

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION

**OFFICIAL
2022**

D513/01

75 minutes

Solution Book

- All Topics

ThrivingScholars 

Paper 1

1 How many real solutions are there to the equation

$$2 \cos^4 \theta - 5 \cos^2 \theta + 3 = 0$$

in the interval $0 \leq \theta \leq 2\pi$?

A 1

B 2

C 3

D 4

E 5

F 6

G 7

H 8

This equation is a quadratic in $\cos^2 \theta$, and it factorises as

$$(2 \cos^2 \theta - 3)(\cos^2 \theta - 1) = 0.$$

Therefore $2 \cos^4 \theta - 5 \cos^2 \theta + 3 = 0$ if and only if $\cos^2 \theta = \frac{3}{2}$ or $\cos^2 \theta = 1$. The first equation has no real solutions, so the original equation holds if and only if $\cos^2 \theta = 1$, or $\cos \theta = \pm 1$.

$\cos \theta = 1$ has the solutions $\theta = 0$ and $\theta = 2\pi$ in the interval $0 \leq \theta \leq 2\pi$, and $\cos \theta = -1$ has the solution $\theta = \pi$ in this interval.

Therefore the original equation has three real solutions in the interval, and the correct option is C.

- 2 Find the complete set of values of p for which the equation

$$x^2 - 2px + y^2 - 6y - p^2 + 8p + 9 = 0$$

describes a circle in the xy -plane.

- A $p < -\frac{9}{4}$
- B $0 < p < 4$
- C $-1 < p < 9$
- D** $p < 0$ or $p > 4$
- E $p < -1$ or $p > 9$
- F all real values of p

We rewrite the equation in completed square form: starting from

$$x^2 - 2px + y^2 - 6y - p^2 + 8p + 9 = 0,$$

completing the square for x and y gives

$$(x - p)^2 - p^2 + (y - 3)^2 - 9 - p^2 + 8p + 9 = 0,$$

which simplifies to

$$(x - p)^2 + (y - 3)^2 - 2p^2 + 8p = 0$$

or

$$(x - p)^2 + (y - 3)^2 = 2p^2 - 8p.$$

This is a circle with squared radius $2p^2 - 8p$, as long as this is positive. So the complete set of values of p for which the original equation describes a circle in the xy -plane are those for which $2p^2 - 8p > 0$, or $p^2 - 4p > 0$.

This factorises as $p(p - 4) > 0$, which has roots $p = 0$, $p = 4$, so the complete set of values in which the inequality holds is $p < 0$ or $p > 4$, which is option D.

3 Given the following statements about a function f

- $f''(x) = a$ for all x
- $f(0) = 1, f(1) = 2$
- $\int_0^1 f(x) dx = 1$

find the value of a .

- A -6
- B -3
- C -2
- D 2
- E 3
- F 6**

Using the first statement, $f''(x) = a$ for all x , we can integrate to get $f'(x) = ax + b$ for some constant b , and a further integration gives $f(x) = \frac{1}{2}ax^2 + bx + c$ for some constant c .

The second statement then yields $f(0) = c = 1$ and $f(1) = \frac{1}{2}a + b + 1 = 2$, so $a + 2b = 2$.

Finally, we can now use the third statement:

$$\begin{aligned}\int_0^1 f(x) dx &= \left[\frac{1}{6}ax^3 + \frac{1}{2}bx^2 + x \right]_0^1 \\ &= \frac{1}{6}a + \frac{1}{2}b + 1 \\ &= 1\end{aligned}$$

so that $a + 3b = 0$.

Combining the simultaneous equations

$$\begin{aligned}a + 2b &= 2 \\ a + 3b &= 0\end{aligned}$$

gives $b = -2$ and $a = 6$, so the correct answer is option F.

4



These sectors of circles are similar.

The arc length of the smaller sector is 6.

The difference between the areas of the sectors is 21.

Find the positive difference between the perimeters of the sectors.

A 4.5

B 7

C 8

D 9

E 10.5

F 14

G 15

Suppose that the angle of the sector is θ (in radians). Then the arc length of the smaller sector is $r\theta$ and the area is $\frac{1}{2}r^2\theta$.

For the smaller sector, this gives $r\theta = 6$, so $\theta = 6/r$. Therefore the area is $\frac{1}{2}r^2(6/r) = 3r$.

Likewise, since the larger sector has the same angle as the smaller one, its arc length is $(r+3)(6/r)$ and its area is $\frac{1}{2}(r+3)^2(6/r) = 3(r+3)^2/r$.

We are told the difference between the areas, so we have

$$\frac{3(r+3)^2}{r} - 3r = 21.$$

Multiplying this by r and rearranging gives

$$3(r+3)^2 - 3r^2 - 21r = 0$$

which we can expand and simplify to give

$$-3r + 27 = 0,$$

so $r = 9$.

We can now calculate the arc length of the larger sector; it is $(r+3)(6/r) = 8$.

Therefore, the perimeter of the first sector is $2r + 6 = 24$ and the perimeter of the second sector is $2(r+3) + 8 = 32$. The positive difference is 8, which is option C.

5 The terms x_n of a sequence follow the rule

$$x_{n+1} = \frac{x_n + p}{x_n + q}$$

where p and q are real numbers.

Given that $x_1 = 3$, $x_2 = 5$, and $x_3 = 7$, find the value of x_4

- A -5
- B 5
- C $\frac{51}{7}$
- D $\frac{15}{2}$
- E $\frac{23}{3}$
- F 9
- G 11
- H** 13

We write out the formulae for x_2 , x_3 and x_4 in terms of p and q , using the values $x_1 = 3$, $x_2 = 5$ and $x_3 = 7$:

$$n = 1 : \quad 5 = \frac{3 + p}{3 + q}$$

$$n = 2 : \quad 7 = \frac{5 + p}{5 + q}$$

$$n = 3 : \quad x_4 = \frac{7 + p}{7 + q}$$

Multiplying the first equation by $3 + q$ and the second by $5 + q$ gives

$$15 + 5q = 3 + p$$

$$35 + 7q = 5 + p$$

These are simultaneous equations for p and q ; subtracting gives $20 + 2q = 2$ so $q = -9$ and therefore $p = -33$.

Substituting these into the expression for x_4 gives

$$x_4 = \frac{7 - 33}{7 - 9} = 13$$

so the answer is option H.

6 Given that

$$\int_{\log_2 5}^{\log_2 20} x \, dx = \log_2 M$$

what is the value of M ?

- A 4
- B 15
- C 16
- D 20
- E 25
- F 100**
- G 10000

We calculate the integral and use the laws of logarithms; we obtain

$$\begin{aligned} \int_{\log_2 5}^{\log_2 20} x \, dx &= \left[\frac{1}{2} x^2 \right]_{\log_2 5}^{\log_2 20} \\ &= \frac{1}{2} (\log_2 20)^2 - \frac{1}{2} (\log_2 5)^2 \\ &= \frac{1}{2} ((\log_2 20)^2 - (\log_2 5)^2) \\ &= \frac{1}{2} ((\log_2 4 + \log_2 5)^2 - (\log_2 5)^2) \\ &= \frac{1}{2} ((2 + \log_2 5)^2 - (\log_2 5)^2) \\ &= \frac{1}{2} (4 + 4 \log_2 5 + (\log_2 5)^2 - (\log_2 5)^2) \\ &= \frac{1}{2} (4 + 4 \log_2 5) \\ &= 2 + 2 \log_2 5 \\ &= \log_2 4 + \log_2 25 \\ &= \log_2 100 \end{aligned}$$

so the answer is option F.

7 Find the finite area enclosed between the line $y = 0$ and the curve $y = x^2 - 4|x| - 12$

A $\frac{128}{3}$

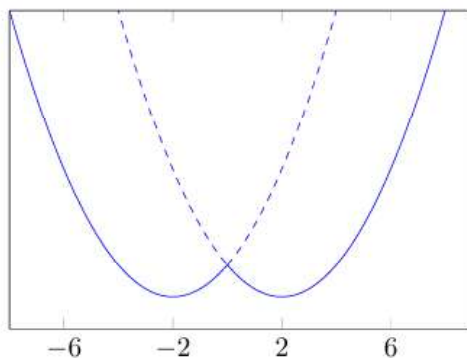
B $\frac{176}{3}$

C $\frac{256}{3}$

D 108

E 144

F 288



The finite area enclosed between the x -axis and the graph $y = x^2 - 4|x| - 12$ is therefore between $x = -6$ and $x = 6$. We could calculate the integrals of the two quadratics, one between $x = -6$ and $x = 0$ and the other between $x = 0$ and $x = 6$. But we could also notice that the graph is symmetrical about the y -axis, so we only need to find one of these and double the answer.

We have

$$\begin{aligned}\int_0^6 x^2 - 4x - 12 \, dx &= \left[\frac{x^3}{3} - 2x^2 - 12x \right]_0^6 \\ &= \frac{6^3}{3} - 2 \times 6^2 - 12 \times 6 - 0 \\ &= -72\end{aligned}$$

so $\int_{-6}^6 x^2 - 4|x| - 12 \, dx = -144$ and the (unsigned) area required is 144, which is option E.

- 8 A geometric sequence has first term a and common ratio r , where a and r are positive integers and r is greater than 1.

The sum of the first n terms of this sequence is denoted by S_n

It is given that the terms of the sequence satisfy

$$S_{30} - S_{20} = kS_{10}$$

for some positive integer k .

What is the smallest possible value of k ?

- A 2^{10}
B 2^{20}
C 2^{30}
D $\frac{2^{10}}{2^{10} - 1}$
E $2^{10}(2^{10} - 1)$

We use the formula $S_n = \frac{a(r^n - 1)}{r - 1}$ and substitute into the given equation:

$$\frac{a(r^{30} - 1)}{r - 1} - \frac{a(r^{20} - 1)}{r - 1} = k \frac{a(r^{10} - 1)}{r - 1}.$$

We can multiply this by $r - 1$ and divide by a to get

$$(r^{30} - 1) - (r^{20} - 1) = k(r^{10} - 1),$$

which simplifies to

$$r^{30} - r^{20} = k(r^{10} - 1).$$

Factorising the left hand side then gives

$$r^{20}(r^{10} - 1) = k(r^{10} - 1);$$

as $r > 1$, we can divide by $r^{10} - 1 \neq 0$ to obtain

$$r^{20} = k.$$

Therefore k is a 20th power, and the smallest possible value is 2^{20} , which is option B.

- 9 This question is about pairs of functions f and g that satisfy

$$f(x) - g(x) = 2 \sin x$$

$$f(x)g(x) = \cos^2 x$$

for all real numbers x .

Across all solutions for $f(x)$, what is the minimum value that $f(x)$ attains for any x ?

- A $1 - \sqrt{2}$
- B $-1 - \sqrt{2}$
- C 0
- D -1
- E -2**
- F -3
- G -4

It may well require reading this question a couple of times to understand what it is asking! It seems like we need to find all possible functions $f(x)$ and $g(x)$, then for each one, to find the minimum value that $f(x)$ takes, and then to find the minimum of all the minima. There might, of course, be some shortcuts!

Let us try to solve the equations to find $f(x)$. The first equation rearranges to give $g(x) = f(x) - 2 \sin x$, so we can substitute this into the second equation to give

$$f(x)(f(x) - 2 \sin x) = \cos^2 x.$$

This is a quadratic in $f(x)$, so we can rearrange it into the standard form and use the quadratic formula:

$$(f(x))^2 - (2 \sin x)f(x) - \cos^2 x = 0,$$

hence

$$\begin{aligned} f(x) &= \frac{2 \sin x \pm \sqrt{4 \sin^2 x + 4 \cos^2 x}}{2} \\ &= \sin x \pm 1. \end{aligned}$$

So there are two possible functions $f(x)$, namely $f(x) = \sin x + 1$ and $f(x) = \sin x - 1$. The minimum value taken by these for any x is 0 for the first function and -2 for the second, so the minimum possible value is -2 , which is option E.

10 A sequence of translations is applied to the graph of $y = x^3$

Which of the following graphs could be the result of this sequence of translations?

I $y = x^3 - 3x^2 + 9x - 27$

II $y = x^3 - 9x^2 + 27x - 3$

III $y = 27x^3 - 9x^2 + x - 3$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Any sequence of translations is equivalent to a translation in the x -direction followed by a translation in the y -direction (where the translations could be zero).

Translating $y = x^3$ by a in the x -direction gives the graph

$$y = (x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3.$$

Then translating by b in the y -direction gives the graph

$$y = x^3 - 3ax^2 + 3a^2x - a^3 + b.$$

Now considering the three graphs:

I The x^2 term is $-3x^2$, which means we require $a = 1$. Substituting this into the above general equation gives $y = x^3 - 3x^2 + 3x - 1 + b$, which does not match the given equation. So this is not a translation of $y = x^3$.

II The x^2 term is $-9x^2$, so we require $a = 3$. Substituting gives $y = x^3 - 9x^2 + 27x - 27 + b$, which matches the equation with $b = 24$, so this is a translation of $y = x^3$.

III The x^3 term does not match here, so this is not a translation of $y = x^3$.

Therefore the answer is option C, II only.

11 Evaluate

$$\sum_{n=1}^{100} \log_{10}(3^{1-n})$$

- A** $-4950 \log_{10} 3$
- B** $4950 \log_{10} 3$
- C** $-5050 \log_{10} 3$
- D** $5050 \log_{10} 3$
- E** $1 - 4950 \log_{10} 3$
- F** $1 + 4950 \log_{10} 3$
- G** $1 - 5050 \log_{10} 3$
- H** $1 + 5050 \log_{10} 3$

We have $\log_{10}(3^{1-n}) = (1-n) \log_{10} 3$, so the sum becomes

$$\begin{aligned} \sum_{n=1}^{100} \log_{10}(3^{1-n}) &= \sum_{n=1}^{100} (1-n) \log_{10} 3 \\ &= (\log_{10} 3) \sum_{n=1}^{100} (1-n). \end{aligned}$$

Now the sum is actually an arithmetic series: expanding it gives

$$\begin{aligned} \sum_{n=1}^{100} (1-n) &= 0 + (-1) + (-2) + \cdots + (-99) \\ &= -(0 + 1 + 2 + \cdots + 99) \\ &= -\frac{1}{2} \times 100(0 + 99 \times 1) \\ &= -4950 \end{aligned}$$

using the formula $S_n = \frac{1}{2}n(2a + (n-1)d)$. (Alternatively, we could have used the formula for the n th triangular number.)

Therefore the original sum evaluates to $-4950 \log_{10} 3$, which is option A.

12 A family of quadratic curves is given by

$$y_k = 2\left(x - \frac{k}{2}\right)^2 + \frac{k^2}{2} + 4k + 3$$

where k is any real number and y_k is a function of x .

All these curves are sketched, and the point with the lowest y -coordinate among all the curves y_k is (a, b) .

Find the value of $a + b$

- A -1
- B -3
- C -5
- D -7**
- E -9

The formula for y_k is given in completed square form. The minimum of the graph $y = a(x-b)^2 + c$, where $a > 0$, is at $x = b$, and the minimum value is c . Therefore the minimum of y_k is at $\left(\frac{k}{2}, \frac{k^2}{2} + 4k + 3\right)$.

The minimum of the minimum y -coordinates is therefore the minimum of $\frac{k^2}{2} + 4k + 3$. We can find this minimum either by completing the square or by differentiating this with respect to k . Let us use the latter method here as it seems simpler. We have

$$\frac{d}{dk} \left(\frac{k^2}{2} + 4k + 3 \right) = k + 4$$

so the minimum occurs at $k = -4$, giving

$$\left(\frac{k}{2}, \frac{k^2}{2} + 4k + 3 \right) = (-2, -5)$$

so that $a + b = -7$, which is option D.

13 Given that

$$\left(a^3 + \frac{2}{b^3}\right)\left(\frac{2}{a^3} - b^3\right) = \sqrt{2}$$

where a and b are real numbers, what is the least value of ab ?

- A** $-\sqrt{2}$
- B** $\sqrt{2}$
- C** $-2\sqrt{2}$
- D** $2\sqrt{2}$
- E** $-\frac{\sqrt{2}}{2}$
- F** $\frac{\sqrt{2}}{2}$
- G** $-2^{\frac{1}{6}}$
- H** $2^{\frac{1}{6}}$

Expanding the brackets gives

$$2 - a^3b^3 + \frac{4}{a^3b^3} - 2 = \sqrt{2}$$

so

$$\frac{4}{a^3b^3} - a^3b^3 = \sqrt{2}.$$

We are interested in the least possible value of ab , so we can find the least possible value of a^3b^3 and take the cube root. Let us simplify the notation by writing $x = a^3b^3$, so $\frac{4}{x} - x = \sqrt{2}$.

Multiplying by x gives a quadratic: $4 - x^2 = \sqrt{2}x$, which we can rearrange to $x^2 + \sqrt{2}x - 4 = 0$ and then solve using the quadratic formula:

$$x = \frac{-\sqrt{2} \pm \sqrt{2+16}}{2} = \frac{-\sqrt{2} \pm 3\sqrt{2}}{2} = -2\sqrt{2} \text{ or } \sqrt{2}.$$

Since we want the least possible value of x , we take $x = a^3b^3 = -2\sqrt{2} = -2^{\frac{3}{2}}$, and so the minimum possible value of ab is $(-2^{\frac{3}{2}})^{\frac{1}{3}} = -2^{\frac{1}{2}} = -\sqrt{2}$, which is option A.

14 A circle has centre O and radius 6.

P , Q and R are points on the circumference with angle $POQ \geq \frac{\pi}{2}$

The area of the triangle POQ is $9\sqrt{3}$

What is the greatest possible area of triangle PRQ ?

A $18 + 9\sqrt{3}$

D $27\sqrt{3}$

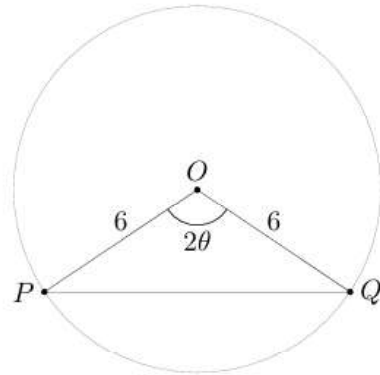
B $18\sqrt{3}$

E $36 + 9\sqrt{3}$

C $27 + 9\sqrt{3}$

F $36\sqrt{3}$

Let us start by drawing the triangle POQ in the circle:



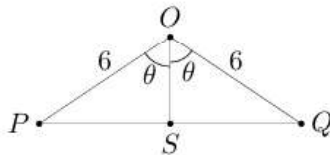
(We have called the angle 2θ as we will shortly want to consider half of the angle.)

We are told that the area of POQ is $9\sqrt{3}$, so we can use the formula for the area of a triangle given two sides and the included angle to determine 2θ : the area formula is $A = \frac{1}{2}ab \sin C$, which in this case gives

$$\frac{1}{2} \times 6 \times 6 \times \sin 2\theta = 9\sqrt{3}$$

so $\sin 2\theta = \frac{\sqrt{3}}{2}$. Since $2\theta \geq \frac{\pi}{2}$, we must have $2\theta = \frac{2}{3}\pi$ (or 120°).

We can now work out the length PQ by dropping a perpendicular from O to PQ :

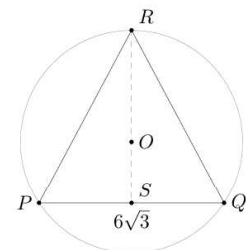


Fixing PQ and allowing R to move, we see that the largest possible area of PQR occurs when R is as far as possible from PQ , as the area of the triangle is $\frac{1}{2} \times \text{base} \times \text{height}$. When R is at the top of the circle (in our diagram), we have this configuration:

We have $RS = RO + OS$. Now $RO = 6$ (it is a radius of the circle) and $OS = 3$ as we calculated earlier, so $RS = 9$. Therefore the maximum possible area of PQR is

$$\frac{1}{2} \times PQ \times RS = \frac{1}{2} \times 6\sqrt{3} \times 9 = 27\sqrt{3},$$

which is option D.



- 15 A rectangle is drawn in the region enclosed by the curves p and q , where

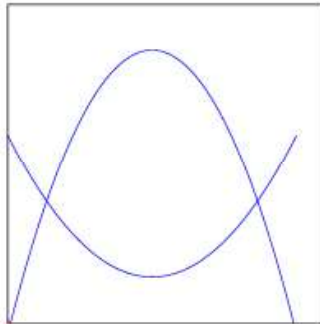
$$p(x) = 8 - 2x^2$$

$$q(x) = x^2 - 2$$

such that the sides of the rectangle are parallel to the x - and y -axes.

What is the maximum possible area of the rectangle?

- A $\frac{26}{9}$
- B $\frac{52}{9}$
- C $\frac{4\sqrt{6}}{3}$
- D $\frac{8\sqrt{6}}{3}$
- E $4\sqrt{2}$
- F $8\sqrt{2}$
- G $\frac{20\sqrt{10}}{9}$
- H** $\frac{40\sqrt{10}}{9}$



Let us suppose that the vertices of the rectangle are at $x = \pm a$, with $0 < a < \sqrt{10/3}$. Then the width of the rectangle is $2a$ and the height is $(8 - 2a^2) - (a^2 - 2) = 10 - 3a^2$. Therefore the area is $A = 2a(10 - 3a^2) = 20a - 6a^3$.

To maximise this area, we differentiate with respect to a to find the stationary points. We have

$$\frac{dA}{da} = 20 - 18a^2$$

so there is a stationary point when this is zero, at $a = \pm\sqrt{20/18} = \pm\sqrt{10/9} = \pm\frac{1}{3}\sqrt{10}$. We are only interested in the positive value, and this is less than $\sqrt{10/3}$, so it is a valid value for our problem. We therefore obtain the maximum possible area: it is

$$\begin{aligned} A &= 2a(10 - 3a^2) \\ &= \frac{2}{3}\sqrt{10}(10 - \frac{1}{3} \times 10) \\ &= \frac{2}{3}\sqrt{10} \times \frac{2}{3} \times 10 \\ &= \frac{40\sqrt{10}}{9} \end{aligned}$$

which is option H.

- 16 The solutions to $7x^4 - 6x^2 + 1 = 0$ are $\pm \cos \theta$ and $\pm \cos \beta$.

Which one of the following equations has solutions $\pm \sin \theta$ and $\pm \sin \beta$?

- A $7x^4 - 8x^2 - 5 = 0$
B $7x^4 - 8x^2 + 2 = 0$
C $7x^4 - 6x^2 - 2 = 0$
D $7x^4 - 6x^2 + 1 = 0$
E $7x^4 + 6x^2 - 1 = 0$
F $7x^4 + 6x^2 + 5 = 0$

The given quartic can be thought of as a quadratic in x^2 , and as such it has roots

$$x^2 = \frac{6 \pm \sqrt{6^2 - 28}}{14} = \frac{3 \pm \sqrt{2}}{7}.$$

Since the roots of the quartic are $\pm \cos \theta$ and $\pm \cos \beta$, these x^2 values are the values of $\cos^2 \theta$ and $\cos^2 \beta$ (in some order).

Now $\sin^2 \theta = 1 - \cos^2 \theta$ and $\sin^2 \beta = 1 - \cos^2 \beta$, so

$$\sin^2 \theta, \sin^2 \beta = 1 - \frac{3 \pm \sqrt{2}}{7} = \frac{4 \mp \sqrt{2}}{7}.$$

Therefore the equations $x^2 - \frac{4+\sqrt{2}}{7} = 0$ and $x^2 - \frac{4-\sqrt{2}}{7} = 0$ have solutions $\pm \sin \theta$ and $\pm \sin \beta$ (again, in some order, and x is not the same as earlier). It follows that the product of these two equations will have the four solutions $\pm \sin \theta$ and $\pm \sin \beta$; this equation is

$$\left(x^2 - \frac{4 + \sqrt{2}}{7}\right) \left(x^2 - \frac{4 - \sqrt{2}}{7}\right) = 0.$$

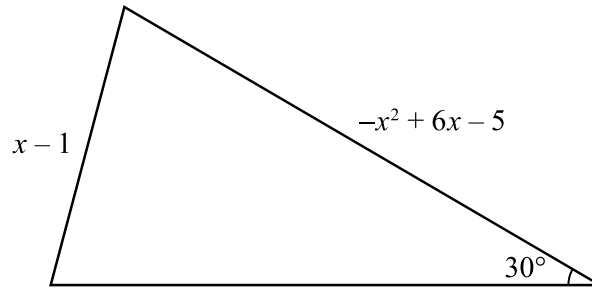
To obtain the form given in the question, we first expand the brackets:

$$x^4 - \left(\frac{4 + \sqrt{2}}{7} + \frac{4 - \sqrt{2}}{7}\right)x^2 + \frac{4 + \sqrt{2}}{7} \cdot \frac{4 - \sqrt{2}}{7} = 0$$

then perform the surd calculations to give

$$x^4 - \frac{8}{7}x^2 + \frac{2}{7} = 0.$$

Finally, multiplying by 7 gives $7x^4 - 8x^2 + 2 = 0$, which is option B.



Find the complete set of values of x for which there are two non-congruent triangles with the side lengths and angle as shown in the diagram.

A $1 < x < 3$

D $3 < x < 4$

B $1 < x < 4$

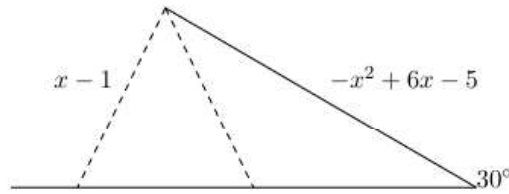
E $3 < x < 5$

C $1 < x < 5$

F $4 < x < 5$

We first require that the lengths shown are both positive; $x - 1 > 0$ if and only if $x > 1$, and $-x^2 + 6x - 5 = -(x - 5)(x - 1) > 0$ if and only if $1 < x < 5$. Therefore we require $1 < x < 5$ to ensure both lengths are positive.

There are two non-congruent triangles when the side marked $x - 1$ can be drawn in two configurations:



This is possible when the following conditions are satisfied:

- $x - 1 < -x^2 + 6x - 5$ (otherwise one of the possibilities will lie to the right of the 30° angle in the above diagram)
- $x - 1 > (-x^2 + 6x - 5) \sin 30^\circ$ (so that the distance from the end of the $-x^2 + 6x - 5$ side to the base is at least $x - 1$, making it possible to form a triangle; if there is equality here, then the dashed circle will be tangent to the base of the triangle, and there will only be one possibility)

The first inequality rearranges as $x^2 - 5x + 4 < 0$, which factorises as $(x - 1)(x - 4) < 0$, giving $1 < x < 4$.

The second equality, using $\sin 30^\circ = \frac{1}{2}$, becomes $2x - 2 > -x^2 + 6x - 5$, so $x^2 - 4x + 3 > 0$. This factorises as $(x - 1)(x - 3) > 0$, so $x < 1$ or $x > 3$.

Requiring both of these inequalities to hold thus gives $3 < x < 4$; this also satisfies the positivity requirement we started with ($1 < x < 5$), and so the answer is option D.

18 It is given that

$$f(x) = x^2(x-1)^2(x-2)$$

$$g(x) = -p(x-q)^2(x-r)^2$$

where p , q and r are positive and $q < r$

Find the set of values of q and r that guarantees the greatest number of distinct real solutions of the equation $f(x) = g(x)$ for all p .

A $q < 1$ and $r < 1$

B $q < 1$ and $1 < r < 2$

C $q < 1$ and $r > 2$

D $1 < q < 2$ and $1 < r < 2$

E $1 < q < 2$ and $r > 2$

F $q > 2$ and $r > 2$

- $f(x) = x^2(x-1)^2(x-2)$ is **positive** for $x < 2$ and **negative** for $x > 2$.
- $g(x) = -p(x-q)^2(x-r)^2 \leq 0$ always.
- To maximize intersections, we want $g(x)$ negative where $f(x)$ is positive (so they cross twice) **and** also intersect where $f(x) < 0$ (cross twice again).
- This happens when $q < 1 < r < 2$:
 - Between q and r , $f(x) > 0$ so there are two crossings.
 - For $x > 2$, $f(x) < 0$ so there can be two more crossings depending on p .
- Other choices give fewer intersections.

Answer: B $q < 1$ and $1 < r < 2$

19 Circle C_1 is defined as $x^2 + y^2 = 25$

A second circle C_2 has radius 4 and centre (a, b) where

$$-2 \leq a \leq 2 \quad \text{and} \quad -3 \leq b \leq 3$$

If the centre of C_2 is equally likely to be located anywhere within the given range, what is the probability that C_2 intersects C_1 ?

- A $\frac{1}{25}$
- B $\frac{9}{25}$
- C $\frac{16}{25}$
- D $\frac{6 - \pi}{6}$
- E $\frac{16 - \pi}{24}$
- F** $\frac{24 - \pi}{24}$

Note first that the region where the centre of C_2 may lie is entirely within the circle C_1 , as $2^2 + 3^2 = 13 < 25$.

As the radius of C_1 is 5, the circle C_2 will intersect C_1 if and only if the centre of C_2 is at least a distance of 1 from the origin (the centre of C_1). This is because in this case, a line from the centre of C_1 passing through the centre of C_2 will reach the circle C_1 before (or as) it reaches the circle C_2 , but a line from the centre of C_1 in the opposite direction will reach C_2 first. If, though, the centre of C_2 is at a distance of less than 1 from the origin, the furthest point on C_2 from the origin will be at a distance of less than $1 + 4 = 5$ from the origin, so C_2 lies entirely inside C_1 .

The area of the allowed (a, b) region is $4 \times 6 = 24$. The (a, b) region where the circles do not intersect is a circle of radius 1, with area π , lying entirely within the allowed region, so the region where the circles do intersect has area $24 - \pi$.

Thus the probability of intersection is $\frac{24 - \pi}{24}$, which is option F.

20 n is the number of points of intersection of the graphs

$$y = |x^2 - a^2| \text{ and } y = a^2|x - 1|$$

where a is a real number.

What is the smallest value of n that is **not** possible?

- A $n = 1$
- B** $n = 2$
- C $n = 3$
- D $n = 4$
- E $n = 5$

- Take $a = 0 \Rightarrow y = x^2, y = 0 \rightarrow$ **1 intersection** ($n=1$ possible).
- Take $a = 1 \Rightarrow y = |x^2 - 1|, y = |x - 1| \rightarrow$ quick sketch shows **3 intersections** ($n=3$ possible).
- Take large $a \rightarrow$ steep V-line meets both parabolas twice \rightarrow **4 or 5 intersections** possible.
- **$n=2$ never occurs** (you can't just hit two points — symmetry forces 0,1,3,4,5).

Answer: B ($n=2$)

Paper 2

1 Determine the number of stationary points on the curve with equation

$$y = 3x^4 + 4x^3 + 6x^2 - 5$$

A 0

B 1

C 2

D 3

E 4

We differentiate to obtain

$$\frac{dy}{dx} = 12x^3 + 12x^2 + 12x.$$

We solve the equation $\frac{dy}{dx} = 0$ to find the number of stationary points:

$$12x^3 + 12x^2 + 12x = 0$$

$$\text{if and only if } 12x(x^2 + x + 1) = 0.$$

So either $x = 0$ or $x^2 + x + 1 = 0$. But this quadratic has discriminant $1^2 - 4 \times 1 = -3 < 0$, so it has no real solutions.

Thus there is only one stationary point, at $x = 0$, and the answer is option B.

2 Find the coefficient of the x^5 term in the expansion of

$$(1+x)^5 \times \sum_{i=0}^5 x^i$$

- A 1
- B 5
- C 16
- D 25
- E 32**

We expand the brackets in the first term using the binomial theorem and write out the sum in full to get

$$(1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5) \times (1 + x + x^2 + x^3 + x^4 + x^5).$$

The x^5 term in the product is obtained by taking all possible pairs of a term in the first bracket and a term in the second whose product is cx^5 for some number c , and then adding these together. Therefore the x^5 term in the product is

$$1 \cdot x^5 + 5x \cdot x^4 + 10x^2 \cdot x^3 + 10x^3 \cdot x^2 + 5x^4 \cdot x + x^5 \cdot 1 = (1 + 5 + 10 + 10 + 5 + 1)x^5 = 32x^5.$$

Therefore the correct answer is option E.

3 Consider the following statement about the positive integer n

if n is prime, then $n^2 + 2$ is not prime

Which of the following is a **counterexample** to this statement?

- I $n = 2$
 - II $n = 3$
 - III $n = 4$
-
- A none of them
 - B I only
 - C** II only
 - D III only
 - E I and II only
 - F I and III only
 - G II and III only
 - H I, II and III

I $n = 2$ is prime and $n^2 + 2 = 6$ is not prime, so this case satisfies the statement and is not a counterexample.

II $n = 3$ is prime but $n^2 + 2 = 11$ is prime, so this is a counterexample as it does not satisfy the statement.

III $n = 4$ is not prime, so it satisfies the statement and is not a counterexample.

Therefore only II provides a counterexample, which is option C.

- 4 The point P has coordinates (p, q) , and the equation of a circle is

$$x^2 + 2fx + y^2 + 2gy + h = 0$$

where f, g, h, p and q are all real constants.

Let L be the distance between the centre of the circle and the point P .

Which one of the following is **sufficient** on its own to be able to calculate L ?

- A the values of f, g and h
- B** the values of f, g, p and q
- C the values of f, h, p and q
- D the values of g, h, p and q
- E none of the options **A-D** is sufficient on its own

We can rewrite the equation of circle by completing the square to find its centre. We get:

$$(x + f)^2 - f^2 + (y + g)^2 - g^2 + h = 0$$

which we can rearrange as

$$(x + f)^2 + (y + g)^2 = f^2 + g^2 - h.$$

Therefore the centre is at $(-f, -g)$ (and the radius is $\sqrt{f^2 + g^2 - h}$, but that is not relevant to us here).

Then the distance between the centre of the circle and the point P is given by

$$L^2 = (p + f)^2 + (q + g)^2$$

so we need the values of f, g, p and q . Hence the correct answer is option B.

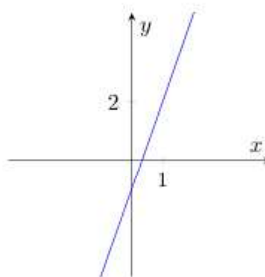
5 A straight line L passes through $(1, 2)$.

Let P be the statement

if the y -intercept of L is negative, **then** the x -intercept of L is positive.

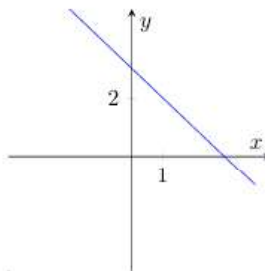
Which of the following statements **must** be true?

- I P
 - II the converse of P
 - III the contrapositive of P
- A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III



It is clear from this sketch that if the line has a negative y -intercept, it must have a positive x -intercept. (A more formal proof goes as follows: since $y < 0$ when $x = 0$ and $y > 0$ when $x = 1$, there must be some value of x with $0 < x < 1$ where $y = 0$. Therefore the x -intercept must lie between 0 and 1.)

II The converse of P reads: If the x -intercept of L is positive, then the y -intercept of L is negative. This is the case if the x -intercept is between 0 and 1, as in the previous sketch, but if the x -intercept is greater than 1, this is no longer true:



So the converse of P is false.

III The contrapositive of P has the same truth value as P itself, so the contrapositive of P is true.

Therefore only I and III are true, and the correct answer is option F.

6 A list consists of n integers.

Consider the following statements:

P: n is odd.

Q: The median of the list is one of the numbers in the list.

Which one of the following is true?

- A P is **necessary and sufficient** for Q.
- B P is **necessary** but **not sufficient** for Q.
- C P is **sufficient** but **not necessary** for Q.
- D P is **not necessary** and **not sufficient** for Q.

Suppose P is true. Since there are an odd number of integers in the list, there is a unique middle one (when they are written in increasing order), so this is the median. Therefore Q is true, and P is sufficient for Q.

Now suppose that Q is true. It does not follow that n is odd; here is a counterexample: if the list is 2, 2, then $n = 2$ and the median is 2, so Q is true, but P is false. (In general, any list with n even and the middle pair of numbers equal is a counterexample.) Therefore P is not necessary for Q.

The correct answer is option C.

7 Consider the following claim:

The difference between two consecutive positive cube numbers is always prime.

Here is an attempted proof of this claim:

- I $(x + 1)^3 = x^3 + 3x^2 + 3x + 1$
- II Taking x to be a positive integer, the difference between two consecutive cube numbers can be expressed as $(x + 1)^3 - x^3 = 3x^2 + 3x + 1$
- III It is impossible to factorise $3x^2 + 3x + 1$ into two linear factors with integer coefficients because its discriminant is negative.
- IV Therefore for every positive integer value of x the integer $3x^2 + 3x + 1$ cannot be factorised.
- V Hence, the difference between two consecutive cube numbers will always be prime.

Which of the following best describes this proof?

- A The proof is completely correct, and the claim is true.
- B The proof is completely correct, but there are counterexamples to the claim.
- C The proof is wrong, and the first error occurs on line I.
- D The proof is wrong, and the first error occurs on line II.
- E The proof is wrong, and the first error occurs on line III.
- F** The proof is wrong, and the first error occurs on line IV.
- G The proof is wrong, and the first error occurs on line V.

Searching for small counterexamples to the claim does not find any; the first few differences are all prime. So we do have to check the proof in detail, as it might be correct.

The algebraic expansion on line I is correct.

The expression in line II is also correct, using the expansion from line I.

For line III, if we could write $3x^2 + 3x + 1 = (ax + b)(cx + d)$ with real coefficients a, b, c and d , then we would have real roots of the equation $3x^3 + 3x + 1 = 0$, namely $x = -\frac{b}{a}$ and $x = -\frac{d}{c}$. But the discriminant is $3^2 - 4 \times 3 \times 1 = -3 < 0$, so this is impossible. So this statement is true.

Line IV is more tricky: just because the algebraic expression is not factorisable into algebraic factors does *not* mean that the integer it represents for any particular value of x is not factorisable in integers. As a simpler example, the algebraic expression $x^2 + 1$ is not factorisable into real algebraic factors, yet substituting integers for x shows that the integer value can sometimes be factorised: taking $x = 3$ gives $3^2 + 1 = 10 = 2 \times 5$, for example. So this line of the argument is not valid.

Line V does follow from lines II and IV.

Therefore the proof is wrong, and the first (and only) error occurs on line IV (option F).

It is worth noting that the 'opposite' argument does work: if an algebraic expression does factorise as $(ax + b)(cx + d)$ with a, b, c and d integers, then the value of original expression always factorises (as an integer) when x is an integer, just by substituting the value of x into the factorised form. (It is possible, though, that one of the factors is 1, so the integer may be prime for some value(s) of x .)

It also turns out that in this case, if we search far enough, we do find a counterexample to the claim: $6^3 - 5^3 = 216 - 125 = 91 = 7 \times 13$. (It might have been the case that the claim was true even though the attempted proof had an error.)

- 8 A selection, S , of n terms is taken from the arithmetic sequence $1, 4, 7, 10, \dots, 70$.

Consider the following statement:

(*) There are two distinct terms in S whose sum is 74.

What is the smallest value of n for which (*) is **necessarily** true?

- A 12
- B 13
- C 14**
- D 21
- E 22
- F 23

We first determine the number of terms in the whole sequence: there are $\frac{70-1}{3} + 1 = 23 + 1 = 24$ terms.

We next determine the number of pairs that sum to 74. The pairs are:

$$\begin{aligned} &4 + 70 \\ &7 + 67 \\ &\quad \vdots \\ &34 + 40 \\ &[37 + 37] \end{aligned}$$

The final pair is excluded because the two terms have to be distinct to fit the rule in (*). The number of valid pairs is therefore $\frac{34-4}{3} + 1 = 10 + 1 = 11$.

If a selection S does *not* satisfy the rule in (*), then it can have at most one term from each of these pairs, so it has to leave out at least 11 terms. So it can have at most $24 - 11 = 13$ terms. Therefore if S has at least 14 terms, it must satisfy (*).

We also need to show that if S only has 13 terms, it might not satisfy (*). We can include 1, 37 and one term from each listed pair, giving $2 + 11 = 13$ terms. These 13 terms do not contain a distinct pair summing to 37.

Hence the smallest value of n which forces (*) to be true is 14, and the correct answer is option C..

9 Consider the following statement:

(*) **For all** real numbers x , **if** $x < k$ **then** $x^2 < k$

What is the complete set of values of k for which (*) is true?

- A** no real numbers
- B** $k > 0$
- C** $k < 1$
- D** $k \leq 1$
- E** $0 < k < 1$
- F** $0 < k \leq 1$
- G** all real numbers

The statement (*) does not restrict us to *positive* real numbers x , so whatever the value of k , we can always find a large negative number x with $x < k$ and $x^2 \geq k$.

We can even be explicit about this (though that is not required to answer this question): if $k \leq 0$, we can take $x = k - 1$ in which case $x^2 > 0 \geq k$, and if $k > 0$, we can take $x = -\sqrt{k}$ so $x < 0 < k$ and $x^2 = k \geq k$.

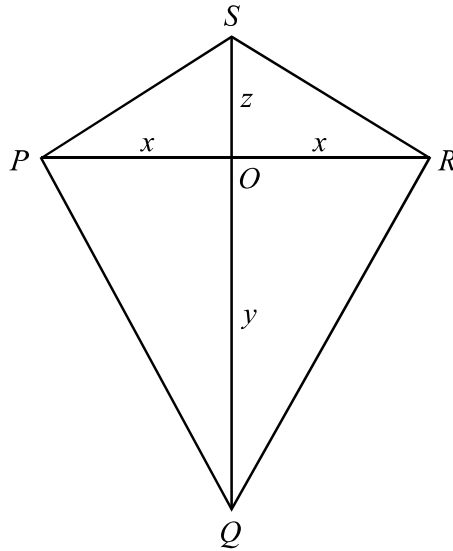
Therefore (*) is true for no real values of k , and the correct option is A.

10 Which of the following statements is/are true?

- I **For all** real numbers x and **for all** positive integers n , $x < n$
 - II **For all** real numbers x , **there exists** a positive integer n such that $x < n$
 - III **There exists** a real number x such that **for all** positive integers n , $x < n$
-
- A none of them
 - B I only
 - C II only
 - D III only
 - E I and II only
 - F I and III only
 - G** II and III only
 - H I, II and III

- I We can find a counterexample to this statement: taking $x = 2$ and $n = 1$ shows that this statement is false.
- II This is true; if $x \leq 0$, we can take $n = 1$, and if $x > 0$, we can take n to be the smallest integer greater than x .
- III This is true; we can take $x = 0$. (If, though, we replaced ‘positive integers’ with ‘integers’, the resulting statement would be false.)

Therefore II and III are true, and the correct option is G.



The diagram shows a kite $PQRS$ whose diagonals meet at O .

$$\begin{aligned} OP &= x \\ OQ &= y \\ OR &= x \\ OS &= z \end{aligned}$$

Which of the following is **necessary and sufficient** for angle SPQ to be a right angle?

A $x = y = z$

B $2x = y + z$

C $x^2 = yz$

D $y = z$

E $y^2 = x^2 + z^2$

The angle SPQ is a right angle if and only if the triangle SPQ satisfies Pythagoras's theorem, that is, if and only if

$$SP^2 + PQ^2 = SQ^2.$$

Since $PQRS$ is a kite, its diagonals intersect at right angles, so both SOP and POQ are right-angled triangles. Therefore, again by Pythagoras,

$$SP^2 = x^2 + z^2$$

$$PQ^2 = x^2 + y^2.$$

Substituting this, along with $SQ = y + z$, into the above equation gives the following necessary and sufficient condition for angle SPQ to be a right angle:

$$x^2 + z^2 + x^2 + y^2 = (y + z)^2.$$

Expanding and simplifying, this is equivalent to

$$2x^2 = 2yz$$

or $x^2 = yz$.

Therefore option C gives a necessary and sufficient condition.

(One should also check that none of the others are necessary and sufficient; we could take $x = 2$, $y = 4$, $z = 1$: this satisfies option C but none of the others.)

12 Place the following integrals in order of size, starting with the smallest.

$$P = \int_0^1 2^{\sqrt{x}} dx$$

$$Q = \int_0^1 2^x dx$$

$$R = \int_0^1 (\sqrt{2})^x dx$$

- A $P < Q < R$
- B $P < R < Q$
- C $Q < P < R$
- D $Q < R < P$
- E $R < P < Q$
- F** $R < Q < P$

We do not know how to integrate any of these, so we will instead compare the integrands as the integrals are all from 0 to 1.

We have $(\sqrt{2})^x = 2^{x/2}$, which makes R look a little simpler. Since $x/2 < x$ for $0 < x < 1$, $2^{x/2} < 2^x$ in this interval, and so $R < Q$.

Now in this interval, we also have $x < \sqrt{x}$ (as $x^2 < x$), so $2^x < 2^{\sqrt{x}}$, hence $Q < P$.

Combining these, we find that $R < Q < P$, so the correct option is F.

13 Consider the statement (*) about a real number x :

(*) **There exists** a real number y such that $x - xy + y$ is negative.

For how many real values of x is (*) true?

- A no values of x
- B exactly one value of x
- C exactly two values of x
- D all except exactly two values of x
- E** all except exactly one value of x
- F all values of x

For each fixed value of x , we can think about this expression as being a function of y . We can write the expression as $(1 - x)y + x$, which is just $my + c$ where $m = 1 - x$ and $c = x$. This is the equation of a straight line graph, so it takes every real value, both positive and negative, as long as $m \neq 0$.

The case $m = 0$ occurs exactly when $x = 1$, and in this case, the expression becomes $(1 - 1)y + 1 = 1$, so it is always 1, for every value of y .

Therefore (*) is true for all except exactly one value of x , namely $x = 1$, and the correct answer is option E.

14 Consider the two inequalities:

$$|x+5| < |x+11|$$

$$|x+11| < |x+1|$$

Which one of the following is correct?

- A There is no real number for which both inequalities are true.
- B There is exactly one real number for which both inequalities are true.
- C The real numbers for which both inequalities are true form an interval of length 1.
- D** The real numbers for which both inequalities are true form an interval of length 2.
- E The real numbers for which both inequalities are true form an interval of length 3.
- F The real numbers for which both inequalities are true form an interval of length 4.
- G The real numbers for which both inequalities are true form an interval of length 5.

To solve this question, we use the fact that $|x - a|$ can be understood as the distance of x from a . The first inequality is true if and only if x is closer to -5 than to -11 , i.e., if and only if $x > -8$. The second inequality is true if and only if x is closer to -11 than to -1 , i.e., if and only if $x < -6$.

Therefore both inequalities are true if and only if $-8 < x < -6$, which is an interval of length 2, which is option D.

One could also do this question by sketching the graphs of $y = |x + 5|$ and $y = |x + 11|$ and working out where they intersect, and so on. But that requires a lot more work than the approach presented here.

- 15 The real numbers x, y and z are all greater than 1, and satisfy the equations

$$\log_x y = z \quad \text{and} \quad \log_y z = x$$

Which one of the following equations for $\log_z x$ **must** be true?

- A $\log_z x = y$
- B $\log_z x = \frac{1}{y}$
- C $\log_z x = xy$
- D $\log_z x = \frac{1}{xy}$
- E $\log_z x = xz$
- F** $\log_z x = \frac{1}{xz}$
- G $\log_z x = yz$
- H $\log_z x = \frac{1}{yz}$

Since $\log_x y = z$, we have $y = x^z$. Likewise $z = y^x$. Combining these gives

$$z = y^x = (x^z)^x = x^{xz}.$$

Therefore raising the equation to the power of $\frac{1}{xz}$ gives $z^{1/xz} = x$, hence $\log_z x = \frac{1}{xz}$, which is option F.

- 16 In this question, a_1, \dots, a_{100} and b_1, \dots, b_{100} and c_1, \dots, c_{100} are three sequences of integers such that

$$a_n \leq b_n + c_n$$

for each n .

Which of the following statements **must** be true?

- I (minimum of a_1, \dots, a_{100}) \leq (minimum of b_1, \dots, b_{100}) + (minimum of c_1, \dots, c_{100})
- II (minimum of a_1, \dots, a_{100}) \geq (minimum of b_1, \dots, b_{100}) + (minimum of c_1, \dots, c_{100})
- III (maximum of a_1, \dots, a_{100}) \leq (maximum of b_1, \dots, b_{100}) + (maximum of c_1, \dots, c_{100})

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

$$A_{\min} = \text{minimum of } a_1, \dots, a_{100}$$

$$B_{\min} = \text{minimum of } b_1, \dots, b_{100}$$

$$C_{\min} = \text{minimum of } c_1, \dots, c_{100}$$

and similarly for A_{\max} etc.

- I Let's suppose that $B_{\min} = 0$ and $C_{\min} = 0$. Can we then make the minimum of the a_n s greater than zero?

Yes, we can: if we take $b_1 = 0$ and the rest of the b_n s to be 100, and we take $c_{100} = 0$ and the rest of the c_n s to be 100, then we can have $a_n = 100$ for every n , so the minimum of the a_n s is 100.

Therefore this statement is not necessarily true.

- II If we try the same example as in I, we find that $A_{\min} = 100$, which is greater than $B_{\min} + C_{\min}$, so the inequality holds in this case.

But to get a counterexample, we want to make A_{\min} *small*. Helpfully, the condition $a_n \leq b_n + c_n$ allows us to make each a_n as small as we like. So if we take $a_n = 0$, $b_n = 1$ and $c_n = 1$ for all n , then the condition will be satisfied for each n , but we will have $A_{\min} = 0$ and $B_{\min} = C_{\min} = 1$, so $A_{\min} < B_{\min} + C_{\min}$ in this case.

Therefore this statement is not necessarily true either.

- III Both of the counterexamples we used for I and II satisfy this statement. It is not obvious how to construct a counterexample: every simple example satisfies this statement. So let us instead try to prove it.

Each a_n satisfies the condition $a_n \leq b_n + c_n$. Now $b_n \leq B_{\max}$ and $c_n \leq C_{\max}$ for each n , so $a_n \leq B_{\max} + C_{\max}$ for each n . But this means that the maximum of all of the a_n s also satisfies this condition, that is $A_{\max} \leq B_{\max} + C_{\max}$, so the given statement must be true.

Therefore only statement III must be true, and the correct answer is option D.

17 A student answered the following question:

a and b are non-zero real numbers.

Prove that the equation $x^3 + ax^2 + b = 0$ has three distinct real roots if

$$27b\left(b + \frac{4a^3}{27}\right) < 0$$

Here is the student's solution:

- I We differentiate $y = x^3 + ax^2 + b$ to get $\frac{dy}{dx} = 3x^2 + 2ax = x(3x + 2a)$
Solving $\frac{dy}{dx} = 0$ shows that the stationary points are at $(0, b)$ and $\left(-\frac{2a}{3}, b + \frac{4a^3}{27}\right)$
- II If $27b\left(b + \frac{4a^3}{27}\right) < 0$, then b and $b + \frac{4a^3}{27}$ must have opposite signs, and so one of the stationary points is above the x -axis and one is below.
- III If the cubic has three distinct real roots, then one of the stationary points is above the x -axis and one is below.
- IV Hence if $27b\left(b + \frac{4a^3}{27}\right) < 0$, then the equation has three distinct real roots.

Which one of the following options best describes the student's solution?

- A It is a completely correct solution.
- B The student has instead proved the converse of the statement in the question.
- C The solution is wrong, because the student should have stated step II after step III.
- D The solution is wrong, because the student should have shown the converse of the result in step II.
- E** The solution is wrong, because the student should have shown the converse of the result in step III.

Considering the offered options, the focus is on the order of the steps and whether the steps prove what they should be proving; there is no requirement for us to check the algebraic calculations themselves.

The task is to prove that **if** $27b\left(b + \frac{4a^3}{27}\right) < 0$, **then** $x^3 + ax^2 + b = 0$ has three distinct real roots.

In step I, the student finds the stationary points of $y = x^3 + ax^2 + b$.

In step II, the student assumes that $27b\left(b + \frac{4a^3}{27}\right) < 0$, which is the correct thing to do to prove an 'if ... then' statement.

In step III, the student says '**if** the cubic has three distinct real roots **then** ...', which is correct but is not useful: we know that one of the stationary points is above the x -axis and the other is below and we wish to deduce that the cubic has three distinct real roots.

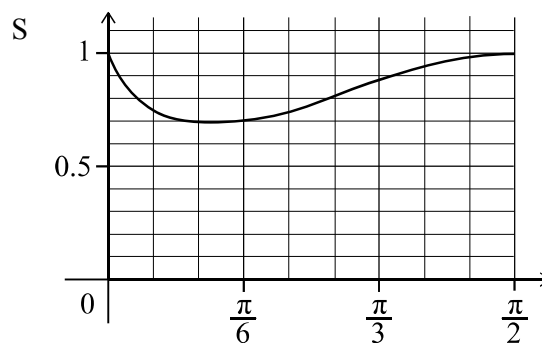
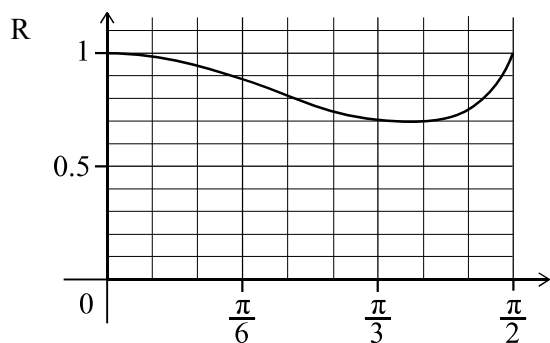
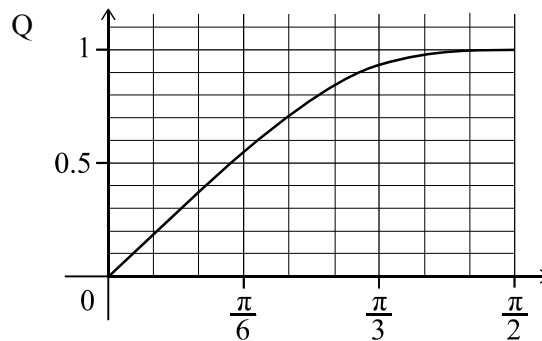
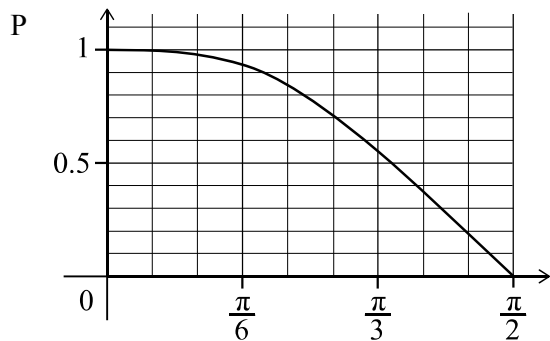
In step IV, the student uses the result of II and the result '**if** one of the stationary points is above the x -axis and one is below, **then** the equation has three distinct real roots', which is the converse of the statement in step III.

Therefore the correct option is E: the student should have shown the converse of the result in step III.

18 P, Q, R and S show the graphs of

$$y = (\cos x)^{\cos x}, y = (\sin x)^{\sin x}, y = (\cos x)^{\sin x} \text{ and } y = (\sin x)^{\cos x}$$

for $0 < x < \frac{\pi}{2}$ in some order.



Which row in the following table correctly identifies the graphs?

	$y = (\cos x)^{\cos x}$	$y = (\sin x)^{\sin x}$	$y = (\cos x)^{\sin x}$	$y = (\sin x)^{\cos x}$
A	P	Q	R	S
B	P	Q	S	R
C	Q	P	R	S
D	Q	P	S	R
E	R	S	P	Q
F	R	S	Q	P
G	S	R	P	Q
H	S	R	Q	P

Let us substitute values into the four equations to begin with.

When $x = 0$, we have

$$\begin{aligned}(\cos x)^{\cos x} &= 1^1 = 1 \\(\sin x)^{\sin x} &= 0^0 = ? \\(\cos x)^{\sin x} &= 1^0 = 1 \\(\sin x)^{\cos x} &= 0^1 = 0\end{aligned}$$

This shows that graph Q is $y = (\sin x)^{\cos x}$. (We do not know what $(\sin x)^{\sin x}$ is when $x = 0$, but from the graphs given, since three of the functions have value 1 at $x = 0$ and only one is 0, it must be 1.)

Next, let us consider the value at $x = \frac{\pi}{2}$:

$$\begin{aligned}(\cos x)^{\cos x} &= 0^0 = ? \\(\sin x)^{\sin x} &= 1^1 = 1 \\(\cos x)^{\sin x} &= 0^1 = 0 \\(\sin x)^{\cos x} &= 1^0 = 1\end{aligned}$$

Again, as only one graph has value 0 at $x = \frac{\pi}{2}$, graph P depicts $(\cos x)^{\sin x}$. (We did not actually need to calculate the final line of this list, as we have already identified this function as being graph Q.)

So we are left with $(\cos x)^{\cos x}$ and $(\sin x)^{\sin x}$ as graphs R and S in some order. Let us put in the value $x = \frac{\pi}{6}$, as that is clearly different between the two functions. We have

$$\begin{aligned}(\cos x)^{\cos x} &= \left(\frac{\sqrt{3}}{2}\right)^{\frac{\sqrt{3}}{2}} \\(\sin x)^{\sin x} &= \left(\frac{1}{2}\right)^{\frac{1}{2}}\end{aligned}$$

The first of these looks horrible, but we can approximate the value of the second of these; it is just $\frac{1}{\sqrt{2}} \approx 0.7$, so this must be graph S. Similarly, $(\cos x)^{\cos x} \approx 0.7$ at $x = \frac{\pi}{3}$, which is graph R.

Therefore the graphs are:

$$\begin{aligned}(\cos x)^{\cos x} &: \text{graph R} \\(\sin x)^{\sin x} &: \text{graph S} \\(\cos x)^{\sin x} &: \text{graph P} \\(\sin x)^{\cos x} &: \text{graph Q}\end{aligned}$$

which is option E.

19 A polygon has n vertices, where $n \geq 3$. It has the following properties:

- Every vertex of the polygon lies on the circumference of a circle C .
- The centre of the circle C is inside the polygon.
- The radii from the centre of the circle C to the vertices of the polygon cut the polygon into n triangles of equal area.

For which values of n are these properties **sufficient** to deduce that the polygon is regular?

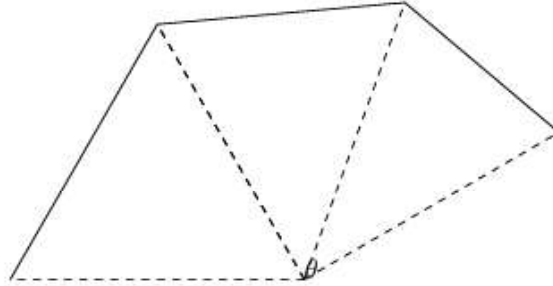
A no values of n

B $n = 3$ only

C $n = 3$ and $n = 4$ only

D $n = 3$ and $n \geq 5$ only

E all values of n



Suppose the angle of one of the triangles is θ , as shown. Then the area of the triangle is given by $\frac{1}{2}r^2 \sin \theta$ where r is the radius of the circle (using the formula $A = \frac{1}{2}ab \sin C$).

Since r is the same for each of the triangles, all the triangles will have equal area if and only if $\sin \theta$ is the same for each triangle. This will certainly be the case if the polygon is regular, but we have to determine if this can be true in any other situation.

If two of the triangles have angles at the centre of the circle of θ and ϕ , then $\sin \phi = \sin \theta$ if and only if $\phi = \theta$ or $\phi = \pi - \theta$. Can we have the latter possibility? If so, it would mean we have some angles being θ and some being $\pi - \theta$.

Suppose then that θ is acute and $\pi - \theta$ is obtuse. (We cannot have a reflex angle as the centre of the circle lies inside the polygon, and if θ is a right angle, then $\pi - \theta = \theta$.) Suppose further that k of the angles of the polygon triangles equal $\pi - \theta$ and the remaining $n - k$ equal θ . Then the sum of these triangle angles is given by

$$(n - k)\theta + k(\pi - \theta) = 2\pi$$

since they form a whole circle. We can rearrange this to $(n - 2k)\theta = (2 - k)\pi$ and hence

$$\theta = \frac{(2 - k)\pi}{n - 2k}.$$

There can be at most three obtuse angled triangles (as four would be more than a whole circle), so let's try $k = 1$, $k = 2$ and $k = 3$ in turn.

If $k = 1$, we have $\theta = \frac{\pi}{n-2}$. It looks as though this will work for every n , but we have to be a little careful as we need θ to be acute. When $n = 3$, we get $\theta = \pi$, which is not allowed. When $n = 4$, we get $\theta = \frac{\pi}{2}$, which is again not allowed. When $n > 4$, $\theta < \frac{\pi}{2}$, so this does work. Therefore when $n \geq 5$, it is possible for the conditions to be satisfied but for the polygon not to be regular.

Now consider $k = 2$, giving $\theta = \frac{0}{n-4}$. We only need to consider $n = 3$ and $n = 4$, as we have dealt with $n \geq 5$ above. This is zero when $n = 3$, so it does not work for a triangle. When $n = 4$, this formula is not meaningful (it gives $\frac{0}{0}$). We can instead go back to the original equation $(n - 2k)\theta = (2 - k)\pi$, which becomes $0 = 0$ in this case, meaning that it works for any value of θ . And indeed, this gives two angles of θ and two of $\pi - \theta$, so we get a non-regular polygon with equal area triangles. The simplest example is a non-square rectangle.

We are left with the case $k = 3$ to consider. When $n = 3$, there would be no acute angles and three obtuse ones, and the formula gives $\theta = \frac{-\pi}{-3} = \frac{\pi}{3}$. And indeed, a triangle with three

obtuse angles of $\frac{\pi}{3}$ is an equilateral triangle satisfying the conditions. Therefore the only triangle ($n = 3$) satisfying the conditions is regular.

We have thus been able to construct non-regular polygons satisfying the conditions for every $n \geq 4$, but not for $n = 3$. Hence the correct answer is option B.

20 The functions f_1 to f_5 are defined on the real numbers by

$$f_1(x) = \cos x$$

$$f_2(x) = \sin(\cos x)$$

$$f_3(x) = \cos(\sin(\cos x))$$

$$f_4(x) = \sin(\cos(\sin(\cos x)))$$

$$f_5(x) = \cos(\sin(\cos(\sin(\cos x))))$$

where all numbers are taken to be in radians.

These functions have maximum values m_1, m_2, m_3, m_4 and m_5 , respectively.

Which one of the following statements is true?

- A** m_1, m_2, m_3, m_4 and m_5 are all equal to 1
- B** $0 < m_5 < m_4 < m_3 < m_2 < m_1 = 1$
- C** $m_1 = m_3 = m_5 = 1$ and $0 < m_2 = m_4 < 1$
- D** $m_1 = m_3 = m_5 = 1$ and $0 < m_4 < m_2 < 1$
- E** $m_1 = m_3 = 1$ and $0 < m_2 = m_4 < 1$ and $0 < m_5 < 1$
- F** $m_1 = m_3 = 1$ and $0 < m_4 < m_2 < 1$ and $0 < m_5 < 1$

- $f_1(x) = \cos x \Rightarrow m_1 = 1$.
- $f_2(x) = \sin(\cos x)$: $\cos x \in [-1, 1] \Rightarrow \max = \sin(1) < 1 \Rightarrow 0 < m_2 < 1$.
- $f_3(x) = \cos(\sin(\cos x))$: \max when $\sin(\cos x) = 0 \Rightarrow m_3 = \cos 0 = 1$.
- $f_4(x) = \sin(\cos(\sin(\cos x)))$: argument in $[0.67, 1] \Rightarrow \sin < \sin 1 < 1 \Rightarrow 0 < m_4 < 1$.
- $f_5(x) = \cos(\sin(\cos(\sin(\cos x))))$: argument never 0 $\Rightarrow \cos < 1 \Rightarrow 0 < m_5 < 1$.

Thus, $m_1 = m_3 = 1, 0 < m_2 < 1, 0 < m_4 < 1, 0 < m_5 < 1 \rightarrow$ Answer: E