# TMUA MOCK FULL TEST 2
# Solution Book

- All Topics

ThrivingScholars

# Paper 1

# Question 1

What is the value of

$$\left(\left((2+1)^{-1}+1\right)^{-1}+1\right)^{-1}+1?$$

$\textbf{(A)}\ \frac{5}{8} \qquad \textbf{(B)}\ \frac{11}{7} \qquad \textbf{(C)}\ \frac{8}{5} \qquad \textbf{(D)}\ \frac{18}{11} \qquad \textbf{(E)}\ \frac{15}{8}$

# Answer B

For all nonzero numbers $a$, recall that $a^{-1}=\frac{1}{a}$ is the reciprocal of $a$.

The original expression becomes

$$\begin{aligned}\left(\left((2+1)^{-1}+1\right)^{-1}+1\right)^{-1}+1 &= \left(\left(3^{-1}+1\right)^{-1}+1\right)^{-1}+1 \\ &= \left(\left(\frac{1}{3}+1\right)^{-1}+1\right)^{-1}+1 \\ &= \left(\left(\frac{4}{3}\right)^{-1}+1\right)^{-1}+1 \\ &= \left(\frac{3}{4}+1\right)^{-1}+1 \\ &= \left(\frac{7}{4}\right)^{-1}+1 \\ &= \frac{4}{7}+1 \\ &= \boxed{\textbf{(B)}\ \frac{11}{7}}.\end{aligned}$$

# Question 2

For how many (not necessarily positive) integer values of $n$ is the value of $4000 \cdot \left(\frac{2}{5}\right)^n$ an integer?

A) $4$

B) $5$

C) $6$

D) $7$

E) $8$

F) $9$

G) $10$

H) $11$

# Answer F

Note that

$$4000 \cdot \left(\frac{2}{5}\right)^n = \left(2^5 \cdot 5^3\right) \cdot \left(2 \cdot 5^{-1}\right)^n = 2^{5+n} \cdot 5^{3-n}.$$

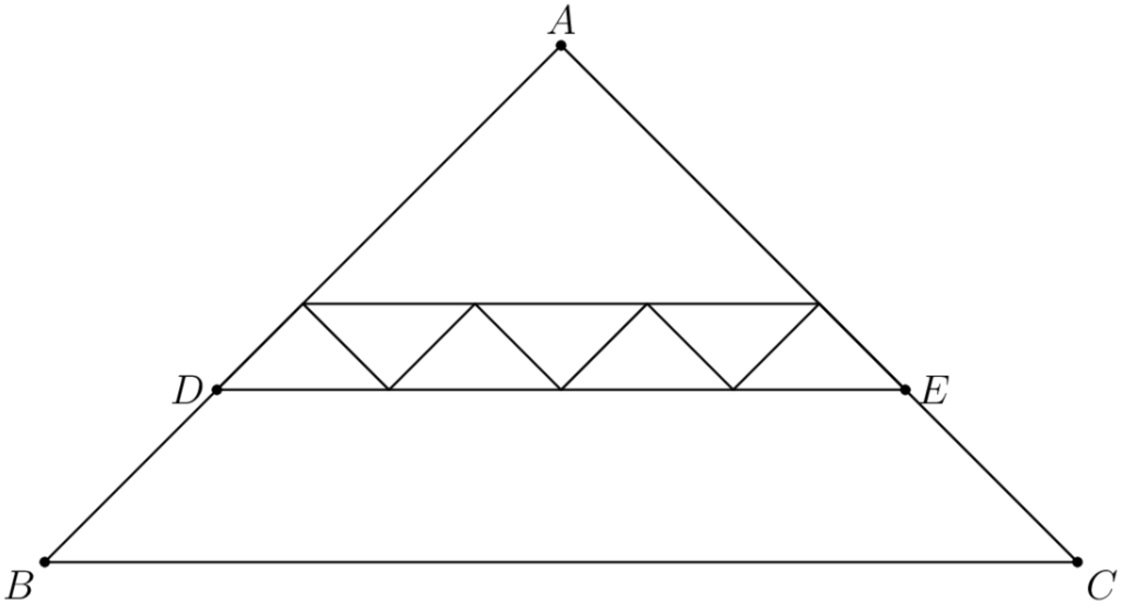
Since this expression is an integer, we need:

1. $5 + n \geq 0,$ from which $n \geq -5.$
2. $3 - n \geq 0,$ from which $n \leq 3.$

Taking the intersection gives $-5 \leq n \leq 3.$ So, there are $3 - (-5) + 1 = \boxed{\text{(F) } 9}$ integer values of $n.$

# Question 3

All of the triangles in the diagram below are similar to isosceles triangle $ABC$, in which $AB = AC$. Each of the $7$ smallest triangles has area $1$, and $\triangle ABC$ has area $40$. What is the area of trapezoid $DBCE$?




$\textbf{(A) }16 \qquad \textbf{(B) }18 \qquad \textbf{(C) }20 \qquad \textbf{(D) }22 \qquad \textbf{(E) }24$

# Answer E

## Solution 1

Let $x$ be the area of $ADE$. Note that $x$ is comprised of the $7$ small isosceles triangles and a triangle similar to $ADE$ with side length ratio $3:4$ (so an area ratio of $9:16$). Thus, we have

$$x = 7 + \frac{9}{16}x.$$

This gives $x = 16$, so the area of $DBCE = 40 - x = \boxed{(E) 24}$.

## Solution 2

Let the base length of the small triangle be $x$. Then, there is a triangle $ADE$ encompassing the 7 small triangles and sharing the top angle with a base length of $4x$. Because the area is proportional to the square of the side, let the base $BC$ be $\sqrt{40}x$. The ratio of the area of triangle $ADE$ to triangle $ABC$ is $\left(\frac{4x}{\sqrt{40}x}\right)^2 = \frac{16}{40}$. The problem says the area of triangle $ABC$ is $40$, so the area of triangle $ADE$ is $16$. So the area of trapezoid $DBCE$ is $40 - 16 = \boxed{24}$.

# Question 4

Suppose that real number $x$ satisfies

$$\sqrt{49-x^2}-\sqrt{25-x^2}=3$$

What is the value of $\sqrt{49-x^2}+\sqrt{25-x^2}$?

$\textbf{(A) }8 \qquad \textbf{(B) }\sqrt{33}+8 \qquad \textbf{(C) }9 \qquad \textbf{(D) }2\sqrt{10}+4 \qquad \textbf{(E) }12$

# Answer A

Let $a=\sqrt{49-x^2}$, and $b=\sqrt{25-x^2}$. Solving for the constants in terms of $x$, $a$ , and $b$, we get $a^2+x^2=49$, and $b^2+x^2=25$. Subtracting the second equation from the first gives us $a^2-b^2=24$. Difference of squares gives us $(a+b)(a-b)=24$. Since we want to find $a+b=\sqrt{49-x^2}+\sqrt{25-x^2}$, and we know $a-b=3$, we get $3(a+b)=24$, so $a+b=\boxed{\textbf{(A) }8}$

# Question 5

$S$ is the complete set of values of $x$ which satisfy **both** the inequalities

$$x^2 - x - 6 < 0 \qquad \text{and} \qquad 4 - 3x > 1$$

The set $S$ can also be represented as a single inequality.

Which of the following single inequalities represents the set $S$?

A $\quad x^2 + x - 2 < 0$

B $\quad x^2 + x - 2 > 0$

C $\quad x^2 - 4x + 3 < 0$

D $\quad x^2 - 4x + 3 > 0$

E $\quad x^2 + 2x < 0$

F $\quad x < 1$

## Answer A

1. Solve each inequality
   - $x^2 - x - 6 < 0 \Rightarrow (x-3)(x+2) < 0 \Rightarrow -2 < x < 3.$
   - $4 - 3x > 1 \Rightarrow -3x > -3 \Rightarrow x < 1.$
2. Combine (AND means intersection):
   $-2 < x < 1.$
3. Write this as a single quadratic inequality.
   An upward-opening quadratic that's negative between roots $-2$ and $1$ is $(x+2)(x-1) < 0 \Rightarrow x^2 + x - 2 < 0.$

Answer: **A.** $x^2 + x - 2 < 0.$

# Question 6

A cylinder has its radius reduced by 20%.
In order to keep the volume of the cylinder the same, the height needs to be increased by:

A 20% B 40% C 56.25% D 62.5% E 64%

Answer C

1. **Volume formula:**

   $$V = \pi r^2 h$$

2. **New radius:**

   $$R = \frac{4}{5}r \Rightarrow R^2 = \frac{16}{25}r^2$$

3. **Equal volumes:**

   $$\pi r^2 h = \pi \frac{16}{25} r^2 H \Rightarrow h = \frac{16}{25} H \Rightarrow H = \frac{25}{16} h$$

4. **Percentage increase:**

   $$\frac{H-h}{h} \times 100 = \left(\frac{25}{16} - 1\right) \times 100 = 56.25\%$$

**Answer: C (56.25%)**

# Question 7

An arithmetic progression and a convergent geometric progression each have first term 1.

The sum of the second terms of the two progressions is $\frac{5}{4}$.

The sum of the third terms of the two progressions is $\frac{3}{4}$.

What is the sum to infinity of the geometric progression?

A $-2$ B $-\frac{3}{2}$ C $-\frac{1}{2}$ D $\frac{1}{2}$ E $2$

# Answer E

AP: $1, 1+d, 1+2d, \ldots$
GP: $1, r, r^2, \ldots$

**Step 1:**
$(1+d)+r=\frac{5}{4} \Rightarrow d=\frac{1}{4}-r$

**Step 2:**
$(1+2d)+r^2=\frac{3}{4}$

Substitute $d$:

$$1+2\left(\frac{1}{4}-r\right)+r^2=\frac{3}{4} \Rightarrow r^2-2r+\frac{3}{4}=0$$

**Step 3:**
Solve: $4r^2-8r+3=0 \Rightarrow r=\frac{1}{2}$ (choose $|r|<1$).

**Step 4:**

$$S_\infty=\frac{1}{1-r}=\frac{1}{1-\frac{1}{2}}=2$$

# Question 8

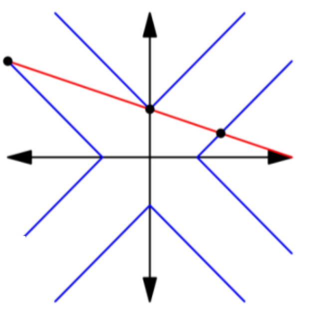
How many ordered pairs of real numbers $(x, y)$ satisfy the following system of equations?

$$x + 3y = 3$$
$$\big||x| - |y|\big| = 1$$

**(A)** 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 8

# Answer C

## Solution 1



We can solve this by graphing the equations. The second equation looks challenging to graph, but start by graphing it in the first quadrant only (which is easy since the inner absolute value signs can be ignored), then simply reflect that graph into the other quadrant

Now, it becomes clear that there are $\boxed{\textbf{(C) } 3}$ intersection points.

## Solution 2

$x + 3y = 3$ can be rewritten to $x = 3 - 3y$. Substituting $3 - 3y$ for $x$ in the second equation will give $||3 - 3y| - y| = 1$. Splitting this question into casework for the ranges of $y$ will give us the total number of solutions.

$\textbf{Case 1:} y > 1$: $3 - 3y$ will be negative so $|3 - 3y| = 3y - 3$. $|3y - 3 - y| = |2y - 3| = 1$

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Subcase 1:
```
$y > \frac{3}{2}$

$2y - 3$ is positive so $2y - 3 = 1$ and $y = 2$ and $x = 3 - 3(2) = -3$

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Subcase 2:
```
$1 < y < \frac{3}{2}$

$2y - 3$ is negative so $|2y - 3| = 3 - 2y = 1$. $2y = 2$ and so there are no solutions ($y$ can't equal to $1$)

$\textbf{Case 2:} y = 1$: It is fairly clear that $x = 0$.

$\textbf{Case 3:} y < 1$: $3 - 3y$ will be positive so $|3 - 3y - y| = |3 - 4y| = 1$

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Subcase 1:
```
$y > \frac{3}{4}$

$3 - 4y$ will be negative so $4y - 3 = 1 \rightarrow 4y = 4$. We already have this solution from Case 2 as $y = 1$.

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Subcase 2:
```
$y < \frac{3}{4}$

$3 - 4y$ will be positive so $3 - 4y = 1 \rightarrow 4y = 2$. $y = \frac{1}{2}$ and $x = \frac{3}{2}$. Thus, the solutions are: $(-3, 2), (0, 1), \left(\frac{3}{2}, \frac{1}{2}\right)$, and the answer is $\boxed{\textbf{(C) } 3}$.

Question 9

The graph of $y = x^2 - 4$ has a series of transformations applied, resulting in the graph of $y = x^2 - 2x$

Which of the following could be the series of transformations?

A a translation two units left followed by a reflection in the y-axis

B a translation one unit left followed by a reflection in the y-axis

C a translation one unit right followed by a translation three units up

D a translation one unit right followed by a translation four units up

E a reflection in the y-axis followed by a translation four units up

Answer C

1. Complete the square:
   $x^2 - 2x = (x-1)^2 - 1.$
2. Start with $f(x) = x^2 - 4.$
   Apply $x \mapsto x - 1$ (translate **one unit right**):
   $f(x-1) = (x-1)^2 - 4.$
3. Then add $3$ (translate **three units up**):
   $f(x-1) + 3 = (x-1)^2 - 1.$

Thus the required series is: **translate 1 right, then 3 up.**

## Question 10

$C_1$ and $C_2$ are circles defined by the equations

$$(x-12)^2+y^2=36 \quad \text{and} \qquad (x+13)^2+y^2=81\ .$$

Find the length of the shortest line segment $PQ$ which is tangent to $C_1$ at $P$ and tangent to $C_2$ at $Q$.

A 15 B 16 C 18 D 20 E $\sqrt{616}$

## Answer D

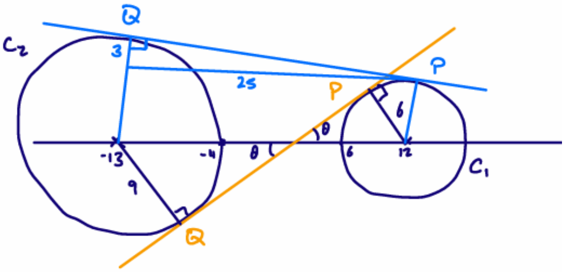
Centres: $C_1(12,0), r_1=6;\ C_2(-13,0), r_2=9;\ C_1C_2=25$.

Join $C_1C_2$ and draw radii $C_1P, C_2Q \perp$ to $PQ$.

By similarity: horizontal segments $=10:15$ (ratio $2:3$).

Right triangle at $C_1$: $6,8,10 \Rightarrow PQ=8+12=20$.

External tangent: $\sqrt{25^2-3^2}=\sqrt{616}>20$.




**Answer: D (20)**

# Question 11

In the simplified expansion of $(2+5x)^9$ .

How many of the terms have a coefficient that is divisible by 60?

A 6 B 7 C 8 D 9 E 10

# Answer B

General term:

$$T_k = \binom{9}{k} 2^{9-k} 5^k x^k$$

We need $T_k$ coefficient divisible by $60 = 2^2 \cdot 3 \cdot 5$.

- $k = 0$: $2^9$ — no factor of $3, 5$ ✗
- $k = 1$: $9 \cdot 2^8 \cdot 5$ — has $5, 3, 2^8$ ✓
- $k = 2$: $\binom{9}{2} 2^7 5^2 = 36 \cdot 2^7 5^2$ — has $3, 5, 2^7$ ✓
- Continue similarly → all $k = 1, \ldots, 7$ work.
- $k = 8, 9$: miss factor $2^2$ ✗

Thus **7 terms** satisfy.

## Question 12

It is given that $y = (1 - 2sinx)\, cos2x$ for $0 < x < 2\pi$ .

The fraction of the interval $0 < x < 2\pi$ for which $y < 0$ is

A $\frac{1}{6}$ B $\frac{1}{4}$ C $\frac{1}{3}$ D $\frac{1}{2}$ E $\frac{2}{3}$

## Answer C

Given $y = (1 - 2\sin x)\cos 2x$.

$y < 0$ when **signs differ**.

**Step 1:** Solve $1 - 2\sin x = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$.

Positive where $x < \frac{\pi}{6}$ or $x > \frac{5\pi}{6}$.

**Step 2:** Solve $\cos 2x = 0 \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Positive where $x \in (0, \frac{\pi}{4}), (\frac{3\pi}{4}, \frac{5\pi}{4}), (\frac{7\pi}{4}, 2\pi)$.

**Step 3:** Find intervals where one positive, one negative →

$$\left(\frac{\pi}{6}, \frac{\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{6}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$$

Total length:

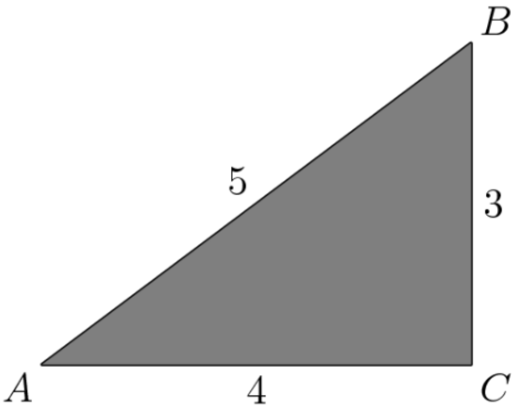
$$\frac{\pi}{12} + \frac{\pi}{12} + \frac{\pi}{2} = \frac{2\pi}{3}$$

Fraction of $0 < x < 2\pi$:

$$\frac{\frac{2\pi}{3}}{2\pi} = \frac{1}{3}$$

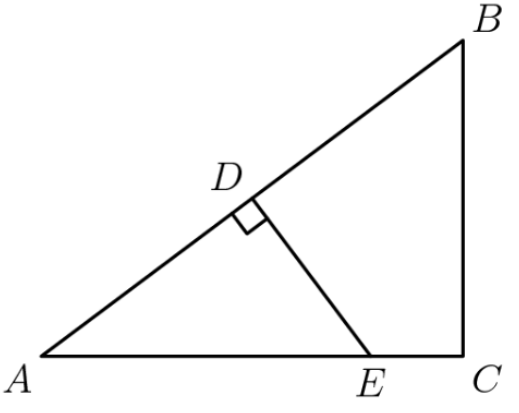
# Question 13

A paper triangle with sides of lengths $3, 4,$ and $5$ inches, as shown, is folded so that point $A$ falls on point $B$. What is the length in inches of the crease?




$\textbf{(A) } 1+\frac{1}{2}\sqrt{2} \qquad \textbf{(B) } \sqrt{3} \qquad \textbf{(C) } \frac{7}{4} \qquad \textbf{(D) } \frac{15}{8} \qquad \textbf{(E) } 2$

# Answer D




First, we need to realize that the crease line is just the perpendicular bisector of side $AB$, the hypotenuse of right triangle $\triangle ABC$. Call the midpoint of $AB$ point $D$. Draw this line and call the intersection point with $AC$ as $E$. Now, $\triangle ACB$ is similar to $\triangle ADE$ by $AA$ similarity. Setting up the ratios, we find that

$$\frac{BC}{AC}=\frac{DE}{AD} \Rightarrow \frac{3}{4}=\frac{DE}{\frac{5}{2}} \Rightarrow DE=\frac{15}{8}.$$

Thus, our answer is $\boxed{\textbf{(D) } \frac{15}{8}}$.

# Question 14

Given that the real numbers $x$ and $y$ satisfy the equations $4^x + 4^y = 10$

and $2(4^x) + 4^{2y} = 20$

What is the value of $x + y$ ?

A $log_4 10$ B 2 C 4 D $4 + log_4 10$ E 10

Answer B

Let $a = 4^x,\ b = 4^y$.

Then

$$a + b = 10, \qquad 2a + b^2 = 20.$$

Eliminate $a$: $b^2 + 2(10 - b) = 20 \Rightarrow b^2 - 2b = 0 \Rightarrow b = 2$ ($b = 0$ impossible since $4^y > 0$).

Hence $a = 8$.

So $4^y = 2 \Rightarrow y = \frac{1}{2}$; $4^x = 8 \Rightarrow x = \frac{3}{2}$.

$$x + y = \frac{3}{2} + \frac{1}{2} = 2.$$

## Question 15

Given that $x^2 + y^2 = 1$, what is the greatest possible value of $2x + 3y$

A $\sqrt{13}$

B $\sqrt{10}$

C $\sqrt{7}$

D $\frac{5\sqrt{2}}{2}$

E 3

## Answer A

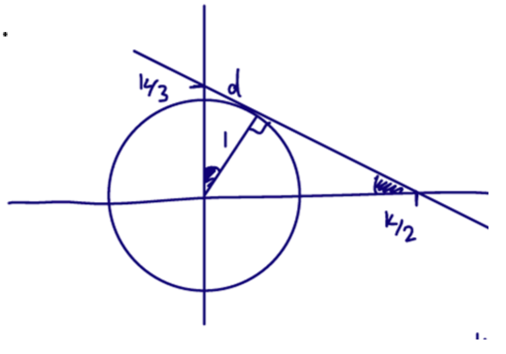
We want maximum $k = 2x + 3y$ subject to $x^2 + y^2 = 1$.
Line: $2x + 3y = k \Rightarrow y = -\frac{2}{3}x + \frac{k}{3}$.

Distance from origin to line:

$$d = \frac{|k|}{\sqrt{2^2 + 3^2}} = \frac{|k|}{\sqrt{13}}$$

For line to touch circle $d = 1 \Rightarrow |k| = \sqrt{13}$.

Greatest value: $k = \sqrt{13}$.



# Question 16

The sum of the maximum and minimum values of the function

$f(x) = (2a)^{sinx}$ where $a > 0$ and $x$ is real, is 4.

Find the sum of the possible values of $a$.

A $2-\sqrt{3}$ B $\sqrt{3}$ C $1+\frac{1}{2}\sqrt{3}$ D 2 E $2+\sqrt{3}$

# Answer D

Given $f(x) = (2a)^{\sin x}$, max occurs at $\sin x = 1 \Rightarrow f_{\max} = 2a$, min at $\sin x = -1 \Rightarrow f_{\min} = \frac{1}{2a}$.

Sum:

$$2a + \frac{1}{2a} = 4 \quad \Rightarrow \quad 4a^2 - 8a + 1 = 0$$

Solve:

$$a = \frac{8 \pm \sqrt{64-16}}{8} = 1 \pm \frac{\sqrt{3}}{2}$$

Sum of possible $a$:

$$\left(1+\frac{\sqrt{3}}{2}\right) + \left(1-\frac{\sqrt{3}}{2}\right) = 2$$

# Question 17

Which of the following describes the set of values of $a$ for which the curves $x^2 + y^2 = a^2$ and $y = x^2 - a$ in the real $xy$-plane intersect at exactly $3$ points?

**(A)** $a = \frac{1}{4}$ **(B)** $\frac{1}{4} < a < \frac{1}{2}$ **(C)** $a > \frac{1}{4}$ **(D)** $a = \frac{1}{2}$ **(E)** $a > \frac{1}{2}$

# Answer E

Substituting $y = x^2 - a$ into $x^2 + y^2 = a^2$, we get

$$x^2 + (x^2 - a)^2 = a^2 \implies x^2 + x^4 - 2ax^2 = 0 \implies x^2(x^2 - (2a - 1)) = 0$$

Since this is a quartic, there are $4$ total roots (counting multiplicity). We see that $x = 0$ always has at least one intersection at $(0, -a)$ (and is in fact a double root).

The other two intersection points have $x$ coordinates $\pm\sqrt{2a - 1}$. We must have $2a - 1 > 0$; otherwise we are in the case where the parabola lies entirely below the circle (tangent at the point $(0, -a)$). This only results in a single intersection point in the real coordinate plane. Thus, we see that $\boxed{\textbf{(E)}\ a > \frac{1}{2}}$.

## Question 18

A function $f$ is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$?

$\textbf{(A)}\ 2016 \qquad \textbf{(B)}\ 2017 \qquad \textbf{(C)}\ 2018 \qquad \textbf{(D)}\ 2019 \qquad \textbf{(E)}\ 2020$

## Answer B

For all integers $n \geq 7$, note that

$$\begin{aligned} f(n) &= f(n-1) - f(n-2) + n \\ &= [f(n-2) - f(n-3) + n - 1] - f(n-2) + n \\ &= -f(n-3) + 2n - 1 \\ &= -[f(n-4) - f(n-5) + n - 3] + 2n - 1 \\ &= -f(n-4) + f(n-5) + n + 2 \\ &= -[f(n-5) - f(n-6) + n - 4] + f(n-5) + n + 2 \\ &= f(n-6) + 6. \end{aligned}$$

It follows that

$$\begin{aligned} f(2018) &= f(2012) + 6 \\ &= f(2006) + 12 \\ &= f(2000) + 18 \\ &\ \vdots \\ &= f(2018 - 6 \cdot 336) + 6 \cdot 336 \\ &= f(2) + 2016 \\ &= \boxed{\textbf{(B)}\ 2017}. \end{aligned}$$

Question 19

In the $xy$-plane, a parabola has vertex (9, –14) and intersects the $x$-axis at two points. If the equation of the parabola is written in the form $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are constants, which of the following could be the value of $a + b + c$ ?

A) –23

B) –19

C) –14

D) –12

E) No upper bound

Answer D

**Choice D** is correct. The equation of a parabola in the $xy$-plane can be written in the form $y = a(x-h)^2 + k$, where $a$ is a constant and $(h, k)$ is the vertex of the parabola. If $a$ is positive, the parabola will open upward, and if $a$ is negative, the parabola will open downward. It's given that the parabola has vertex $(9, -14)$. Substituting 9 for $h$ and $-14$ for $k$ in the equation $y = a(x-h)^2 + k$ gives $y = a(x-9)^2 - 14$, which can be rewritten as $y = a(x-9)(x-9) - 14$, or $y = a(x^2 - 18x + 81) - 14$. Distributing the factor of $a$ on the right-hand side of this equation yields $y = ax^2 - 18ax + 81a - 14$. Therefore, the equation of the parabola, $y = ax^2 - 18ax + 81a - 14$, can be written in the form $y = ax^2 + bx + c$, where $a = a$, $b = -18a$, and $c = 81a - 14$. Substituting $-18a$ for $b$ and $81a - 14$ for $c$ in the expression $a + b + c$ yields $(a) + (-18a) + (81a - 14)$, or $64a - 14$. Since the vertex of the parabola, $(9, -14)$, is below the $x$-axis, and it's given that the parabola intersects the $x$-axis at two points, the parabola must open upward. Therefore, the constant $a$ must have a positive value. Setting the expression $64a - 14$ equal to the value in choice D yields $64a - 14 = -12$. Adding 14 to both sides of this equation yields $64a = 2$. Dividing both sides of this equation by 64 yields $a = \frac{2}{64}$, which is a positive value. Therefore, if the equation of the parabola is written in the form $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are constants, the value of $a + b + c$ could be $-12$.

# Question 20

What is

$$\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j)?$$

$\textbf{(A)}\ 100{,}100 \qquad \textbf{(B)}\ 500{,}500 \qquad \textbf{(C)}\ 505{,}000 \qquad \textbf{(D)}\ 1{,}001{,}000 \qquad \textbf{(E)}\ 1{,}010{,}000$

# Answer E

Recall that the sum of the first $100$ positive integers is $\sum_{k=1}^{100} k = \frac{101\cdot 100}{2} = 5050.$ It follows that

$$\begin{aligned}\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j) &= \sum_{i=1}^{100}\left(\sum_{j=1}^{100} i + \sum_{j=1}^{100} j\right) \\ &= \sum_{i=1}^{100}(100i+5050) \\ &= 100\sum_{i=1}^{100} i + \sum_{i=1}^{100} 5050 \\ &= 100\cdot 5050 + 5050\cdot 100 \\ &= \boxed{\textbf{(E)}\ 1{,}010{,}000}.\end{aligned}$$

# Paper 2

# Question 1

The sum of two nonzero real numbers is $4$ times their product. What is the sum of the reciprocals of the two numbers?

$\textbf{(A)}\ 1\qquad\textbf{(B)}\ 2\qquad\textbf{(C)}\ 4\qquad\textbf{(D)}\ 8\qquad\textbf{(E)}\ 12$

# Answer C

## Solution

Let the two real numbers be $x, y$. We are given that $x+y=4xy,$ and dividing both sides by $xy$, $\frac{x}{xy}+\frac{y}{xy}=4.$

$$\frac{1}{y}+\frac{1}{x}=\boxed{\textbf{(C)}\ 4}.$$

Note: we can easily verify that this is the correct answer; for example, $\left(\frac{1}{2}, \frac{1}{2}\right)$ works, and the sum of their reciprocals is $4$.

## Solution 2

Instead of using algebra, another approach at this problem would be to notice the fact that one of the nonzero numbers has to be a fraction. See for yourself. And by looking into fractions, we immediately see that $\frac{1}{3}$ and $1$ would fit the rule. $\boxed{\textbf{(C)} 4}$.

# Question 2

Alice, Bob, and Charlie were on a hike and were wondering how far away the nearest town was. When Alice said, "We are at least $6$ miles away," Bob replied, "We are at most $5$ miles away." Charlie then remarked, "Actually the nearest town is at most $4$ miles away." It turned out that none of the three statements were true. Let $d$ be the distance in miles to the nearest town. Which of the following intervals is the set of all possible values of $d$?

$\textbf{(A) } (0,4) \qquad \textbf{(B) } (4,5) \qquad \textbf{(C) } (4,6) \qquad \textbf{(D) } (5,6) \qquad \textbf{(E) } (5,\infty)$

# Answer D

## Solution 1

For each of the false statements, we identify its corresponding true statement. Note that:

1. $\text{False} \cap \text{True} = \varnothing.$
2. $\text{False} \cup \text{True} = [0,\infty).$

We construct the following table:

| **Hiker** | **False Statement** | **True Statement** |
|---|---|---|
| **Alice** | $[6,\infty)$ | $[0,6)$ |
| **Bob** | $[0,5]$ | $(5,\infty)$ |
| **Charlie** | $[0,4]$ | $(4,\infty)$ |

Taking the intersection of the true statements, we have

$$[0,6)\cap(5,\infty)\cap(4,\infty)=(5,6)\cap(4,\infty)=\boxed{\textbf{(D) } (5,6)}.$$

~MRENTHUSIASM

## Solution 2

Think of the distances as if they are on a number line. Alice claims that $d>6$, Bob says $d<5$, while Charlie thinks $d<4$. This means that all possible numbers less than $5$ and greater than $6$ are included. However, since the three statements are actually false, the distance to the nearest town is one of the numbers not covered. Therefore, the answer is $\boxed{\textbf{(D) } (5,6)}$.

# Question 3

Ms. Carroll promised that anyone who got all the multiple choice questions right on the upcoming exam would receive an A on the exam. Which one of these statements necessarily follows logically?

**(A)** If Lewis did not receive an A, then he got all of the multiple choice questions wrong.
**(B)** If Lewis did not receive an A, then he got at least one of the multiple choice questions wrong.
**(C)** If Lewis got at least one of the multiple choice questions wrong, then he did not receive an A.
**(D)** If Lewis received an A, then he got all of the multiple choice questions right.
**(E)** If Lewis received an A, then he got at least one of the multiple choice questions right.

# Answer B

## Solution (Elimination)

Note: An A is usually 90%-100% of the questions correct.

**(A)** False. If Lewis did not receive an A, that does not mean he got *all* the multiple-choice questions wrong. For example, he might get 19/20 or 18/20, which still accounts for an A.

**(B)** True. If Lewis did not receive an A, then he must have got at least one wrong. Otherwise, Lewis would have gotten an A.

**(C)** False. Again, Lewis can get 19/20 or 18/20, which is still an A.

**(D)** False. The above situation can happen.

**(E)** False. Lewis can get 17/20 or less but it is not an A.

Therefore, our answer is $\boxed{\textbf{(B)}}$.

# Question 4

At a gathering of $30$ people, there are $20$ people who all know each other and $10$ people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur?

$\textbf{(A)}\ 240 \qquad \textbf{(B)}\ 245 \qquad \textbf{(C)}\ 290 \qquad \textbf{(D)}\ 480 \qquad \textbf{(E)}\ 490$

# Answer B

Each one of the ten people has to shake hands with all the $20$ other people they don't know. So $10 \cdot 20 = 200$. From there, we calculate how many handshakes occurred between the people who don't know each other. This is simply counting how many ways to choose two people to shake hands from $10$, or $\binom{10}{2} = 45$. Thus the answer is $200 + 45 = \boxed{\textbf{(B)}\ 245}$.

## Question 5

Find the coefficient of $x^3y^2$ in the expansion of $(1 + x + y^2)^5$

A 5 B 10 C 15 D 20 E 30

## Answer D

In $(1 + x + y^2)^5$, to get $x^3y^2$ choose:

- $x$ from 3 factors,
- $y^2$ from 1 factor,
- $1$ from 1 factor.

Coefficient is the multinomial:

$$\frac{5!}{3!\,1!\,1!} = 20.$$

# Question 6

Consider the following statement:

If a positive integer $N$ has the property that $N^2$ is divisible by 9, then $N$ is divisible by 6.

Which of the following is a counterexample to this statement?

| | | | | | |
|---|---|---|---|---|---|
| I | $N = 15$ | II | $N = 18$ | III | N = 21 |

A none of them

B I only

C II only

D III only

E I and II only

F II and III only

G I and III only

H I, II and III

Answer G

We need $N^2$ divisible by $9$ but $N$ **not** divisible by $6$.

- I: $N = 15 \Rightarrow N^2 = 225$ divisible by $9$, but $15 \not\div 6$ ✓
- II: $N = 18 \Rightarrow N^2$ divisible by $9$, and $18 \div 6$ ✗ (not a counterexample)
- III: $N = 21 \Rightarrow N^2 = 441$ divisible by $9$, but $21 \not\div 6$ ✓

Thus counterexamples: **I and III only**

# Question 7

Use the trapezium rule approximation with four strips to estimate:

$$\int_0^1 16^x \ dx$$

A $\frac{31}{8}$ B $\frac{45}{8}$ C $\frac{15}{2}$ D $\frac{31}{4}$ E $\frac{45}{4}$

## Answer B

We approximate

$$\int_0^1 16^x dx$$

using 4 strips, $h = \frac{1-0}{4} = \frac{1}{4}$.

Values:

$$f(0) = 1,\ f\left(\tfrac{1}{4}\right) = 2,\ f\left(\tfrac{1}{2}\right) = 4,\ f\left(\tfrac{3}{4}\right) = 8,\ f(1) = 16$$

Trapezium rule:

$$\frac{h}{2}\left[f(0) + f(1) + 2(f(\tfrac{1}{4}) + f(\tfrac{1}{2}) + f(\tfrac{3}{4}))\right] = \frac{1}{8}(1 + 16 + 2(2 + 4 + 8)) = \frac{45}{8}$$

Question 8

Consider the following statements for real values of $x$.

$$A: \quad \int_0^{2\pi} sin\ kx\ dx = 0$$

$B$: $k$ is an even integer

Which one of the following is correct?

A $A$ is **necessary** but **not sufficient** for $B$

B $A$ is **sufficient** but **not necessary** for $B$

C $A$ is **necessary** and **sufficient** for $B$

D $A$ is **not necessary** and **not sufficient** for $B$

Answer A

$$A: \int_0^{2\pi} \sin kx\, dx = 0 \qquad B:\ k \text{ is even integer}$$

For **any integer** $k$, $\int_0^{2\pi} \sin kx\, dx = 0$, so $A$ holds for both even and odd $k$.
Thus $A$ is true **whenever** $B$ is true (necessary),
but $A$ can hold even when $k$ is odd (not sufficient).

# Question 9

A student attempts to prove that there are infinitely many prime numbers as follows:

Assume that there are finitely many prime numbers, and let the largest be $p$ (I)

Let $n = 2p + 1$ so that $n > p$ (II)

Consider $n = 2p + 1$. This is not divisible by $p$ (III)

Therefore $n$ is also a prime number (IV)

This contradicts the initial assumption so we conclude that there are infinitely many prime numbers (V)

Which of the following best describes this proof?

A The statement is not true and there is an error in the proof in line (I)

B The statement is not true and there is an error in the proof in line (II)

C The statement is true but there is an error in the proof in line (III)

D The statement is true but there is an error in the proof in line (IV)

E The statement is true but there is an error in the proof in line (V)

F The statement is true and the proof is completely correct.

# Answer D

Statement (I)–(III) are valid: $n = 2p + 1$ is not divisible by $p$.

**Error in (IV):** $n$ not divisible by $p$ does **not** imply $n$ is prime (e.g. $p = 7 \Rightarrow n = 15$ which is composite).

Thus statement ("infinitely many primes") is true, but proof fails at line (IV).

# Question 10

Simplify the following expression $log_a 2 \times log_a 4 \times log_a 8 \times \ldots \times log_a 2^n$

A $log_a(2(2^n - 1))$

B $log_a(2^{n-1} + 1)$

C $n!(log_a 2)^n$

D $n!(log_a 2^n)$

E $n(n!)(log_a 2)$

Answer C

$$\log_a 2 \times \log_a 4 \times \log_a 8 \times \cdots \times \log_a 2^n = \prod_{k=1}^{n} \log_a 2^k = \prod_{k=1}^{n} k \log_a 2$$

Factor terms:

$$= (\log_a 2)^n \times (1 \cdot 2 \cdot 3 \ldots n) = n!(\log_a 2)^n$$

# Question 11

Consider the set of integers of the form $3^n - 2n - 1$, where $n$ is a positive integer greater than 1.

Which of the following statements are necessarily true?

I Integers in this set are even only if $n$ is odd.

II Integers in this set are always 2 more or 2 less than a multiple of 6.

III Integers in this set are always a multiple of 4.

A none of them

B I only

C II only

D III only

E I and II only

F II and III only

G I and III only

H I, II and III

# Answer D

Numbers are $3^n - 2n - 1 = (2+1)^n - 2n - 1$.

Binomial expansion:

$$(2+1)^n = 2^n + n2^{n-1} + \binom{n}{2} 2^{n-2} + \cdots + 1$$

Hence

$$3^n - 2n - 1 = 2^n + n2^{n-1} + \binom{n}{2} 2^{n-2} + \ldots$$

Factor $4:\ 3^n - 2n - 1 = 4[\ldots]$, so expression is always a multiple of $4$.

- **I:** False (counterexample $n = 2 \Rightarrow 4$ even though $n$ even).
- **II:** False (counterexample $n = 4 \Rightarrow 236 \not\equiv \pm 2 \mod 6$).
- **III:** True.

Question 12

Given that $f\left(x - f(x)\right) = x$ and $f(a) = b$

which of the following is true?

A $f(b) = a - b$

B $f(-a) = a - b$

C $f(-b) = -a$

D $f(-a) = -b$

E $f(-b) = b - a$

Answer D

Given $f(x - f(x)) = x$ and $f(a) = b$.

1. Put $x = a$:
   $f(a - f(a)) = a \Rightarrow f(a - b) = a.$
2. Put $x = a - b$:
   $f((a - b) - f(a - b)) = a - b \Rightarrow f(-b) = a - b.$
3. Put $x = -b$:
   $f(-b - f(-b)) = -b \Rightarrow f(-b - (a - b)) = f(-a) = -b.$

Thus $f(-a) = -b.$

# Question 13

Let $P$ be the set of prime numbers greater than 3.

Consider the following assertion:

All members of $P$ can be represented in the form $6n \pm 1$ (*)

Which of the following statements, taken individually is equivalent to (*)?

I A number is **not** a member of $P$ **only if** it can **not** be written in the form $6n \pm 1$

II A **sufficient** condition for a number to be in $P$ is that it can be written in the form $6n \pm 1$

III **If** a number can **not** be written in the form $6n \pm 1$ then it is **not** a member of P

| | Statement I is equivalent to (*) | Statement II is equivalent to (*) | Statement III is equivalent to (*) |
|---|---|---|---|
| A | Yes | Yes | Yes |
| B | Yes | Yes | No |
| C | Yes | No | Yes |
| D | Yes | No | No |
| E | No | Yes | Yes |
| F | No | Yes | No |
| G | No | No | Yes |
| H | No | No | No |

# Answer G

- **Statement I:** $P' \Rightarrow (6n \pm 1)'$ — **not equivalent** to (*).
- **Statement II:** $(6n \pm 1) \Rightarrow P$ — **not equivalent** to (*).
- **Statement III:** $(6n \pm 1)' \Rightarrow P'$ — **equivalent** to (*) (contrapositive).

Correct row: **No, No, Yes**

# Question 14

Arrange the following integrals in order from smallest to largest:

P: $\int_{0}^{4} x\left|(x-2)\right| \; dx$ Q: $\int_{-4}^{0} x\left|(x+2)\right| \; dx$ R: $\int_{-4}^{4} x\left|x\right| \; dx$

| | |
|---|---|
| A | P Q R |
| B | P R Q |
| C | Q P R |
| D | Q R P |
| E | R P Q |
| F | R Q P |

## Answer D

- $P = \displaystyle\int_0^4 x|x-2|\,dx.$
  Split at $x = 2$: on $[0, 2]$, $x(2-x) \geq 0$; on $[2, 4]$, $x(x-2) \geq 0$. Hence $P > 0$.
- $Q = \displaystyle\int_{-4}^0 x|x+2|\,dx.$
  Split at $x = -2$: on $[-4, -2]$, integrand $-x(x+2) < 0$; on $[-2, 0]$, integrand $x(x+2) < 0$. Hence $Q < 0$.
- $R = \displaystyle\int_{-4}^4 x|x|\,dx.$
  $x|x|$ is odd $\Rightarrow$ integral over symmetric limits is $0$. So $R = 0$.

Therefore $Q < R < P$.

# Question 15

Suppose that n is an integer such that $n^3$ is divisible by 5.

Which of the following statements are necessarily true?

| | |
|---|---|
| I | 5 is a factor of $n$ |
| II | $n^3$ is divisible by 125 |
| III | If $k$ is a multiple of 5 which divides $n^3$, then $\frac{n^3}{k}$ is a multiple of 5 |

| | |
|---|---|
| A | none of them |
| B | I only |
| C | II only |
| D | III only |
| E | I and II only |
| F | II and III only |
| G | I and III only |
| H | I, II and III |

# Answer E

Given $n^3$ divisible by $5 \Rightarrow n$ must be divisible by $5$.

- **I:** True (if $5 \mid n^3 \Rightarrow 5 \mid n$).
- **II:** True ($n = 5k \Rightarrow n^3 = 125k^3$, so divisible by $125$).
- **III:** False (counterexample $n = 10, n^3 = 1000, k = 125 \Rightarrow n^3/k = 8$, not multiple of $5$).

# Question 16

Let $x$ be a real number.

Which **one** of the following statements is a **sufficient** condition for **all** of the other four statements?

| | |
|---|---|
| A | $x = -1$ or $x = 1$ |
| B | $x \geq -1$ and $x \leq 1$ |
| C | $x \geq -1$ or $x \leq 1$ |
| D | $x \leq 1$ |
| E | $x^2 \leq 1$ |

# Answer A

If $x = -1$ or $x = 1$:

- Then $-1 \leq x \leq 1 \Rightarrow$ **B** holds.
- $x \leq 1 \Rightarrow$ **D** holds.
- $x^2 = 1 \leq 1 \Rightarrow$ **E** holds.
- **C** ("$x \geq -1$ or $x \leq 1$") is a tautology, so it holds for any $x$.

Thus **A** implies all the other four statements.

Why the others aren't sufficient:

- **B** allows $x = 0$, which does **not** satisfy **A**.
- **D** allows $x = -2$, which violates **E**.
- **E** allows $x = 0$, which does **not** satisfy **A**.
- **C** is always true and implies nothing stronger.

Therefore, **A** is the sufficient condition for all the others.

# Question 17

Joey and Chloe and their daughter Zoe all have the same birthday. Joey is $1$ year older than Chloe, and Zoe is exactly $1$ year old today. Today is the first of the $9$ birthdays on which Chloe's age will be an integral multiple of Zoe's age. What will be the sum of the two digits of Joey's age the next time his age is a multiple of Zoe's age?

$\textbf{(A) }7 \qquad \textbf{(B) }8 \qquad \textbf{(C) }9 \qquad \textbf{(D) }10 \qquad \textbf{(E) }11$

# Answer E

Suppose that Chloe is $c$ years old today, so Joey is $c+1$ years old today. After $n$ years, Chloe and Zoe will be $n+c$ and $n+1$ years old, respectively. We are given that

$$\frac{n+c}{n+1}=1+\frac{c-1}{n+1}$$

is an integer for $9$ nonnegative integers $n$. It follows that $c-1$ has $9$ positive divisors. The prime factorization of $c-1$ is either $p^8$ or $p^2q^2$. Since $c-1<100,$ the only possibility is $c-1=2^2\cdot3^2=36,$ from which $c=37.$ We conclude that Joey is $c+1=38$ years old today.

Suppose that Joey's age is a multiple of Zoe's age after $k$ years, in which Joey and Zoe will be $k+38$ and $k+1$ years old, respectively. We are given that

$$\frac{k+38}{k+1}=1+\frac{37}{k+1}$$

is an integer for some positive integer $k$. It follows that $37$ is divisible by $k+1,$ so the only possibility is $k=36$. We conclude that Joey will be $k+38=74$ years old then, from which the answer is $7+4=\boxed{\textbf{(E) }11}$.

Question 18

An infinite sequence of integers is such that $u_1 = -17$

and $u_{n-1} - 7 < u_n < u_{n-1} - 2$ for $n > 1$

Given that $u_k = -53$ how many different values can $k$ take.

A 1

B 6

C 7

D 12

E 15

Answer C

Given $u_1 = -17$ and $u_n \in (u_{n-1} - 7,\ u_{n-1} - 2)$.

- Lower bound: $u_n > u_{n-1} - 7 \Rightarrow u_n \geq u_{n-1} - 6$ (integer).
- Upper bound: $u_n < u_{n-1} - 2 \Rightarrow u_n \leq u_{n-1} - 3$.

Thus each term satisfies

$$u_n \in [u_{n-1} - 6,\ u_{n-1} - 3].$$

Compute range step by step:

$$u_2 \in [-23, -20], \quad u_3 \in [-29, -23], \quad \ldots, \quad u_7 \in [-53, -35].$$

Hence $u_7 = -53$ is possible, and so are $u_8, \ldots, u_{13} = -53$.

Total $k$-values: **7**.

## Question 19

Which of the following statements, taken independently, is/are true?

| | |
|---|---|
| I | **For all** real $x$, **there exists** a real $y$ such that $y^2 = x$ |
| II | **There exists** a real $y$ such that **for all** positive real $x$, $y^2 = x$ |
| III | **For all** real $y$, **there exists** a positive real $x$ such that $y^2 = x$ |

| | |
|---|---|
| A | none of them |
| B | I only |
| C | II only |
| D | III only |
| E | I and II only |
| F | II and III only |
| G | I and III only |
| H | I, II and III |

Answer D

**I:** $y^2 = x$ not possible for $x < 0$. **False**

**II:** No single $y$ satisfies $y^2 = x$ for all $x > 0$. **False**

**III:** For any real $y$, choose $x = y^2 > 0$. Always possible. **True**

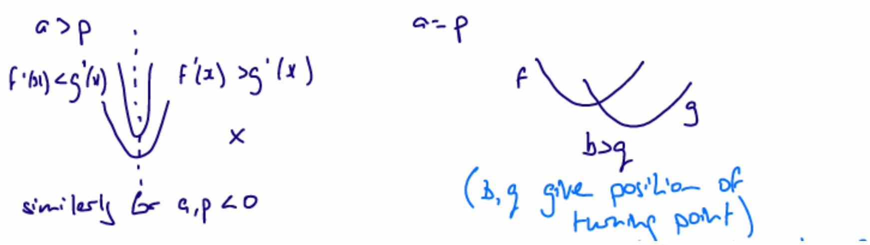
# Question 20

Consider the quadratic graphs given by $f(x) = ax^2 + bx + c$ and $g(x) = px^2 + qx + r$

If $f'(x) > g'(x)$ for all real $x$, which of the following is / are **necessarily** true?

| | |
|---|---|
| I | $a > p$ and $b > q$ |
| II | $a = p$ and $b > q$ |
| III | $b > q$ and $c > r$ |

| | |
|---|---|
| A | none of them |
| B | I only |
| C | II only |
| D | III only |
| E | I and II only |
| F | II and III only |
| G | I and III only |
| H | I, II and III |




Answer C

Given $f'(x) = 2ax + b,\ g'(x) = 2px + q$ and $f'(x) > g'(x)\ \forall x$:

$$2ax + b > 2px + q \quad \forall x$$

$$(2a - 2p)x > q - b \quad \forall x$$

- If $a > p$, LHS grows with $x$ and will fail for large negative $x$.
- If $a < p$, fails for large positive $x$.
- Hence must have $a = p$, so inequality becomes $b > q$.

Constant $c, r$ irrelevant.