field alone reverses the torque. Reversing both reverses it twice, so the direction is unchanged. The answer is $\boxed{\mathrm{A}}$.

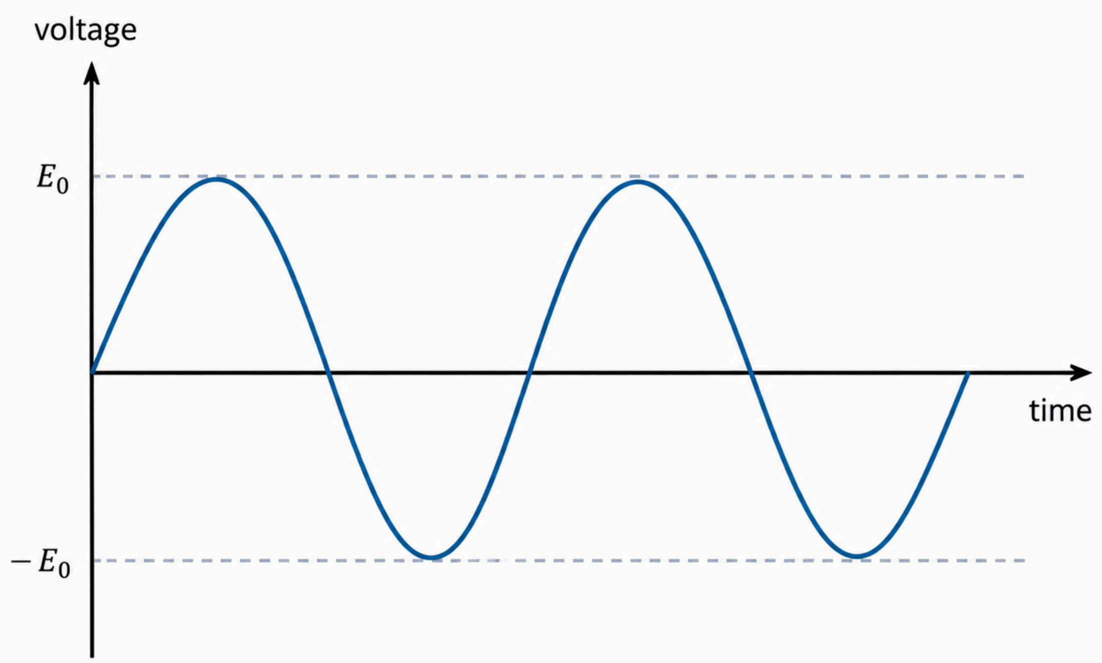
P-07

Electromagnetic induction

Answer E

Question 7

A simple generator produces the voltage shown when a coil of N turns rotates with period T. Its peak voltage is E_0.



The coil is replaced by one with $3N$ turns and is rotated at twice the original rotational speed in the same magnetic field.

What are the new peak voltage and period?

A Peak $= \frac{1}{2}E_0$, period $= \frac{1}{6}T$

B Peak $= 2E_0$, period $= \frac{1}{3}T$

C Peak $= 3E_0$, period $= \frac{1}{2}T$

D Peak $= 6E_0$, period $= T$

E Peak $= 6E_0$, period $= \frac{1}{2}T$

WORKED SOLUTION

The induced voltage increases in proportion to the number of turns and to the rate at which the magnetic field through the coil changes.

Tripling the turns multiplies the peak voltage by 3, and doubling the rotational speed multiplies it by a further factor of 2. Thus the peak becomes $6E_0$.

Doubling the rotational speed halves the time for one revolution, so the period becomes $T/2$.

The answer is $\boxed{\text{E}}$.

P-08

Transformer ratio, efficiency and energy

Answer D

Question 8

A transformer has 1200 turns on its primary coil and 100 turns on its secondary coil. The primary is connected to a 240 V alternating supply, and the secondary is connected to a $4.0\,\Omega$ resistor.

The transformer is 80% efficient. How much electrical energy is drawn from the supply in 2.0 minutes?

- **A** 10 kJ
- **B** 12 kJ
- **C** 12.5 kJ
- **D** 15 kJ
- **E** 18.75 kJ

WORKED SOLUTION

The secondary voltage is

$$V_s = 240\left(\frac{100}{1200}\right) = 20\text{ V}.$$

Thus $I_s = 20/4 = 5\text{ A}$, so the output power is 100 W.

With efficiency 0.80, the input power is

$$P_{\text{in}} = \frac{100}{0.80} = 125\text{ W}.$$

In 120 s, the energy drawn is

$$E = 125(120) = 15000\text{ J} = \boxed{15\text{ kJ}}.$$

P-09

Velocity–time graphs

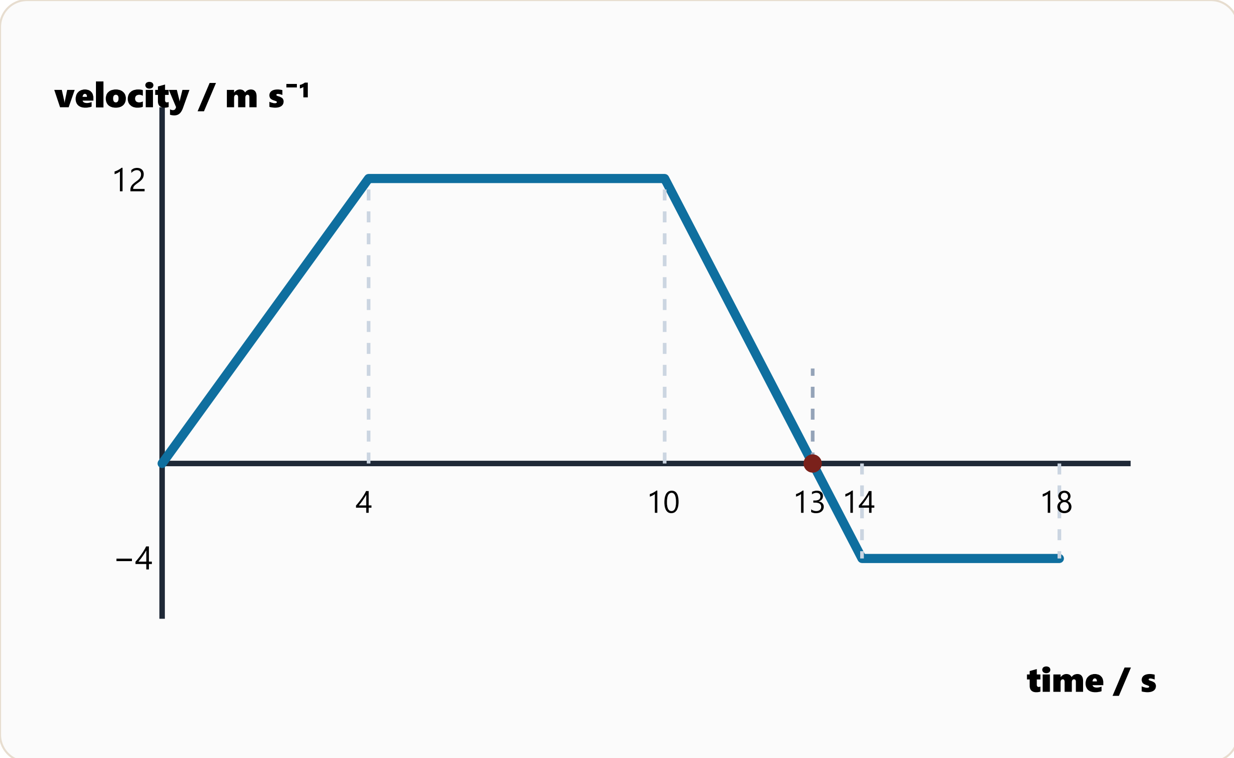
Answer **B**

Question 9

The velocity–time graph of an object is shown. The object comes to a momentary rest at the 13s mark.

What is the ratio

$$\frac{\text{total distance travelled in 18 s}}{\text{magnitude of the displacement in 18 s}}?$$



A $\frac{5}{4}$

B $\frac{11}{8}$

C $\frac{3}{2}$

D $\frac{13}{8}$

E $\frac{7}{4}$

WORKED SOLUTION

From 0 to 4 s, the positive area is

$$\frac{1}{2}(4)(12) = 24 \text{ m}.$$

From 4 to 10 s, it is $6(12) = 72$ m.

From 10 to 14 s, the velocity falls from 12 to -4 m s^{-1}, crossing zero at 13 s. The positive area is

$$\frac{1}{2}(3)(12) = 18 \text{ m},$$

and the negative area has magnitude

$$\frac{1}{2}(1)(4) = 2 \text{ m}.$$

From 14 to 18 s, the negative area has magnitude $4(4) = 16 \text{ m}$.

Therefore

$$\text{distance} = 24 + 72 + 18 + 2 + 16 = 132 \text{ m},$$

while

$$|\text{displacement}| = |24 + 72 + 18 - 2 - 16| = 96 \text{ m}.$$

The ratio is

$$\frac{132}{96} = \boxed{\frac{11}{8}}.$$

P-10

Forces, motion and efficiency

Answer C

Question 10

An elevator of total mass $800\ \mathrm{kg}$ starts from rest and accelerates vertically upwards at $1.5\ \mathrm{m\ s^{-2}}$ for $4.0\ \mathrm{s}$. It then continues upwards at constant speed for a further $6.0\ \mathrm{s}$.

Take $g = 10\ \mathrm{N\ kg^{-1}}$. The motor is 80% efficient throughout. Neglect friction and air resistance.

How much electrical energy is supplied to the motor during the full $10\ \mathrm{s}$?

- **A** $438\ \mathrm{kJ}$
- **B** $468\ \mathrm{kJ}$
- **C** $498\ \mathrm{kJ}$
- **D** $512\ \mathrm{kJ}$
- **E** $552\ \mathrm{kJ}$

WORKED SOLUTION

During acceleration, the cable force is

$$T = m(g + a) = 800(11.5) = 9200\ \mathrm{N}.$$

The distance travelled in the first $4\ \mathrm{s}$ is $\frac{1}{2}at^2 = 12\ \mathrm{m}$, so the useful work is $9200(12) = 110.4\ \mathrm{kJ}$.

The speed reached is $at = 6\ \mathrm{m\ s^{-1}}$. In the next $6\ \mathrm{s}$, the elevator travels $36\ \mathrm{m}$ at constant speed, with cable force $mg = 8000\ \mathrm{N}$. The useful work is $8000(36) = 288\ \mathrm{kJ}$.

Total useful energy is $398.4\ \mathrm{kJ}$, so the electrical input is

$$\frac{398.4}{0.80} = \boxed{498\ \mathrm{kJ}}.$$

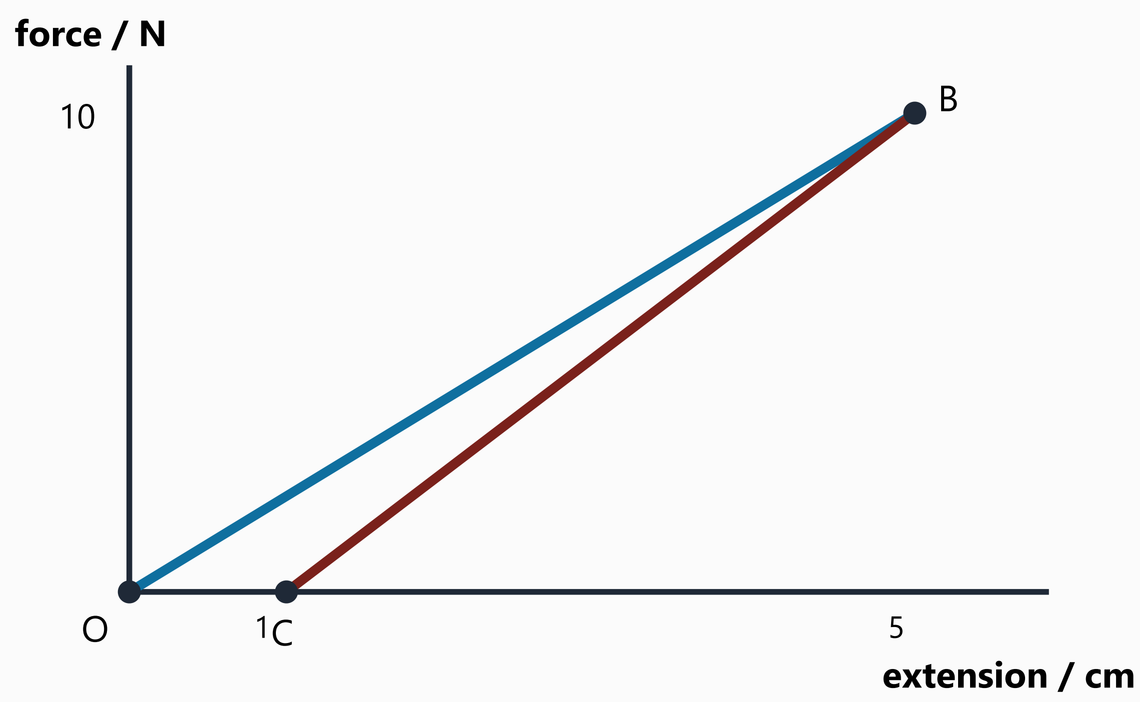
P-11

Force–extension graphs

Answer C

Question 11

A material is loaded and then unloaded. Its force–extension graph is shown. The loading path is $O \to B$, and the unloading path is $B \to C$.



How much energy is dissipated in one loading–unloading cycle?

- **A** $0.025\ \mathrm{J}$
- **B** $5\ \mathrm{J}$
- **C** $0.050\ \mathrm{J}$
- **D** $0.20\ \mathrm{J}$
- **E** $0.25\ \mathrm{J}$

WORKED SOLUTION

The work done during loading is the area under OB:

$$E_{\text{in}} = \frac{1}{2}(10)(0.050) = 0.25\ \mathrm{J}.$$

During unloading, the extension falls from $5\ \mathrm{cm}$ to the permanent extension $1\ \mathrm{cm}$. The recovered energy is the area under BC:

$$E_{\text{recovered}} = \frac{1}{2}(10)(0.050 - 0.010) = 0.20\ \mathrm{J}.$$

The dissipated energy is

$$0.25 - 0.20 = \boxed{0.050 \text{ J}}.$$

P-12

Momentum, gravitational energy and spring energy

Answer **A**

Question 12

A 1.0 kg trolley moving at 6.0 m s^{-1} collides with a stationary 2.0 kg trolley. The trolleys stick together.

They then move on a frictionless track, rise through a vertical height of 0.10 m, and compress an initially uncompressed spring of force constant 150 N m^{-1}.

Take $g = 10 \text{ N kg}^{-1}$. What is the maximum spring compression?

- **A** 0.20 m
- **B** 0.10 m
- **C** 0.15 m
- **D** 0.25 m
- **E** 0.30 m

WORKED SOLUTION

Momentum conservation in the collision gives

$$(1.0)(6.0) = (3.0)v,$$

so $v = 2.0 \text{ m s}^{-1}$. The kinetic energy immediately after collision is

$$\frac{1}{2}(3.0)(2.0)^2 = 6.0 \text{ J}.$$

Rising 0.10 m increases gravitational energy by

$$(3.0)(10)(0.10) = 3.0 \text{ J}.$$

The remaining 3.0 J becomes spring energy:

$$\frac{1}{2}(150)x^2 = 3.0.$$

Thus $x^2 = 0.040$, giving $\boxed{x = 0.20 \text{ m}}$.

P-13

Momentum transfer and work against friction

Answer D

Question 13

A $20\ \mathrm{g}$ pellet travels at $300\ \mathrm{m\ s^{-1}}$ into a stationary $0.80\ \mathrm{kg}$ block on a horizontal track. The pellet leaves the block in the same direction at $100\ \mathrm{m\ s^{-1}}$.

Immediately after the pellet has left, the block enters a rough section where a constant frictional force of $10\ \mathrm{N}$ brings it to rest.

How far does the block slide?

- **A** $0.40\ \mathrm{m}$
- **B** $0.50\ \mathrm{m}$
- **C** $0.80\ \mathrm{m}$
- **D** $1.0\ \mathrm{m}$
- **E** $2.0\ \mathrm{m}$

WORKED SOLUTION

The pellet loses momentum

$$0.020(300-100)=4.0\ \mathrm{kg\ m\ s^{-1}}.$$

This is the momentum gained by the block, so its speed is

$$v=\frac{4.0}{0.80}=5.0\ \mathrm{m\ s^{-1}}.$$

The block's kinetic energy is

$$\frac{1}{2}(0.80)(5.0)^2=10\ \mathrm{J}.$$

Equating this to the work done against friction, $10s=10$, so $\boxed{s=1.0\ \mathrm{m}}$.

P-14

Terminal speed, upthrust and scaling

Answer B

Question 14

A fully submerged sphere of weight $12\ \mathrm{N}$ falls at terminal speed v through a liquid. At this speed, the upthrust is $4\ \mathrm{N}$.

The sphere is transferred to a second liquid of half the density. For the fixed submerged volume, upthrust is proportional to liquid density. In both liquids, assume the drag force is proportional to the square of speed with the same constant of proportionality.

What is the terminal speed in the second liquid?

- **A** v
- **B** $\frac{\sqrt{5}}{2}v$
- **C** $\frac{5}{4}v$
- **D** $\sqrt{\frac{3}{2}}v$
- **E** $\frac{3}{2}v$

WORKED SOLUTION

At terminal speed in the first liquid, the drag is

$$12 - 4 = 8\ \mathrm{N}.$$

In the second liquid, the upthrust halves to $2\ \mathrm{N}$, so the terminal drag must be

$$12 - 2 = 10\ \mathrm{N}.$$

Since drag is proportional to speed squared,

$$\left(\frac{v_2}{v}\right)^2 = \frac{10}{8} = \frac{5}{4}.$$

Therefore $\boxed{v_2 = (\sqrt{5}/2)v}$.

P-15

Thermal conduction and junction temperature

Answer E

Question 15

Two sections of the same material are joined in series between ends maintained at $100°\mathrm{C}$ and $20°\mathrm{C}$.

Section 1, next to the $100°\mathrm{C}$ end, has length L and area A. Section 2 has length $2L$ and area $3A$. For a section, the steady rate of energy transfer is proportional to $A\Delta T/L$.

Which option gives both the rate through the composite rod as a fraction of the rate through section 1 alone between the same end temperatures, and the junction temperature?

- **A** $\frac{2}{5}$; $52°\mathrm{C}$
- **B** $\frac{3}{5}$; $68°\mathrm{C}$
- **C** $\frac{2}{3}$; $52°\mathrm{C}$
- **D** $\frac{3}{5}$; $48°\mathrm{C}$
- **E** $\frac{3}{5}$; $52°\mathrm{C}$

WORKED SOLUTION

Thermal resistance is proportional to L/A. Relative to section 1, the two resistances are

$$R_1 = 1, \qquad R_2 = \frac{2}{3}.$$

Thus the total resistance is $5/3$, so the rate is $1/(5/3) = 3/5$ of the rate through section 1 alone.

The $80°\mathrm{C}$ total temperature drop divides in the ratio $R_1 : R_2 = 1 : 2/3 = 3 : 2$. The drop across section 1 is therefore

$$80 \cdot \frac{3}{5} = 48°\mathrm{C}.$$

The junction temperature is $100 - 48 = 52°\mathrm{C}$. The correct pair is $\boxed{3/5;\ 52°\mathrm{C}}$.

P-16

Thermal equilibrium and inferred loss coefficients

Answer **A**

Question 16

Two otherwise identical objects are in the same room. One is matt black and the other is shiny. Each contains a $12\ \mathrm{W}$ heater.

At equilibrium, the matt black object is $15^\circ\mathrm{C}$ above room temperature and the shiny object is $30^\circ\mathrm{C}$ above room temperature.

For either object, conduction and convection together remove $0.20\Delta T\ \mathrm{W}$, where ΔT is the temperature rise. Radiative loss is also proportional to ΔT and to the area of each surface finish.

A third identical object has half its surface matt black and half shiny, and contains the same heater. What is its equilibrium temperature rise?

- **A** $20^\circ\mathrm{C}$
- **B** $15^\circ\mathrm{C}$
- **C** $18^\circ\mathrm{C}$
- **D** $24^\circ\mathrm{C}$
- **E** $30^\circ\mathrm{C}$

WORKED SOLUTION

The total loss coefficient of the black object is $12/15 = 0.80\ \mathrm{W\ K^{-1}}$, so its radiative coefficient is $0.80 - 0.20 = 0.60$.

For the shiny object, the total coefficient is $12/30 = 0.40$, so its radiative coefficient is $0.40 - 0.20 = 0.20$.

With half of each finish, the radiative coefficient is the average:

$$\frac{0.60 + 0.20}{2} = 0.40.$$

Including conduction and convection gives a total coefficient of $0.60\ \mathrm{W\ K^{-1}}$. Hence

$$\Delta T = \frac{12}{0.60} = \boxed{20^\circ\mathrm{C}}.$$

P-17

Specific heat and latent heat in stages

Answer **D**

Question 17

A $0.20\ \mathrm{kg}$ block of ice initially at $-10^\circ\mathrm{C}$ is heated until it has completely melted and the resulting water reaches $20^\circ\mathrm{C}$.

A $500\ \mathrm{W}$ heater transfers 80% of its energy to the ice and water. Use

$$c_{\text{ice}} = 2.0 \times 10^3\ \mathrm{J\ kg^{-1}\ {}^\circ C^{-1}}, \quad L_f = 3.0 \times 10^5\ \mathrm{J\ kg^{-1}}, \quad c_{\text{water}} = 4.0 \times 10^3\ \mathrm{J\ kg^{-1}\ {}^\circ C^{-1}}.$$

How long does the heating take?

- **A** $160\ \mathrm{s}$
- **B** $180\ \mathrm{s}$
- **C** $190\ \mathrm{s}$
- **D** $200\ \mathrm{s}$
- **E** $240\ \mathrm{s}$

WORKED SOLUTION

The required energies are

$$E_1 = 0.20(2.0 \times 10^3)(10) = 4000\ \mathrm{J},$$

$$E_2 = 0.20(3.0 \times 10^5) = 60000\ \mathrm{J},$$

and

$$E_3 = 0.20(4.0 \times 10^3)(20) = 16000\ \mathrm{J}.$$

Total useful energy is $80000\ \mathrm{J}$. The useful power is $0.80(500) = 400\ \mathrm{W}$, so

$$t = \frac{80000}{400} = \boxed{200\ \mathrm{s}}.$$

P-18

Boyle's law with an inferred pressure

Answer C

Question 18

An air bubble has volume $2.0\ \mathrm{cm}^3$ at the bottom of a lake and $5.0\ \mathrm{cm}^3$ at the surface. The water has uniform density, the temperature is constant, and the surface pressure is atmospheric pressure.

What is the bubble's volume when it is halfway from the bottom to the surface?

A $\frac{5}{2}\ \mathrm{cm}^3$

B $\frac{8}{3}\ \mathrm{cm}^3$

C $\frac{20}{7}\ \mathrm{cm}^3$

D $\frac{10}{3}\ \mathrm{cm}^3$

E $\frac{7}{2}\ \mathrm{cm}^3$

WORKED SOLUTION

Let atmospheric pressure be P. Boyle's law between the bottom and surface gives

$$P_{\text{bottom}}(2.0) = P(5.0),$$

so $P_{\text{bottom}} = 2.5P$. Therefore the water contributes $1.5P$ at the bottom.

Halfway up, the water contribution is $0.75P$, so the total pressure is $1.75P = 7P/4$.

Using the surface state,

$$P(5.0) = \frac{7P}{4}V,$$

giving $\boxed{V = 20/7\ \mathrm{cm}^3}$.

P-19

Layered-fluid pressure and equivalent depth

Answer E

Question 19

A small vertical hatch of area $0.020\ \mathrm{m}^2$ separates a tank from the atmosphere. Its centre is below $0.40\ \mathrm{m}$ of oil and a further $0.20\ \mathrm{m}$ of water.

The oil density is $800\ \mathrm{kg\ m^{-3}}$, the water density is $1000\ \mathrm{kg\ m^{-3}}$, and $g = 10\ \mathrm{N\ kg^{-1}}$. Pressure is effectively uniform over the hatch.

Which option gives both the resultant force due to the liquids and the depth below a pure-water surface that would produce the same force?

A $104\ \mathrm{N};\ 0.40\ \mathrm{m}$

B $80\ \mathrm{N};\ 0.52\ \mathrm{m}$

C $100\ \mathrm{N};\ 0.50\ \mathrm{m}$

D $104\ \mathrm{N};\ 0.60\ \mathrm{m}$

E $104\ \mathrm{N};\ 0.52\ \mathrm{m}$

WORKED SOLUTION

The gauge pressure is

$$p = (800)(10)(0.40) + (1000)(10)(0.20) = 5200\ \mathrm{Pa}.$$

Hence the force is

$$F = pA = 5200(0.020) = 104\ \mathrm{N}.$$

For pure water, $1000(10)h = 5200$, so $h = 0.52\ \mathrm{m}$. The correct pair is $\boxed{104\ \mathrm{N};\ 0.52\ \mathrm{m}}$.

P-20

Wave refraction and wave properties

Answer B

Question 20

Plane waves travel from region X into region Y. Their wavelength is $0.60\ \mathrm{m}$ in X and $0.40\ \mathrm{m}$ in Y. The waves cross the boundary at a non-zero angle to the normal.

Which option gives v_Y/v_X, f_Y/f_X, and the direction in which the waves bend on entering region Y?

- **A** $\frac{3}{2}$; 1; towards the normal
- **B** $\frac{2}{3}$; 1; towards the normal
- **C** $\frac{2}{3}$; $\frac{2}{3}$; towards the normal
- **D** $\frac{2}{3}$; 1; away from the normal
- **E** $\frac{3}{2}$; 1; away from the normal

WORKED SOLUTION

The frequency is fixed by the source, so $f_Y/f_X = 1$.

Using $v = f\lambda$,

$$\frac{v_Y}{v_X} = \frac{\lambda_Y}{\lambda_X} = \frac{0.40}{0.60} = \frac{2}{3}.$$

The wave is slower in region Y, so on entering Y it bends **towards the normal**.

The correct row is $\boxed{\frac{2}{3};\ 1;\ \text{towards the normal}}$, so the answer is $\boxed{\mathrm{B}}$.

P-21

Doppler effect from pulse spacing

Answer **A**

Question 21

A sound source emits one pulse every $0.10\ \mathrm{s}$ while moving at $20\ \mathrm{m\ s^{-1}}$ through still air. The speed of sound is $100\ \mathrm{m\ s^{-1}}$.

A stationary observer is first directly in front of the approaching source and later directly behind the receding source.

Which option gives the two observed frequencies, approaching first and receding second?

A $\frac{25}{2}\ \mathrm{Hz};\ \frac{25}{3}\ \mathrm{Hz}$

B $12\ \mathrm{Hz};\ 8\ \mathrm{Hz}$

C $15\ \mathrm{Hz};\ 10\ \mathrm{Hz}$

D $\frac{25}{3}\ \mathrm{Hz};\ \frac{25}{2}\ \mathrm{Hz}$

E $20\ \mathrm{Hz};\ 5\ \mathrm{Hz}$

WORKED SOLUTION

In $0.10\ \mathrm{s}$, a sound pulse travels $10\ \mathrm{m}$, while the source moves $2\ \mathrm{m}$.

Ahead of the source, pulse spacing is $10 - 2 = 8\ \mathrm{m}$, so

$$f_{\text{app}} = 100/8 = 25/2\ \mathrm{Hz}.$$

Behind the source, pulse spacing is $10 + 2 = 12\ \mathrm{m}$, so

$$f_{\text{rec}} = 100/12 = 25/3\ \mathrm{Hz}.$$

The correct pair is $\boxed{25/2\ \mathrm{Hz};\ 25/3\ \mathrm{Hz}}$.

P-22

Ultrasound time-of-flight and wavelength

Answer **D**

Question 22

An ultrasound scanner emits waves of frequency $4.0\ \mathrm{MHz}$ into tissue in which the wave speed is $1600\ \mathrm{m\ s^{-1}}$.

An echo from a boundary returns to the probe $150\ \mu s$ after the pulse is emitted.

Which option gives both the boundary depth and the number of wavelengths travelled on the complete outward-and-return journey?

- **A** $0.24\ \mathrm{m};\ 600$
- **B** $0.12\ \mathrm{m};\ 300$
- **C** $0.24\ \mathrm{m};\ 1200$
- **D** $0.12\ \mathrm{m};\ 600$
- **E** $0.060\ \mathrm{m};\ 300$

WORKED SOLUTION

The round-trip distance is

$$d = vt = 1600(150 \times 10^{-6}) = 0.24\ \mathrm{m},$$

so the boundary depth is $0.12\ \mathrm{m}$.

The wavelength is

$$\lambda = \frac{1600}{4.0 \times 10^{6}} = 4.0 \times 10^{-4}\ \mathrm{m}.$$

The number of wavelengths in the round trip is

$$\frac{0.24}{4.0 \times 10^{-4}} = 600.$$

The correct pair is $\boxed{0.12\ \mathrm{m};\ 600}$.

P-23

Electromagnetic spectrum and frequency ratio

Answer C

Question 23

Electromagnetic radiation X has frequency $3.0 \times 10^{10}\ \mathrm{Hz}$. Radiation Y, produced by a nuclear transition, has wavelength $1.0 \times 10^{-12}\ \mathrm{m}$.

Take the speed of electromagnetic waves in vacuum as $3.0 \times 10^{8}\ \mathrm{m\ s^{-1}}$.

Which statement is correct?

- **A** X is infrared, Y is X-radiation, and $f_Y = 10^8 f_X$.
- **B** X is radio radiation, Y is gamma radiation, and $f_Y = 10^{10} f_X$.
- **C** X is microwave radiation, Y is gamma radiation, and $f_Y = 10^{10} f_X$.
- **D** X is microwave radiation, Y is X-radiation, and $f_Y = 10^8 f_X$.
- **E** X is visible light, Y is gamma radiation, and $f_Y = 10^{12} f_X$.

WORKED SOLUTION

A frequency of $3.0 \times 10^{10}\ \mathrm{Hz}$ is in the microwave region.

For Y,

$$f_Y = \frac{3.0 \times 10^8}{1.0 \times 10^{-12}} = 3.0 \times 10^{20}\ \mathrm{Hz},$$

The nuclear-transition origin identifies Y as gamma radiation. The ratio is

$$\frac{f_Y}{f_X} = \frac{3.0 \times 10^{20}}{3.0 \times 10^{10}} = 10^{10}.$$

The answer is $\boxed{\mathrm{C}}$.

P-24

Nuclear changes and neutron number

Answer E

Question 24

An unknown nuclide undergoes, in some order, two alpha decays, two beta-minus decays and one gamma emission. The final nuclide is

$$^{211}_{84}Y.$$

Which option gives both the original nuclide and the change in the number of neutrons from original to final?

- **A** $^{219}_{84}X;$ decrease of 8
- **B** $^{219}_{86}X;$ decrease of 8
- **C** $^{215}_{86}X;$ decrease of 4
- **D** $^{219}_{85}X;$ decrease of 7
- **E** $^{219}_{86}X;$ decrease of 6

WORKED SOLUTION

Two alpha decays reduce A by 8 and Z by 4. Two beta-minus decays then increase Z by 2. Gamma emission changes neither.

Thus the original values were

$$A = 211 + 8 = 219, \qquad Z = 84 + 2 = 86.$$

The original neutron number was $219 - 86 = 133$, while the final neutron number is $211 - 84 = 127$. The number of neutrons decreases by 6.

The correct pair is $\boxed{^{219}_{86}X;\ \text{decrease of } 6}$.

P-25

Half-life with unknown background

Answer **B**

Question 25

A detector records a radioactive source together with a constant background count rate.

The total count rates are:

Time	Total count rate
$0\ \mathrm{h}$	$680\ \mathrm{min}^{-1}$
$12\ \mathrm{h}$	$200\ \mathrm{min}^{-1}$
$24\ \mathrm{h}$	$80\ \mathrm{min}^{-1}$

What total count rate is expected after $30\ \mathrm{h}$?

A $45\ \mathrm{min}^{-1}$

B $60\ \mathrm{min}^{-1}$

C $55\ \mathrm{min}^{-1}$

D $80\ \mathrm{min}^{-1}$

E $120\ \mathrm{min}^{-1}$

WORKED SOLUTION

Let the background rate be B. Over equal 12-hour intervals, the source count rates form a geometric sequence, so

$$(200 - B)^2 = (680 - B)(80 - B).$$

Expanding gives $B = 40\ \mathrm{min}^{-1}$.

The source rates at 0, 12 and 24 hours are therefore $640, 160, 40$, so the source falls by a factor of 4 every 12 hours. Its half-life is 6 hours.

After 30 hours, five half-lives have elapsed, so the source rate is $640/2^5 = 20\ \mathrm{min}^{-1}$. Adding background gives

$$20 + 40 = \boxed{60\ \mathrm{min}^{-1}}.$$

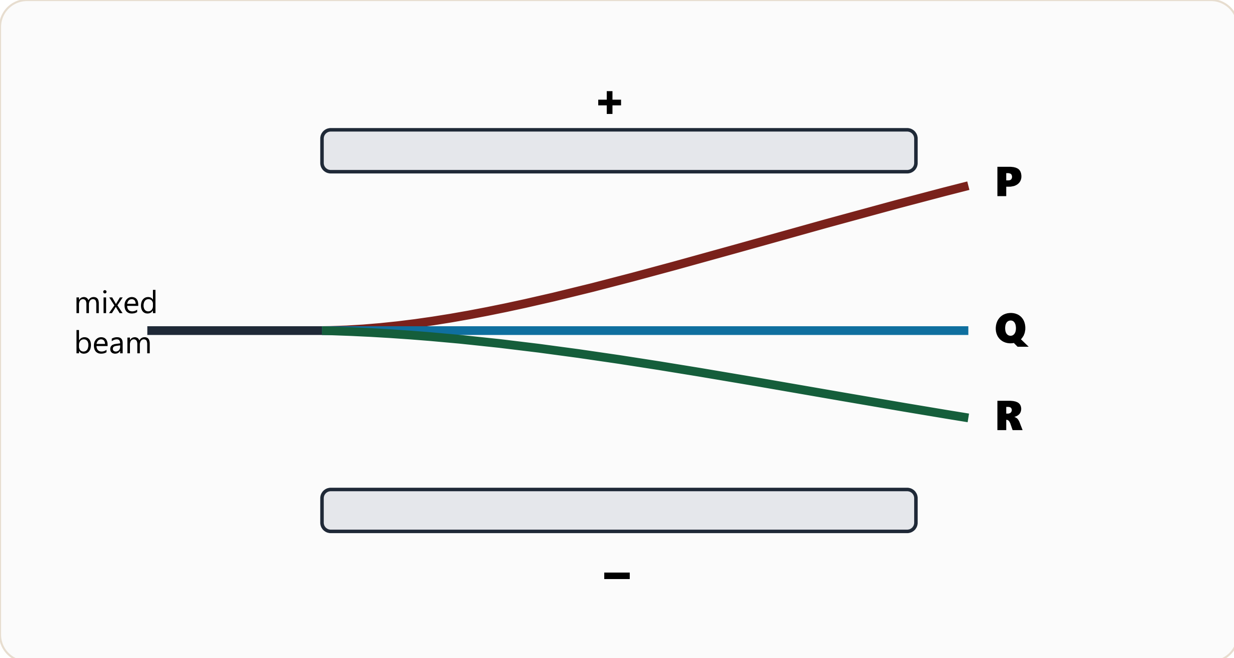
P-26

Ionising radiation

Answer B

Question 26

A narrow beam containing alpha particles, beta-minus particles and gamma radiation passes between charged parallel plates. The upper plate is positive and the lower plate is negative.



A $5\ \mathrm{mm}$ aluminium sheet is then placed after the plates.

Which identification and prediction are correct?

A $P = \alpha,\ Q = \gamma,\ R = \beta$; only Q reaches the detector.

B $P = \beta,\ Q = \gamma,\ R = \alpha$; only Q reaches the detector.

C $P = \beta,\ Q = \alpha,\ R = \gamma$; P and R reach the detector.

D $P = \gamma,\ Q = \beta,\ R = \alpha$; only P reaches the detector.

E $P = \alpha,\ Q = \beta,\ R = \gamma$; Q and R reach the detector.

WORKED SOLUTION

Beta-minus particles are negatively charged, so they are deflected towards the positive upper plate. Their small mass gives the greatest deflection, so P is beta.

Gamma radiation is uncharged, so it is undeflected; Q is gamma.

Alpha particles are positively charged, so they are deflected towards the negative lower plate. Their much larger mass gives a smaller deflection; R is alpha.

A $5\ \mathrm{mm}$ aluminium sheet stops alpha and beta radiation, while gamma radiation is much more penetrating. Thus only path Q continues to the detector.

The answer is $\boxed{B}$.

P-27

Radiation pressure, energy and kinematics

Answer **D**

Question 27

A spacecraft of mass $1.5\ \mathrm{kg}$ carries a solar sail of area $1000\ \mathrm{m}^2$. The radiation pressure on the sail is constant at $3.0 \times 10^{-6}\ \mathrm{N\,m^{-2}}$.

The spacecraft is already moving in the direction of the force at $2.0\ \mathrm{km\,s^{-1}}$. Ignore all other forces and assume the mass remains constant.

How far does the spacecraft travel while its kinetic energy doubles?

- **A** $2.5 \times 10^8\ \mathrm{m}$
- **B** $5.0 \times 10^8\ \mathrm{m}$
- **C** $2.0 \times 10^9\ \mathrm{m}$
- **D** $1.0 \times 10^9\ \mathrm{m}$
- **E** $4.0 \times 10^9\ \mathrm{m}$

WORKED SOLUTION

The force is

$$F = pA = (3.0 \times 10^{-6})(1000) = 3.0 \times 10^{-3}\ \mathrm{N},$$

so the acceleration is

$$a = F/m = 2.0 \times 10^{-3}\ \mathrm{m\,s^{-2}}.$$

Doubling kinetic energy doubles v^2. Thus, if the initial speed is $u = 2000\ \mathrm{m\,s^{-1}}$, then $v^2 - u^2 = u^2 = 4.0 \times 10^6\ \mathrm{m^2s^{-2}}$.

Using $v^2 - u^2 = 2as$,

$$s = \frac{4.0 \times 10^6}{2(2.0 \times 10^{-3})} = \boxed{1.0 \times 10^9\ \mathrm{m}}.$$

ANSWER KEY

ESAT Mock Test — Mathematics 2

Hard ESAT-style Mathematics 2 questions designed to reward structural insight, domain awareness and efficient exact reasoning.

Mathematics 2 answer key

1 D	2 C	3 B	4 A	5 E	6 A	7 E	8 D	9 B
10 C	11 C	12 B	13 E	14 A	15 E	16 B	17 C	18 D
19 A	20 B	21 D	22 C	23 B	24 C	25 D	26 E	27 A

ESAT MOCK TEST

Mathematics 2

Twenty-seven challenging questions designed to reward structural insight, domain awareness and efficient exact reasoning rather than long routine calculation.

M2-01

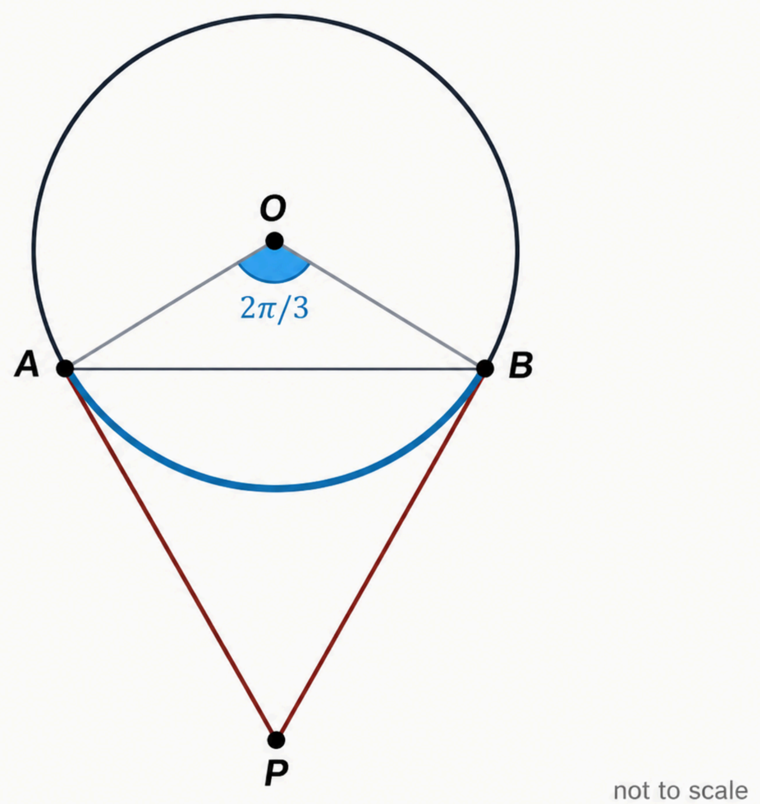
Radians, sectors and exact area

Answer D

Question 1

A circle has centre O and radius 2. The chord AB subtends an angle $\frac{2\pi}{3}$ at O.

An equilateral triangle ABP is constructed on the side of AB opposite O. Let R be the region bounded by the minor arc AB and the line segments AP and BP.



What is the exact area of R?

A $\frac{4\pi}{3} - 2\sqrt{3}$

B $\frac{2\pi}{3} + 2\sqrt{3}$

C $\frac{4\pi}{3} + \sqrt{3}$

D $4\sqrt{3} - \frac{4\pi}{3}$

E $2\pi + \sqrt{3}$

WORKED SOLUTION

The chord length is

$$AB = 2(2)\sin\left(\frac{\pi}{3}\right) = 2\sqrt{3}.$$

Hence the area of equilateral triangle ABP is

$$\frac{\sqrt{3}}{4}(2\sqrt{3})^2 = 3\sqrt{3}.$$

The area of sector AOB is

$$\frac{1}{2}(2^2)\left(\frac{2\pi}{3}\right) = \frac{4\pi}{3}.$$

The area of triangle AOB is

$$\frac{1}{2}(2^2)\sin\left(\frac{2\pi}{3}\right) = \sqrt{3}.$$

Therefore the minor segment between chord AB and the minor arc has area

$$\frac{4\pi}{3} - \sqrt{3}.$$

This minor circular segment lies inside the equilateral triangle. Hence region R is the area of the equilateral triangle with this segment removed.

$$\begin{aligned}\text{area}(R) &= 3\sqrt{3} - \left(\frac{4\pi}{3} - \sqrt{3}\right)\\ &= 4\sqrt{3} - \frac{4\pi}{3}.\end{aligned}$$

Therefore

$$\boxed{4\sqrt{3} - \frac{4\pi}{3}}.$$

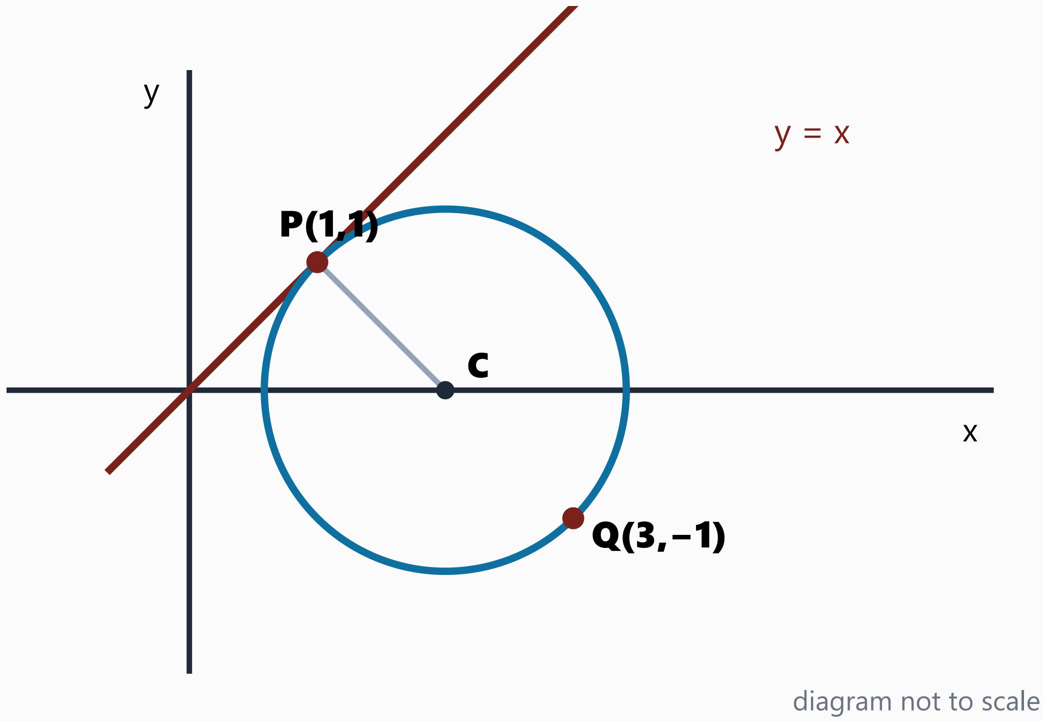
M2-02

Coordinate geometry of a circle

Answer C

Question 2

A circle is tangent to the line $y = x$ at $P = (1, 1)$ and passes through $Q = (3, -1)$.



What is the smaller x-coordinate at which the circle meets the x-axis?

A $1 - \sqrt{2}$

B $\sqrt{2}$

C $2 - \sqrt{2}$

D $2 + \sqrt{2}$

E $3 - \sqrt{2}$

WORKED SOLUTION

The radius at P is perpendicular to the tangent $y = x$, so the centre $C = (h, k)$ lies on the line through P with gradient -1:

$$k - 1 = -(h - 1), \qquad \text{so} \qquad h + k = 2.$$

Since P and Q lie on the same circle, $CP = CQ$:

$$(h - 1)^2 + (k - 1)^2 = (h - 3)^2 + (k + 1)^2.$$

Simplifying gives $h - k = 2$. Solving with $h + k = 2$,

$$h = 2, \qquad k = 0.$$

The radius squared is

$$(2-1)^2+(0-1)^2=2.$$

Thus the circle is

$$(x-2)^2+y^2=2.$$

On the x-axis, $y=0$, so $x=2\pm\sqrt{2}$. The smaller value is

$$\boxed{2-\sqrt{2}}.$$

M2-03

Prisms, volume and lateral area

Answer B

Question 3

A right prism S has a square base of side a and height h_S. A right prism H has a regular hexagonal base of side b and height h_H.

The prisms have equal volumes and equal lateral surface areas.

What is $\dfrac{h_H}{h_S}$?

- A $\frac{\sqrt{3}}{3}$
- B $\frac{2\sqrt{3}}{3}$
- C $\sqrt{3}$
- D $\frac{3\sqrt{3}}{2}$
- E $2\sqrt{3}$

WORKED SOLUTION

Equal lateral surface areas give

$$4ah_S = 6bh_H,$$

so

$$h_H = \frac{2a}{3b}h_S.$$

The area of a regular hexagon of side b is $\frac{3\sqrt{3}}{2}b^2$. Equal volumes therefore give

$$a^2h_S = \frac{3\sqrt{3}}{2}b^2h_H.$$

Substituting the expression for h_H,

$$a^2h_S = \frac{3\sqrt{3}}{2}b^2\left(\frac{2a}{3b}h_S\right) = \sqrt{3}\,ab\,h_S.$$

Hence $a = \sqrt{3}\,b$. Therefore

$$\frac{h_H}{h_S} = \frac{2a}{3b} = \boxed{\frac{2\sqrt{3}}{3}}.$$

M2-04

Three-dimensional cutting

Answer A

Question 4

A cube is cut by two planes parallel to each pair of opposite faces. The six planes divide the cube into 27 cuboids. The planes need not be equally spaced.

What is the ratio

$$\frac{\text{sum of the surface areas of all 27 cuboids}}{\text{surface area of the original cube}}\,?$$

- **A** 3
- **B** $\frac{3}{2}$
- **C** 2
- **D** $\frac{5}{2}$
- **E** 4

WORKED SOLUTION

Let the cube have side length s, so its original surface area is $6s^2$.

Each cutting plane has area s^2. A cut creates two new exposed faces, so each plane adds $2s^2$ to the total surface area of all pieces.

There are 6 cutting planes, so the added area is

$$6(2s^2) = 12s^2.$$

The total surface area is therefore

$$6s^2 + 12s^2 = 18s^2.$$

The required ratio is

$$\frac{18s^2}{6s^2} = \boxed{3}.$$

The spacing of the planes is irrelevant because every complete cut has area s^2.

M2-05

Cone geometry and cosine rule

Answer E

Question 5

A right circular cone has height $2r$ and base radius r. Let θ be the angle at the apex in an axial cross-section of the cone.

What is $\cos\theta$?

A $\frac{1}{5}$

B $\frac{1}{2}$

C $\frac{4}{5}$

D $\frac{\sqrt{5}}{3}$

E $\frac{3}{5}$

WORKED SOLUTION

The axial cross-section is an isosceles triangle with base $2r$, height $2r$, and equal sides

$$\sqrt{(2r)^2 + r^2} = \sqrt{5}\,r.$$

Applying the cosine rule to the side opposite θ,

$$(2r)^2 = (\sqrt{5}r)^2 + (\sqrt{5}r)^2 - 2(\sqrt{5}r)(\sqrt{5}r)\cos\theta.$$

Thus

$$4r^2 = 10r^2 - 10r^2\cos\theta,$$

so

$$\boxed{\cos\theta = \frac{3}{5}}.$$

M2-06

Non-linear scale and radians

Answer A

Question 6

On a semicircular speedometer, the angle θ moved by the pointer from the zero mark is proportional to $\sqrt{v}$, where v is the speed.

The pointer moves through π radians when $v = 200$.

Through what angle does the pointer move as the speed increases from 50 to 128?

A $\frac{3\pi}{10}$

B $\frac{\pi}{3}$

C $\frac{2\pi}{5}$

D $\frac{\pi}{2}$

E $\frac{3\pi}{5}$

WORKED SOLUTION

Since $\theta \propto \sqrt{v}$ and $\theta = \pi$ at $v = 200$,

$$\theta(v) = \pi\sqrt{\frac{v}{200}}.$$

At $v = 50$,

$$\theta_{50} = \pi\sqrt{\frac{50}{200}} = \frac{\pi}{2}.$$

At $v = 128$,

$$\theta_{128} = \pi\sqrt{\frac{128}{200}} = \pi\sqrt{\frac{16}{25}} = \frac{4\pi}{5}.$$

The increase is

$$\frac{4\pi}{5} - \frac{\pi}{2} = \boxed{\frac{3\pi}{10}}.$$

M2-07

Similar rhombi and exact trigonometry

Answer **E**

Question 7

Rhombi A and B are similar. Their areas are in the ratio

$$A : B = 16 : 25.$$

The sum of the two diagonals of A is equal to the longer diagonal of B.

If θ is the acute angle of either rhombus, what is $\cos\theta$?

A $\frac{7}{17}$

B $\frac{8}{17}$

C $\frac{12}{17}$

D $\frac{13}{17}$

E $\frac{15}{17}$

WORKED SOLUTION

The linear scale factor from A to B is

$$\sqrt{\frac{25}{16}} = \frac{5}{4}.$$

Let the shorter and longer diagonals of A be p and q. The longer diagonal of B is $\frac{5}{4}q$. Therefore

$$p + q = \frac{5}{4}q,$$

so $p = \frac{1}{4}q$.

The diagonals of a rhombus bisect each other at right angles. If its side length is s, then

$$s^2 = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 = \frac{p^2 + q^2}{4}.$$

Using the cosine rule on the triangle formed by two adjacent sides and the shorter diagonal,

$$p^2 = 2s^2(1 - \cos\theta).$$

Hence

$$\cos\theta = 1 - \frac{p^2}{2s^2} = 1 - \frac{2p^2}{p^2 + q^2} = \frac{q^2 - p^2}{q^2 + p^2}.$$

With $p/q = 1/4$,

$$\cos\theta = \frac{1 - \frac{1}{16}}{1 + \frac{1}{16}} = \boxed{\frac{15}{17}}.$$

M2-08

Functions and one-to-one restrictions

Answer D

Question 8

Consider

$$f(x) = \log_2(2x^2).$$

Which restriction of the domain makes f one-to-one and strictly increasing, with range $[3, \infty)$?

- **A** $x \le -2$
- **B** $|x| \ge 2$
- **C** $x \ge 1$
- **D** $x \ge 2$
- **E** $0 < x \le 2$

WORKED SOLUTION

For $x > 0$, both x^2 and $\log_2(2x^2)$ increase as x increases. Therefore a positive half-line gives an increasing one-to-one restriction.

To make the least output equal to 3, the least allowed positive input must satisfy

$$\log_2(2x^2) = 3.$$

Thus $2x^2 = 8$, so $x = 2$ on the positive branch.

Therefore the required restriction is

$$\boxed{x \ge 2}.$$

The restriction $x \le -2$ gives the same range but is strictly decreasing, while $|x| \ge 2$ is not one-to-one.

M2-09

Ordering logarithms

Answer **B**

Question 9

Let

$$p = \log_{1/2} 3, \qquad q = -1, \qquad r = \log_3\left(\frac{1}{2}\right).$$

Which ordering is correct?

- **A** $p < r < q$
- **B** $p < q < r$
- **C** $q < p < r$
- **D** $q < r < p$
- **E** $r < q < p$

WORKED SOLUTION

Let $a = \log_2 3$. Since $3 > 2$, $a > 1$.

Now

$$p = \log_{1/2} 3 = -\log_2 3 = -a,$$

and

$$r = \log_3\left(\frac{1}{2}\right) = -\log_3 2 = -\frac{1}{a}.$$

Since $a > 1$,

$$-a < -1 < -\frac{1}{a}.$$

Therefore

$$\boxed{p < q < r}.$$

M2-10

Compound proportionality

Answer C

Question 10

All quantities are positive. Suppose

$$x \propto y^{2/3}, \qquad y \propto z^{-3/2}, \qquad w \propto x^2\sqrt{z}.$$

By what factor is w multiplied when z is multiplied by 8?

A $\frac{1}{32}$

B $\frac{1}{16}$

C $\frac{1}{16\sqrt{2}}$

D $\frac{1}{8\sqrt{2}}$

E $\frac{1}{8}$

WORKED SOLUTION

From

$$y \propto z^{-3/2},$$

we obtain

$$x \propto y^{2/3} \propto \left(z^{-3/2}\right)^{2/3} = z^{-1}.$$

Therefore

$$w \propto x^2\sqrt{z} \propto z^{-2}z^{1/2} = z^{-3/2}.$$

Multiplying z by 8 multiplies w by

$$8^{-3/2} = \frac{1}{8\sqrt{8}} = \boxed{\frac{1}{16\sqrt{2}}}.$$

M2-11

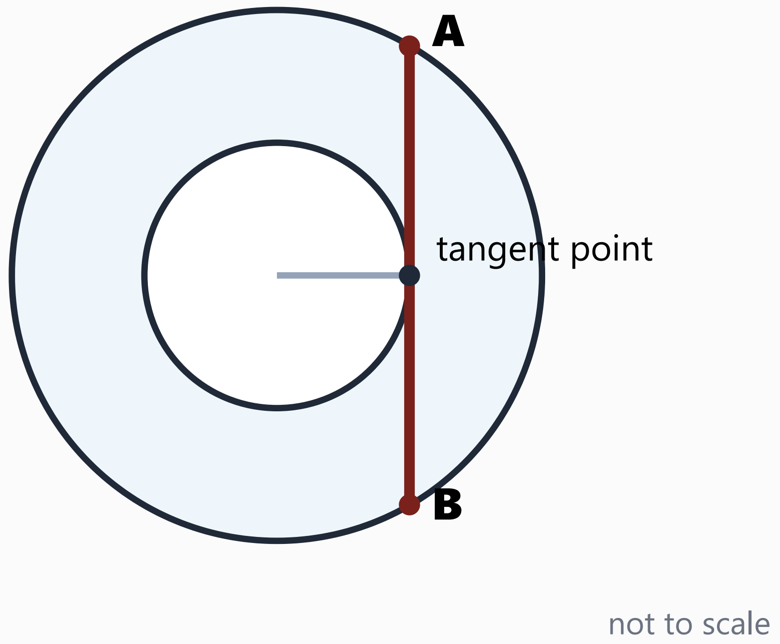
Annulus and chord geometry

Answer C

Question 11

An annulus has inner radius r and outer radius R. Its area is three times the area of the inner circle.

A chord AB of the outer circle is tangent to the inner circle.



What is the ratio

$$\frac{AB}{R-r}?$$

A $\sqrt{2}$

B $\sqrt{3}$

C $2\sqrt{3}$

D 3

E 4

WORKED SOLUTION

The annulus condition gives

$$\pi(R^2 - r^2) = 3\pi r^2.$$

Hence $R^2 = 4r^2$, so $R = 2r$.

The perpendicular from the centre to chord AB meets it at its midpoint. Since the chord is tangent to the inner circle, the distance from the centre to the chord is r.

Therefore half the chord has length

$$\sqrt{R^2 - r^2} = \sqrt{4r^2 - r^2} = \sqrt{3}\,r.$$

Thus

$$AB = 2\sqrt{3}\,r, \qquad R - r = r.$$

The required ratio is

$$\boxed{2\sqrt{3}}.$$

M2-12

Quadratic inequality with a parameter

Answer **B**

Question 12

For a constant k, the solution set of

$$(x-1)(x-4) \le k$$

is a single closed interval of length 6.

What is k?

- **A** $\frac{9}{4}$
- **B** $\frac{27}{4}$
- **C** 9
- **D** $\frac{27}{2}$
- **E** 18

WORKED SOLUTION

Rearrange the inequality:

$$x^2 - 5x + (4-k) \le 0.$$

Because the quadratic opens upwards, the solution interval lies between its two real roots.

The discriminant is

$$\Delta = 25 - 4(4-k) = 9 + 4k.$$

For a monic quadratic, the distance between the roots is $\sqrt{\Delta}$. Hence

$$\sqrt{9+4k} = 6.$$

Therefore

$$9 + 4k = 36, \qquad k = \boxed{\frac{27}{4}}.$$

M2-13

Exponential and quadratic inequalities

Answer E

Question 13

Solve

$$2^{x^2-3x} < 4.$$

A $x < \frac{3-\sqrt{17}}{2}$

B $x > \frac{3+\sqrt{17}}{2}$

C $x < \frac{3-\sqrt{17}}{2}$ or $x > \frac{3+\sqrt{17}}{2}$

D $-2 < x < 1$

E $\frac{3-\sqrt{17}}{2} < x < \frac{3+\sqrt{17}}{2}$

WORKED SOLUTION

Since $4 = 2^2$ and the function 2^t is strictly increasing,

$$x^2 - 3x < 2.$$

Thus

$$x^2 - 3x - 2 < 0.$$

The roots are

$$x = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}.$$

The quadratic opens upwards, so it is negative between its roots. Hence

$$\boxed{\frac{3-\sqrt{17}}{2} < x < \frac{3+\sqrt{17}}{2}}.$$

M2-14

Repeated difference of squares

Answer A

Question 14

Let $a = \sqrt{2}$. Evaluate

$$(a-1)(a+1)(a^2+1)(a^4+1)(a^8+1).$$

- **A** 255
- **B** 127
- **C** 256
- **D** 511
- **E** 1023

WORKED SOLUTION

Repeatedly use the difference-of-squares identity:

$$(a-1)(a+1) = a^2 - 1,$$

so

$$(a^2-1)(a^2+1) = a^4 - 1.$$

Continuing,

$$(a^4-1)(a^4+1) = a^8 - 1,$$

and

$$(a^8-1)(a^8+1) = a^{16} - 1.$$

Since $a = \sqrt{2}$,

$$a^{16} = (\sqrt{2})^{16} = 2^8 = 256.$$

Therefore the product is

$$\boxed{255}.$$

M2-15

Logarithmic equation and domain

Answer E

Question 15

Solve

$$\log_2(x-1)+\log_2(x+3)=3+\log_2 x.$$

A $3-2\sqrt{3}$

B $2+\sqrt{3}$

C $3+\sqrt{3}$

D $4+\sqrt{3}$

E $3+2\sqrt{3}$

WORKED SOLUTION

The logarithms require $x>1$.

Combine the logarithms:

$$\log_2\big((x-1)(x+3)\big)=\log_2(8x).$$

Therefore

$$(x-1)(x+3)=8x.$$

Expanding,

$$x^2+2x-3=8x,$$

so

$$x^2-6x-3=0.$$

Thus

$$x=3\pm 2\sqrt{3}.$$

The value $3-2\sqrt{3}$ is negative and violates $x>1$. Hence

$$\boxed{x=3+2\sqrt{3}}.$$

M2-16

Modulus graph and minimisation

Answer **B**

Question 16

For real x, define

$$g(x) = |2x - 1| + |x + 2|.$$

At which point $(x, g(x))$ does g attain its minimum value?

A $(-2, 5)$

B $(\frac{1}{2}, \frac{5}{2})$

C $(0, 3)$

D $(1, 4)$

E $(-\frac{1}{2}, \frac{7}{2})$

WORKED SOLUTION

The expressions inside the modulus signs change sign at

$$x = -2 \quad \text{and} \quad x = \frac{1}{2}.$$

For $x < -2$,

$$g(x) = -(2x - 1) - (x + 2) = -3x - 1,$$

which decreases as x increases.

For $-2 \le x \le \frac{1}{2}$,

$$g(x) = -(2x - 1) + (x + 2) = -x + 3,$$

which also decreases as x increases.

For $x \ge \frac{1}{2}$,

$$g(x) = (2x - 1) + (x + 2) = 3x + 1,$$

which increases.

Therefore the unique minimum occurs at $x = \frac{1}{2}$, where

$$g\left(\frac{1}{2}\right) = 0 + \frac{5}{2} = \frac{5}{2}.$$

The point is

$$\boxed{\left(\frac{1}{2}, \frac{5}{2}\right)}.$$

M2-17

Intersecting exponential graphs

Answer C

Question 17

The curves

$$y = 2^x - 3 \qquad \text{and} \qquad y = 5 - 2^{\,2-x}$$

intersect at two points whose x-coordinates are x_1 and x_2.

What is $x_1 + x_2$?

A -2

B 1

C 2

D 3

E 4

WORKED SOLUTION

At an intersection,

$$2^x - 3 = 5 - 2^{2-x}.$$

Let $t = 2^x$, so $t > 0$ and $2^{2-x} = 4/t$. Then

$$t - 3 = 5 - \frac{4}{t}.$$

Multiplying by t,

$$t^2 - 8t + 4 = 0.$$

If the two positive roots are t_1, t_2, their product is 4. Since $t_i = 2^{x_i}$,

$$2^{x_1+x_2} = t_1 t_2 = 4 = 2^2.$$

Therefore

$$\boxed{x_1 + x_2 = 2}.$$

M2-18

Trigonometric equation and root counting

Answer D

Question 18

How many solutions does

$$4\cos^2 x + 2\sin x = 3$$

have in the interval $0 \le x < 2\pi$?

A 1

B 2

C 3

D 4

E 6

WORKED SOLUTION

Use $\cos^2 x = 1 - \sin^2 x$. Let $s = \sin x$. Then

$$4(1 - s^2) + 2s = 3,$$

so

$$4s^2 - 2s - 1 = 0.$$

The roots are

$$s = \frac{1 \pm \sqrt{5}}{4}.$$

Both values lie strictly between -1 and 1. For each such value of $\sin x$, there are exactly two solutions in $0 \le x < 2\pi$.

Hence the total number of solutions is

$$\boxed{4}.$$

M2-19

Trigonometric simplification and exclusions

Answer A

Question 19

For real x, consider

$$E(x) = \frac{\cos^4 x - \sin^4 x}{1 - 2\sin^2 x}.$$

Which statement is correct?

- **A** $E(x) = 1$, except when $x = \frac{\pi}{4} + \frac{k\pi}{2}$, $k \in \mathbb{Z}$, where E is undefined.
- **B** $E(x) = 1$ for every real x.
- **C** $E(x) = \cos 2x$ for every real x.
- **D** $E(x) = 1$, except when $x = k\pi$, $k \in \mathbb{Z}$.
- **E** $E(x) = \cos^2 x - \sin^2 x$, except when $x = \frac{\pi}{2} + k\pi$.

WORKED SOLUTION

Factor the numerator:

$$\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x).$$

Since $\cos^2 x + \sin^2 x = 1$,

$$\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x.$$

Thus $E(x) = 1$ wherever the original denominator is non-zero.

The denominator is zero when

$$\sin^2 x = \frac{1}{2},$$

that is, when

$$x = \frac{\pi}{4} + \frac{k\pi}{2}, \qquad k \in \mathbb{Z}.$$

At these values the original expression is undefined, even though the common factor cancels algebraically. The answer is $\boxed{\text{A}}$.

M2-20

Polynomial factorisation and graph behaviour

Answer B

Question 20

Consider

$$P(x) = x^5 - 3x^4 - 2x^3 + 6x^2 + x - 3.$$

At how many distinct points does the graph $y = P(x)$ cross the x-axis?

- A 0
- B 1
- C 2
- D 3
- E 5

WORKED SOLUTION

Factor by grouping:

$$P(x) = x^4(x-3) - 2x^2(x-3) + (x-3).$$

Therefore

$$P(x) = (x-3)(x^4 - 2x^2 + 1) = (x-3)(x^2-1)^2.$$

Hence

$$P(x) = (x-3)(x-1)^2(x+1)^2.$$

The roots $x = 1$ and $x = -1$ have even multiplicity, so the graph touches the axis there without crossing it. The root $x = 3$ has odd multiplicity, so the graph crosses there.

Thus the graph crosses the x-axis at exactly

$$\boxed{1}$$

distinct point.

M2-21

Recurrence and geometric series

Answer D

Question 21

A sequence is defined by

$$u_1 = 1, \qquad u_{n+1} = 2u_n + 3.$$

What is

$$\sum_{n=1}^{6} u_n\ ?$$

A 126

B 189

C 216

D 234

E 252

WORKED SOLUTION

Define $v_n = u_n + 3$. Then

$$v_{n+1} = u_{n+1} + 3 = 2u_n + 6 = 2(u_n + 3) = 2v_n.$$

Also $v_1 = 4$. Hence

$$v_n = 4 \cdot 2^{n-1}, \qquad u_n = 4 \cdot 2^{n-1} - 3.$$

Therefore

$$\sum_{n=1}^{6} u_n = 4 \sum_{n=1}^{6} 2^{n-1} - 18.$$

Using the geometric-series sum,

$$\sum_{n=1}^{6} 2^{n-1} = 2^6 - 1 = 63.$$

Thus

$$4(63) - 18 = \boxed{234}.$$

M2-22

Arithmetic series constraints

Answer C

Question 22

An arithmetic progression has first term a, common difference d, and fifth term 14.

The sum of its first 10 terms is equal to the sum of its 11th to 15th terms inclusive.

What is the sum of its first 20 terms?

A 450

B 520

C 500

D 540

E 600

WORKED SOLUTION

The first 10 terms have sum

$$S_{10} = 5(2a + 9d) = 10a + 45d.$$

The sum of terms 11 to 15 is

$$\frac{5}{2}\big((a + 10d) + (a + 14d)\big) = 5a + 60d.$$

Equating these sums gives

$$10a + 45d = 5a + 60d,$$

so $a = 3d$.

The fifth term is

$$a + 4d = 7d = 14,$$

hence $d = 2$ and $a = 6$.

Therefore

$$S_{20} = 10(2a + 19d) = 10(12 + 38) = \boxed{500}.$$

M2-23

Binomial expansion by restructuring

Answer **B**

Question 23

What is the coefficient of x^7 in

$$(1+x)^8(1-x)^5\,?$$

A -40

B -20

C 0

D 20

E 40

WORKED SOLUTION

Rewrite the expression as

$$(1+x)^8(1-x)^5 = (1+x)^3(1-x^2)^5.$$

To obtain x^7, the possible combinations are:

- x^1 from $(1+x)^3$ and x^6 from $(1-x^2)^5$;
- x^3 from $(1+x)^3$ and x^4 from $(1-x^2)^5$.

The first contribution is

$$\binom{3}{1}\left[-\binom{5}{3}\right] = 3(-10) = -30.$$

The second contribution is

$$\binom{3}{3}\binom{5}{2} = 1(10) = 10.$$

Therefore the coefficient is

$$-30+10 = \boxed{-20}.$$

M2-24

Definite integration and total area

Answer C

Question 24

What is the total area between the curve

$$y = x(x-2)$$

and the x-axis for $0 \le x \le 4$?

A $\frac{16}{3}$

B $\frac{20}{3}$

C 8

D $\frac{28}{3}$

E $\frac{32}{3}$

WORKED SOLUTION

The curve is below the axis on $0 < x < 2$ and above it on $2 < x \le 4$.

An antiderivative is

$$F(x) = \frac{x^3}{3} - x^2.$$

The signed integral from 0 to 2 is

$$F(2) - F(0) = \frac{8}{3} - 4 = -\frac{4}{3},$$

so the corresponding area is $\frac{4}{3}$.

The integral from 2 to 4 is

$$F(4) - F(2) = \left(\frac{64}{3} - 16\right) - \left(-\frac{4}{3}\right) = \frac{20}{3}.$$

Hence the total area is

$$\frac{4}{3} + \frac{20}{3} = \boxed{8}.$$

M2-25

Differentiation and stationary-point classification

Answer D

Question 25

Consider the curve

$$y = 5x^3 + 27x^2 + 27x.$$

Which statement about its stationary points is correct?

- **A** It has only one stationary point, at $x = -3$, which is a local maximum.
- **B** It has only one stationary point, at $x = -\frac{3}{5}$, which is a local minimum.
- **C** It has a local minimum at $x = -3$ and a local maximum at $x = -\frac{3}{5}$.
- **D** It has a local maximum at $x = -3$ and a local minimum at $x = -\frac{3}{5}$.
- **E** It has local minima at both $x = -3$ and $x = -\frac{3}{5}$.

WORKED SOLUTION

Differentiate term by term:

$$\frac{dy}{dx} = 15x^2 + 54x + 27.$$

Factorising,

$$\frac{dy}{dx} = 3(5x^2 + 18x + 9) = 3(x+3)(5x+3).$$

Thus the stationary points occur at

$$x = -3 \qquad \text{and} \qquad x = -\frac{3}{5}.$$

For $x < -3$, both linear factors are negative, so $dy/dx > 0$. For $-3 < x < -\frac{3}{5}$, $x + 3 > 0$ but $5x + 3 < 0$, so $dy/dx < 0$. Therefore $x = -3$ is a local maximum.

For $x > -\frac{3}{5}$, both factors are positive, so $dy/dx > 0$. Therefore $x = -\frac{3}{5}$ is a local minimum.

The answer is $\boxed{\text{D}}$.

M2-26

Trapezium rule and exact comparison

Answer E

Question 26

The trapezium rule with two equal strips is used to estimate

$$\int_0^2 x^4\,dx.$$

Which option gives the ratio

$$\frac{\text{trapezium-rule estimate}}{\text{exact integral}}$$

and correctly states whether the estimate is an overestimate or an underestimate?

- **A** $\frac{32}{45}$, underestimate
- **B** $\frac{45}{32}$, underestimate
- **C** $\frac{32}{45}$, overestimate
- **D** $\frac{5}{4}$, overestimate
- **E** $\frac{45}{32}$, overestimate

WORKED SOLUTION

The strip width is 1. The trapezium-rule estimate is

$$T = \frac{1}{2}[f(0) + 2f(1) + f(2)] = \frac{1}{2}(0 + 2 + 16) = 9.$$

The exact integral is

$$\int_0^2 x^4\,dx = \left[\frac{x^5}{5}\right]_0^2 = \frac{32}{5}.$$

Therefore

$$\frac{T}{\text{exact}} = \frac{9}{32/5} = \boxed{\frac{45}{32}}.$$

Since this ratio is greater than 1, the estimate is an overestimate. This is also consistent with the curve being convex upwards on the interval.

M2-27

Differential equation and stationary points

Answer A

Question 27

A curve satisfies

$$\frac{dy}{dx} = (x-1)(x-3)$$

and passes through the origin.

What is the vertical distance between its two stationary points?

A $\frac{4}{3}$

B $\frac{1}{3}$

C $\frac{2}{3}$

D 1

E $\frac{8}{3}$

WORKED SOLUTION

Expand and integrate:

$$\frac{dy}{dx} = x^2 - 4x + 3,$$

so

$$y = \frac{x^3}{3} - 2x^2 + 3x + C.$$

The curve passes through $(0,0)$, hence $C = 0$.

The stationary points occur where

$$(x-1)(x-3) = 0,$$

so $x = 1$ and $x = 3$.

The corresponding y-coordinates are

$$y(1) = \frac{1}{3} - 2 + 3 = \frac{4}{3},$$

and

$$y(3) = 9 - 18 + 9 = 0.$$

The vertical distance is therefore

$$\boxed{\frac{4}{3}}.$$