Pure

Further Mathematics 1

Revision Notes

Further Pure 1

1 Complex Numbers

Definitions and arithmetical operations

$i = \sqrt{-1}$, so $\sqrt{-16} = 4i$, $\sqrt{-11} = \sqrt{11}\,i$, etc.
These are called *imaginary* numbers

Complex numbers are written as $z = a + bi$, where a and $b \in \mathbb{R}$.
a is the *real part* and b is the *imaginary part*.

+, –, × are defined in the 'sensible' way; division is more complicated.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

$$(a + bi) \times (c + di) = ac + bdi^2 + adi + bci$$

$$= (ac - bd) + (ad + bc)i \qquad \text{since } i^2 = -1$$

So $(3 + 4i) - (7 - 3i) = -4 + 7i$
and $(4 + 3i)(2 - 5i) = 23 - 14i$

Division – this is just rationalising the denominator.

$$\frac{3+4i}{5+2i} = \frac{3+4i}{5+2i} \times \frac{5-2i}{5-2i} \qquad \text{multiply top and bottom by the complex conjugate}$$

$$= \frac{23+14i}{25+4} = \frac{23}{29} + \frac{14}{29}i$$

Complex conjugate

$z = a + bi$
The *complex conjugate* of z is $z^* = \bar{z} = a - bi$

Properties

If $z = a + bi$ and $w = c + di$, then

(i) $\{(a + bi) + (c + di)\}^* = \{(a + c) + (b + d)i\}^*$

$$= \{(a + c) - (b + d)i\}$$

$$= (a - bi) + (c - di)$$

$\Leftrightarrow \quad (z + w)^* = z^* + w^*$

(ii) $\{(a + bi)(c + di)\}^*$ $= \{(ac - bd) + (ad + bc)i\}^*$

$= \{(ac - bd) - (ad + bc)i\}$

$= (a - bi)(c - di)$

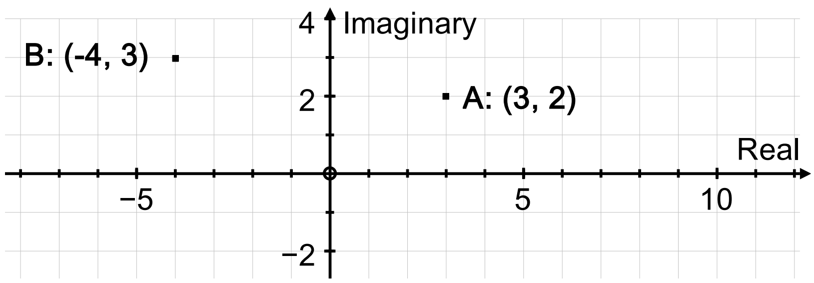
$= (a + bi)^*(c + di)^*$

$\Leftrightarrow \quad (zw)^* = z^* w^*$

Complex number plane, or Argand diagram

We can represent complex numbers as points on the complex number plane:

$3 + 2i$ as the point A (3, 2), and $-4 + 3i$ as the point (–4, 3).



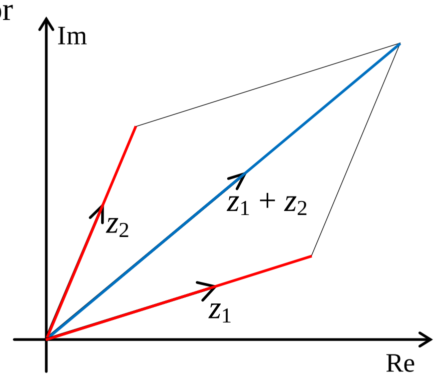
Complex numbers and vectors

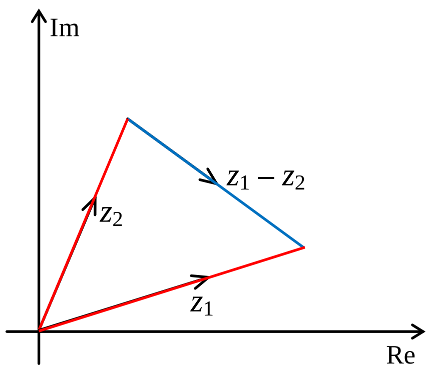
Complex numbers under addition (or subtraction) behave just like vectors under addition (or subtraction). We can show complex numbers on the Argand diagram as **either** points **or** vectors.

$(a + bi) + (c + di) = (a + c) + (b + d)\,i \quad \Leftrightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$

$(a + bi) - (c + di) = (a - c) + (b - d)\,i \quad \Leftrightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a - c \\ b - d \end{pmatrix}$

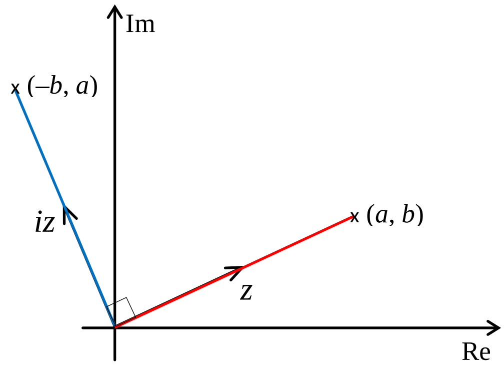
or





Multiplication by *i*

$i(3 + 4i) = -4 + 3i$ – on an Argand diagram this would have the effect of a positive quarter turn about the origin.

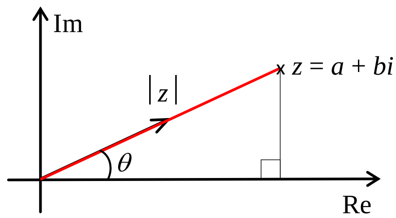


In general;

$i(a + bi) = -b + ai$

Modulus of a complex number

This is just like polar co-ordinates.



The modulus of z is $|z|$ and

is the length of the complex number

$$|z| = \sqrt{a^2 + b^2}.$$

$$z\,z^* = (a + bi)(a - bi) = a^2 + b^2$$

$$\Rightarrow \quad z\,z^* = |z|^2.$$

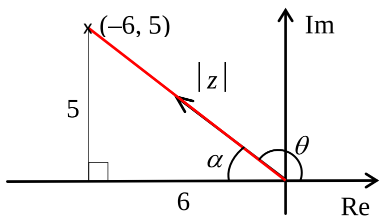
Argument of a complex number

The argument of z is $\arg z$ = the angle made by the complex number with the positive x-axis.
By convention, $-\pi < \arg z \le \pi$.

N.B. **Always draw a diagram when finding arg *z*.**

Example: Find the modulus and argument of $z = -6 + 5i$.

Solution: First sketch a diagram (it is easy to get the argument wrong if you don't).



$$|z| = \sqrt{6^2 + 5^2} = \sqrt{61}$$

and $\tan\alpha = \frac{5}{6} \Rightarrow \alpha = 0{\cdot}694738276$

$\Rightarrow \quad \arg z = \theta = \pi - \alpha = 2.45$ to 3 S.F.

Equality of complex numbers

$$a + bi = c + di \quad \Rightarrow \quad a - c = (d - b)i$$

$\Rightarrow$ $(a - c)^2 = (d - b)^2 i^2 = -(d - b)^2$ squaring both sides

But $(a - c)^2 \geq 0$ and $-(d - b)^2 \leq 0$

$\Rightarrow$ $(a - c)^2 = -(d - b)^2 = 0$

$\Rightarrow$ $a = c$ and $b = d$

Thus $a + bi = c + di$

$\Rightarrow$ real parts are equal ($a = c$), and imaginary parts are equal ($b = d$).

Square roots

Example: Find the square roots of $5 + 12i$, in the form $a + bi$, $a, b \in \mathbb{R}$.

Solution: Let $\sqrt{5 + 12i} = a + bi$

$\Rightarrow$ $5 + 12i = (a + bi)^2 = a^2 - b^2 + 2abi$

Equating real parts $\Rightarrow$ $a^2 - b^2 = 5$, **I**

equating imaginary parts $\Rightarrow$ $2ab = 12$ $\Rightarrow$ $a = \frac{6}{b}$

Substitute in **I** $\Rightarrow$ $\left(\frac{6}{b}\right)^2 - b^2 = 5$

$\Rightarrow$ $36 - b^4 = 5b^2$ $\Rightarrow$ $b^4 + 5b^2 - 36 = 0$

$\Rightarrow$ $(b^2 - 4)(b^2 + 9) = 0$ $\Rightarrow$ $b^2 = 4$

$\Rightarrow$ $b = \pm 2$, and $a = \pm 3$

$\Rightarrow$ $\sqrt{5 + 12i} = 3 + 2i$ or $-3 - 2i$.

Roots of equations

(a) **Any** polynomial equation with complex coefficients has a complex solution.

The is *The Fundamental Theorem of Algebra*, and is too difficult to prove at this stage.

Corollary: Any complex polynomial can be factorised into linear factors over the complex numbers.

(*b*) **If** $z = a + bi$ is a root of $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \alpha_{n-2} z^{n-2} + \ldots + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$,
and if all the α_i are **real**,
then the conjugate, $z^* = a - bi$ is also a root.

The proof of this result is in the appendix.

(*c*) For any polynomial with zeros $a + bi$, $a - bi$,
$(z - (a + bi))(z - (a - bi)) = z^2 - 2az + a^2 - b^2$ will be a quadratic factor in which the coefficients are all **real**.

(*d*) Using (*a*), (*b*), (c) we can see that any polynomial with **real** coefficients can be factorised into a mixture of linear and quadratic factors, all of which have **real** coefficients.

Example: Show that $3 - 2i$ is a root of the equation $z^3 - 8z^2 + 25z - 26 = 0$.
Find the other two roots.

Solution: Put $z = 3 - 2i$ in $z^3 - 8z^2 + 25z - 26$

$= (3 - 2i)^3 - 8(3 - 2i)^2 + 25(3 - 2i) - 26$

$= 27 - 54i + 36i^2 - 8i^3 - 8(9 - 12i + 4i^2) + 75 - 50i - 26$

$= 27 - 54i - 36 + 8i - 72 + 96i + 32 + 75 - 50i - 26$

$= 27 - 36 - 72 + 32 + 75 - 26 + (-54 + 8 + 96 - 50)i$

$= 0 + 0i$

$\Rightarrow$ $3 - 2i$ is a root

$\Rightarrow$ the conjugate, $3 + 2i$, is also a root — since all coefficients are **real**

$\Rightarrow$ $(z - (3 + 2i))(z - (3 - 2i)) = z^2 - 6z + 13$ is a factor.

Factorising, by inspection,

$$z^3 - 8z^2 + 25z - 26 = (z^2 - 6z + 13)(z - 2) = 0$$

$\Rightarrow$ roots are $z = 3 \pm 2i$, or 2

2 Numerical solutions of equations

Accuracy of solution

When asked to show that a solution is accurate to n D.P., you must look at the value of $f(x)$ 'half' below and 'half' above, and conclude that

there is a **change of sign** in the **interval**, and the function is **continuous**, therefore there is a **solution in the interval correct to n D.P.**

Example: Show that $\alpha = 2\cdot0946$ is a root of the equation $f(x) = x^3 - 2x - 5 = 0$, accurate to 4 D.P.

Solution:

$f(2.09455) = -0\cdot0000165\ldots$, and $f(2.09465) = +0\cdot00997$

There is a **change of sign** and f is **continuous**

$\Rightarrow$ there is a **root** in **[2·09455, 2·09465]** $\Rightarrow$ **root is** $\alpha = 2\cdot0946$ **to 4 D.P.**

Interval bisection

(i) Find an interval $[a, b]$ which contains the root of an equation $f(x) = 0$.

(ii) $x = \frac{a+b}{2}$ is the mid-point of the interval $[a, b]$

Find $f\left(\frac{a+b}{2}\right)$ to decide whether the root lies in $\left[a, \frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2}, b\right]$.

(iii) Continue finding the mid-point of each subsequent interval to narrow the interval which contains the root.

Example: (i) Show that there is a root of the equation $f(x) = x^3 - 2x - 7 = 0$ in the interval $[2, 3]$.

(ii) Find an interval of width 0·25 which contains the root.

Solution: (i) $f(2) = 8 - 4 - 7 = -3$, and $f(3) = 27 - 6 - 7 = 14$

There is a **change of sign** and f is **continuous** $\Rightarrow$ there is a root in $[2, 3]$.

(ii) Mid-point of $[2, 3]$ is $x = 2\cdot5$, and $f(2\cdot5) = 15\cdot625 - 5 - 7 = 3\cdot625$

$\Rightarrow$ change of sign between $x = 2$ and $x = 2\cdot5$

$\Rightarrow$ root in $[2, 2\cdot5]$

Mid-point of [2, 2·5] is $x = 2{\cdot}25$,
and $f(2{\cdot}25) = 11{\cdot}390625 - 4{\cdot}5 - 7 = -0{\cdot}109375$

$\Rightarrow$ change of sign between $x = 2{\cdot}25$ and $x = 2{\cdot}5$

$\Rightarrow$ root in [2·25, 2·5], which is an interval of width 0·25

Linear interpolation

To solve an equation $f(x)$ using linear interpolation.

First, find an interval which contains a root,

second, assume that the curve is a straight line and use similar triangles to find where the line crosses the x-axis,

third, repeat the process as often as necessary.

Example: (i) Show that there is a root, α, of the equation

$$f(x) = x^3 - 2x - 9 = 0 \text{ in the interval } [2, 3].$$

(ii) Use linear interpolation once to find an approximate value of α.
Give your answer to 3 D.P.

Solution: (i) $f(2) = 8 - 4 - 9 = -5$, and $f(3) = 27 - 6 - 9 = 12$

There is a **change of sign** and f is **continuous** $\Rightarrow$ there is a root in [2, 3].

(ii) From (i), curve passes through (2, –5) and (3, 12), and we assume that the curve is a straight line between these two points.

Let the line cross the x-axis at $(\alpha, 0)$

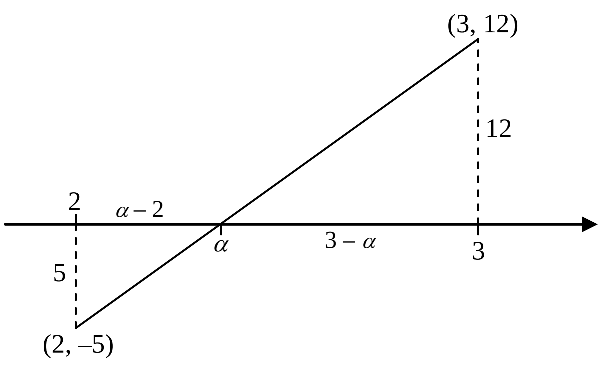
Using similar triangles

$$\frac{3-\alpha}{\alpha-2} = \frac{12}{5}$$

$\Rightarrow$ $15 - 5\alpha = 12\alpha - 24$

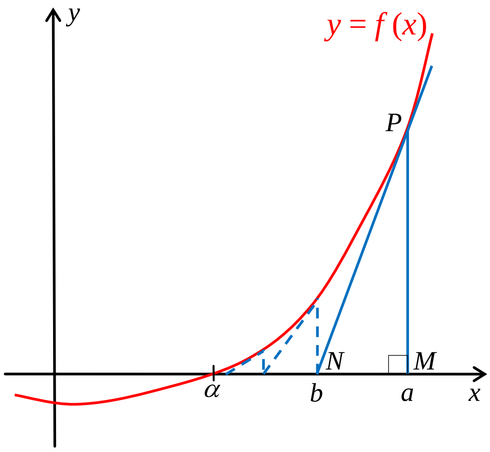
$\Rightarrow$ $\alpha = \frac{39}{17} = 2\frac{5}{17}$

$\Rightarrow$ $\alpha = 2{\cdot}294$ to 3 D.P.



Repeating the process will improve accuracy.

Newton-Raphson



Suppose that the equation $f(x) = 0$ has a root at $x = \alpha$, $\Rightarrow f(\alpha) = 0$

To find an approximation for this root, we first find a value $x = a$ near to $x = \alpha$ (decimal search).

In general, the point where the tangent at P, $x = a$, meets the x-axis, $x = b$, will give a better approximation.

At P, $x = a$, the gradient of the tangent is $f'(a)$,

and the gradient of the tangent is also $\frac{PM}{NM}$.

$PM = y = f(a)$ and $NM = a - b$

$$\Rightarrow \quad f'(a) = \frac{PM}{NM} = \frac{f(a)}{a-b} \quad \Rightarrow \quad b = a - \frac{f(a)}{f'(a)}.$$

Further approximations can be found by repeating the process, which would follow the dotted line converging to the point $(\alpha, 0)$.

This formula can be written as the iteration $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Example: (i) Show that there is a root, α, of the equation $f(x) = x^3 - 2x - 5 = 0$ in the interval $[2, 3]$.

(ii) Starting with $x_0 = 2$, use the Newton-Raphson formula to find x_1, x_2 and x_3, giving your answers to 3 D.P. where appropriate.

Solution: (i) $f(2) = 8 - 4 - 5 = -1$, and $f(3) = 27 - 6 - 5 = 16$

There is a **change of sign** and f is **continuous** $\Rightarrow$ there is a root in $[2, 3]$.

(ii) $f(x) = x^3 - 2x - 5 \quad \Rightarrow \quad f'(x) = 3x^2 - 2$

$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{8-4-5}{12-2} = 2{\cdot}1$$

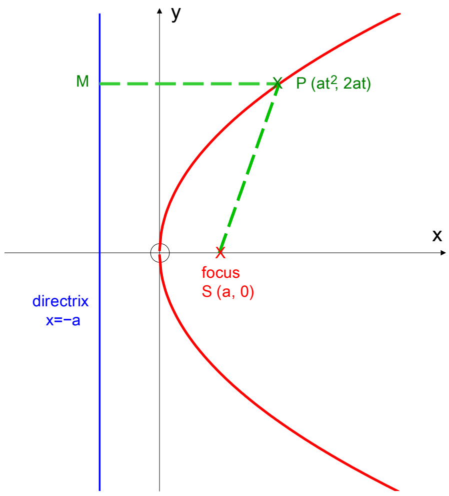
$$\Rightarrow \quad x_2 = 2{\cdot}094568121 = 2{\cdot}095$$

$$\Rightarrow \quad x_3 = 2{\cdot}094551482 = 2{\cdot}095$$

3 Coordinate systems

Parabolas

$y^2 = 4ax$ is the equation of a parabola which passes through the origin and has the x-axis as an axis of symmetry.



Parametric form

$x = at^2$, $y = 2at$ satisfy the equation for all values of t.
t is a parameter, and these equations are the parametric equations of the parabola $y^2 = 4ax$.

Focus and directrix

The point S $(a, 0)$ is the *focus*, and

the line $x = -a$ is the *directrix*.

Any point P of the curve is equidistant from the focus and the directrix, $PM = PS$.

Proof: $PM = at^2 - (-a) = at^2 + a$

$$PS^2 = (at^2 - a)^2 + (2at)^2 = a^2t^4 - 2a^2t^2 + a^2 + 4a^2t^2$$

$$= a^2t^4 + 2a^2t^2 + a^2 = (at^2 + a)^2 = PM^2$$

$\Rightarrow$ $PM = PS$.

Gradient

For the parabola $y^2 = 4ax$, with general point P, $(at^2, 2at)$, we can find the gradient in two ways:

1. $y^2 = 4ax$
   $\Rightarrow 2y\dfrac{dy}{dx} = 4a \Rightarrow \dfrac{dy}{dx} = \dfrac{2a}{y}$, which we can write as $\dfrac{dy}{dx} = \dfrac{2a}{2at} = \dfrac{1}{t}$
2. At P, $x = at^2$, $y = 2at$
   $\Rightarrow \dfrac{dy}{dt} = 2a, \quad \dfrac{dx}{dt} = 2at$
   $\Rightarrow \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2a}{2at} = \dfrac{1}{t}$

Tangents and normals

Example: Find the equations of the tangents to $y^2 = 8x$ at the points where $x = 18$, and show that the tangents meet on the x-axis.

Solution: $x = 18 \quad \Rightarrow \quad y^2 = 8 \times 18 \quad \Rightarrow \quad y = \pm 12$

$2y\frac{dy}{dx} = 8 \quad \Rightarrow \quad \frac{dy}{dx} = \pm\frac{1}{3}$ since $y = \pm 12$

$\Rightarrow$ tangents are $y - 12 = \frac{1}{3}(x - 18) \quad \Rightarrow \quad x - 3y + 18 = 0$ at (18, 12)

and $y + 12 = -\frac{1}{3}(x - 18) \quad \Rightarrow \quad x + 3y + 18 = 0.$ at (18, –12)

To find the intersection, add the equations to give

$2x + 36 = 0 \quad \Rightarrow \quad x = -18 \quad \Rightarrow \quad y = 0$

$\Rightarrow$ tangents meet at (–18, 0) on the x-axis.

Example: Find the equation of the normal to the parabola given by $x = 3t^2$, $y = 6t$.

Solution: $x = 3t^2,\ y = 6t \Rightarrow \quad \frac{dx}{dt} = 6t, \quad \frac{dy}{dt} = 6,$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6}{6t} = \frac{1}{t}$

$\Rightarrow$ gradient of the normal is $\frac{-1}{\frac{1}{t}} = -t$

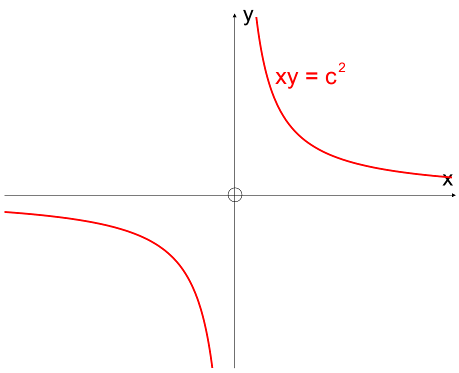
$\Rightarrow$ equation of the normal is $y - 6t = -t(x - 3t^2)$.

Notice that this 'general equation' gives the equation of the normal for any particular value of t:– when $t = -3$ the normal is $y + 18 = 3(x - 27) \Leftrightarrow y = 3x - 99$.

Rectangular hyperbolas

A *rectangular* hyperbola is a hyperbola in which the asymptotes meet at 90°.

$xy = c^2$ is the equation of a rectangular hyperbola in which the x-axis and y-axis are perpendicular asymptotes.



Parametric form

$x = ct$, $y = \frac{c}{t}$ are parametric equations of the hyperbola $xy = c^2$.

Tangents and normals

Example: Find the equation of the tangent to the hyperbola $xy = 36$ at the point where $x = 3$.

Solution: $x = 3 \Rightarrow 3y = 36 \Rightarrow y = 12$

$y = \frac{36}{x} \Rightarrow \frac{dy}{dx} = -\frac{36}{x^2} = -4$ when $x = 3$

$\Rightarrow$ tangent is $y - 12 = -4(x - 3) \Rightarrow 4x + y - 24 = 0$.

Example: Find the equation of the normal to the hyperbola given by $x = 3t$, $y = \frac{3}{t}$.

Solution: $x = 3t$, $y = \frac{3}{t} \Rightarrow \frac{dx}{dt} = 3$, $\frac{dy}{dt} = \frac{-3}{t^2}$

$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-3}{t^2}}{3} = \frac{-1}{t^2}$

$\Rightarrow$ gradient of the normal is $\frac{-1}{\frac{-1}{t^2}} = t^2$

$\Rightarrow$ equation of the normal is $y - \frac{3}{t} = t^2(x - 3t)$

$\Rightarrow t^3x - ty = 3t^4 - 3$.

4 Matrices

You must be able to add, subtract and multiply matrices.

Order of a matrix

An $r \times c$ matrix has r rows and c columns;

the fi**R**st number is the number of **R**ows

the se**C**ond number is the number of **C**olumns.

Identity matrix

The identity matrix is $\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note that $\boldsymbol{MI} = \boldsymbol{IM} = \boldsymbol{M}$ for *any* matrix $\boldsymbol{M}$.

Determinant and inverse

Let $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the *determinant* of $\boldsymbol{M}$ is

Det $\boldsymbol{M} = |\boldsymbol{M}| = ad - bc$.

To find the *inverse* of $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Note that $\boldsymbol{M}^{-1}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{M}^{-1} = \boldsymbol{I}$

(i) Find the determinant, $ad - bc$.
If $ad - bc = 0$, there is no inverse.

(ii) Interchange a and d (the leading diagonal)
Change sign of b and c, (the other diagonal)
Divide all elements by the determinant, $ad - bc$.

$$\Rightarrow \quad \boldsymbol{M}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Check:

$$\boldsymbol{M}^{-1}\boldsymbol{M} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} da-bc & 0 \\ 0 & -cb+ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}$$

Similarly we could show that $\boldsymbol{M}\boldsymbol{M}^{-1} = \boldsymbol{I}$.

Example: $\boldsymbol{M} = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$ and $\boldsymbol{MN} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$. Find $\boldsymbol{N}$.

Solution: Notice that $\boldsymbol{M}^{-1}(\boldsymbol{MN}) = (\boldsymbol{M}^{-1}\boldsymbol{M})\boldsymbol{N} = \boldsymbol{IN} = \boldsymbol{N}$ multiplying on the **left** by $\boldsymbol{M}^{-1}$

But $\boldsymbol{MNM}^{-1} \neq \boldsymbol{IN}$ we can**not** multiply on the **right** by $\boldsymbol{M}^{-1}$

First find $\boldsymbol{M}^{-1}$

Det $\boldsymbol{M} = 4 \times 3 - 2 \times 5 = 2 \quad \Rightarrow \quad \boldsymbol{M}^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}$

Using $\boldsymbol{M}^{-1}(\boldsymbol{MN}) = \boldsymbol{IN} = \boldsymbol{N}$

$$\Rightarrow \quad \boldsymbol{N} = \frac{1}{2}\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -7 & 4 \\ 13 & -6 \end{pmatrix} = \begin{pmatrix} -3 \cdot 5 & 2 \\ 6 \cdot 5 & -3 \end{pmatrix}.$$

Singular and non-singular matrices

If det $\boldsymbol{A} = 0$, then $\boldsymbol{A}$ is a *singular matrix*, and $\boldsymbol{A}^{-1}$ does not exist.

If det $\boldsymbol{A} \neq 0$, then $\boldsymbol{A}$ is a *non-singular matrix*, and $\boldsymbol{A}^{-1}$ exists

Linear Transformations

A matrix can represent a transformation, but the point must be written as a column vector before multiplying by the matrix.

Example: The image of (2, 3) under $\boldsymbol{T} = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}$ is given by $\begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 23 \\ 8 \end{pmatrix}$

$\Rightarrow$ the image of (2, 3) is (23, 8).

Note that the image of (0, 0) is always (0, 0)

$\Leftrightarrow$ the **origin never moves** under a matrix (linear) transformation

Basis vectors

The vectors $\underline{\boldsymbol{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{\boldsymbol{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are called *basis* vectors, and are particularly important in describing the geometrical effect of a matrix, and in finding the matrix for a particular geometric transformation.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

$\underline{\boldsymbol{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} a \\ c \end{pmatrix}$, the *first* column, and $\underline{\boldsymbol{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} b \\ d \end{pmatrix}$, the *second* column

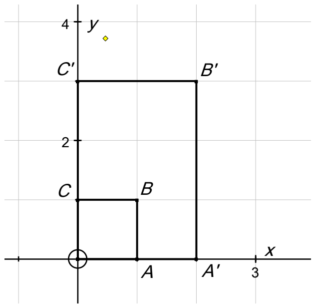
This is a more important result than it seems!

Finding the geometric effect of a matrix transformation

We can easily write down the images of $\underline{i}$ and $\underline{j}$, sketch them and find the geometrical transformation.

Example: Find the transformation represented by the matrix $\boldsymbol{T} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

Solution: Find images of $\underline{i}, \underline{j}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and show on a sketch. Make sure that you letter the points



$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 3 & 3 \end{pmatrix}$$

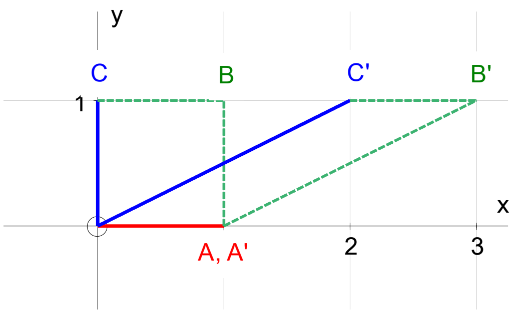
From sketch we can see that the transformation is a two-way stretch, of factor 2 parallel to the x-axis and of factor 3 parallel to the y-axis.

Finding the matrix of a given transformation.

Example: Find the matrix for a shear with factor 2 and invariant line the x-axis.

Solution: Each point is moved in the x-direction by a distance of (2 × its y-coordinate).

$\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (does not move as it is on the invariant line).
This will be the first column of the matrix $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$

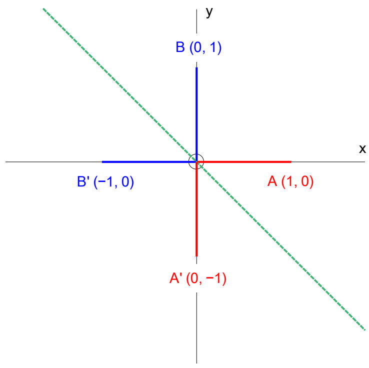


$\underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. This will be the second column of the matrix $\begin{pmatrix} * & 2 \\ * & 1 \end{pmatrix}$

$\Rightarrow$ Matrix of the shear is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Example: Find the matrix for a reflection in $y = -x$.

Solution: First find the images of $\underline{i}$ and $\underline{j}$. These will be the two columns of the matrix.



$A \to A' \quad \Rightarrow \quad \underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

This will be the first column of the matrix $\begin{pmatrix} 0 & * \\ -1 & * \end{pmatrix}$

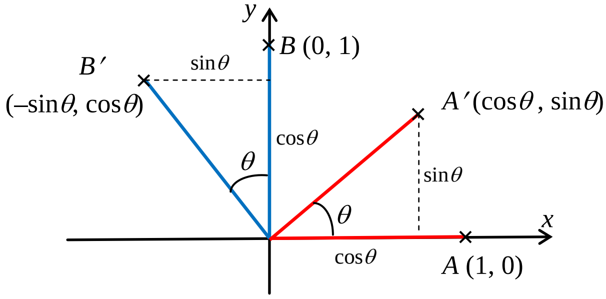
$B \to B' \quad \Rightarrow \quad \underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$

This will be the second column of the matrix $\begin{pmatrix} * & -1 \\ * & 0 \end{pmatrix}$

$\Rightarrow$ Matrix of the reflection is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

Rotation matrix

From the diagram we can see that



$\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$,

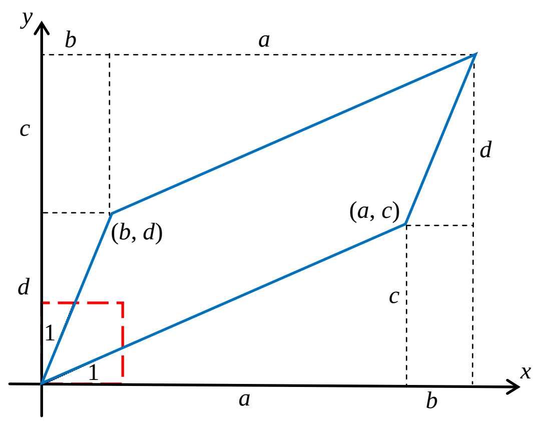
$\underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

These will be the first and second columns of the matrix

$\Rightarrow$ matrix is $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Determinant and area factor

For the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$



$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$

$\Rightarrow$ the unit square is mapped on to the parallelogram as shown in the diagram.

The area of the unit square = 1.

The area of the parallelogram $= (a + b)(c + d) - 2 \times (bc + \frac{1}{2}ac + \frac{1}{2}bd)$

$= \quad ac + ad + bc + bd - 2bc - ac - bd$

$= \quad ad - bc \quad = \det A.$

All squares of the grid are mapped onto congruent parallelograms

$\Rightarrow$ area factor of the transformation is $\det A = ad - bc$.

5 Series

You need to know the following sums

$$\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$$

$$= \left(\tfrac{1}{2}n(n+1)\right)^2 = \left(\sum_{r=1}^{n} r\right)^2$$ a fluke, but it helps to remember it

Example: Find $\sum_{r=1}^{n} r(r^2-3)$.

Solution: $\sum_{r=1}^{n} r(r^2-3) = \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r$

$= \quad \tfrac{1}{4}n^2(n+1)^2 - 3 \times \tfrac{1}{2}n(n+1)$

$= \quad \tfrac{1}{4}n(n+1)\{n(n+1) - 6\}$

$= \quad \tfrac{1}{4}n(n+1)(n+3)(n-2)$

Example: Find $S_n = 2^2 + 4^2 + 6^2 + \ldots + (2n)^2$.

Solution: $S_n = 2^2 + 4^2 + 6^2 + \ldots + (2n)^2 = 2^2(1^2 + 2^2 + 3^2 + \ldots + n^2)$

$= \quad 4 \times \tfrac{1}{6}n(n+1)(2n+1) = \tfrac{2}{3}n(n+1)(2n+1)$.

Example: Find $\sum_{r=5}^{n+2} r^2$

Solution: $\sum_{r=5}^{n+2} r^2 = \sum_{r=1}^{n+2} r^2 - \sum_{r=1}^{4} r^2$ notice that the top limit is 4 **not** 5

$= \quad \tfrac{1}{6}(n+2)(n+2+1)(2(n+2)+1) - \tfrac{1}{6} \times 4 \times 5 \times 9$

$= \quad \tfrac{1}{6}(n+2)(n+3)(2n+5) - 30$.

6 Proof by induction

1. Show that the result/formula is true for $n = 1$ (and sometimes $n = 2$, 3 ..).
   Conclude
   "therefore the result/formula ……… is true for $n = 1$".

2. Make induction assumption
   "Assume that the result/formula ……… is true for $n = k$".
   Show that the result/formula must then be true for $n = k + 1$
   Conclude
   "therefore the result/formula ……… is true for $n = k + 1$".

3. Final conclusion
   "therefore the result/formula ……….. is true for all positive integers, n, by mathematical induction".

Summation

Example: Use mathematical induction to prove that

$$S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

Solution: When $n = 1$, $S_1 = 1^2 = 1$ and $S_1 = \tfrac{1}{6} \times 1(1+1)(2 \times 1 + 1) = \tfrac{1}{6} \times 1 \times 2 \times 3 = 1$

$\Rightarrow$ $S_n = \tfrac{1}{6}n(n+1)(2n+1)$ is true for $n = 1$.

Assume that the formula is true for $n = k$

$\Rightarrow$ $S_k = 1^2 + 2^2 + 3^2 + \ldots + k^2 = \tfrac{1}{6}k(k+1)(2k+1)$

$\Rightarrow$ $S_{k+1} = 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \tfrac{1}{6}k(k+1)(2k+1) + (k+1)^2$

$$= \tfrac{1}{6}(k+1)\{k(2k+1) + 6(k+1)\}$$

$$= \tfrac{1}{6}(k+1)\{2k^2 + 7k + 6\} = \tfrac{1}{6}(k+1)(k+2)(2k+3)$$

$$= \tfrac{1}{6}(k+1)\{(k+1)+1\}\{2(k+1)+1\}$$

$\Rightarrow$ The formula is true for $n = k + 1$

$\Rightarrow$ $S_n = \tfrac{1}{6}n(n+1)(2n+1)$ is true for all positive integers, n, by mathematical induction.

Recurrence relations

Example: A sequence, 4, 9, 19, 39, … is defined by the recurrence relation

$u_1 = 4$, $u_{n+1} = 2u_n + 1$. Prove that $u_n = 5 \times 2^{n-1} - 1$.

Solution: When $n = 1$, $u_1 = 4$, and $u_1 = 5 \times 2^{1-1} - 1 = 5 - 1 = 4$, $\Rightarrow$ formula true for $n = 1$.

Assume that the formula is true for $n = k$, $\Rightarrow$ $u_k = 5 \times 2^{k-1} - 1$.

From the recurrence relation,

$$u_{k+1} = 2u_k + 1 = 2(5 \times 2^{k-1} - 1) + 1$$

$$\Rightarrow \quad u_{k+1} = 5 \times 2^k - 2 + 1 = 5 \times 2^{(k+1)-1} - 1$$

$\Rightarrow$ the formula is true for $n = k + 1$

$\Rightarrow$ the formula is true for all positive integers, n, by mathematical induction.

Divisibility problems

Considering $f(k + 1) - f(k)$, will lead to a proof which sometimes has hidden difficulties,

and a more reliable way is to consider $f(k + 1) - m \times f(k)$, where m is chosen to eliminate the exponential term.

Example: Prove that $f(n) = 5^n - 4n - 1$ is divisible by 16 for all positive integers, n.

Solution: When $n = 1$, $f(1) = 5^1 - 4 - 1 = 0$, which is divisible by 16, and so $f(n)$ is divisible by 16 when $n = 1$.

Assume that the result is true for $n = k$, $\Rightarrow$ $f(k) = 5^k - 4k - 1$ is divisible by 16.

Considering $f(k + 1) - 5 \times f(k)$ we will eliminate the 5^k term.

$$f(k+1) - 5 \times f(k) = (5^{k+1} - 4(k+1) - 1) - 5 \times (5^k - 4k - 1)$$

$$= 5^{k+1} - 4k - 4 - 1 - 5^{k+1} + 20k + 5 = 16k$$

$$\Rightarrow \quad f(k+1) = 5 \times \boldsymbol{f(k)} + \mathbf{16}k$$

Since $\boldsymbol{f(k)}$ is divisible by 16 (induction assumption), and $\mathbf{16}k$ is divisible by 16, then $f(k + 1)$ must be divisible by 16,

$\Rightarrow$ $f(n) = 5^n - 4n - 1$ is divisible by 16 for $n = k + 1$

$\Rightarrow$ $f(n) = 5^n - 4n - 1$ is divisible by 16 for all positive integers, n, by mathematical induction.

Example: Prove that $f(n) = 2^{2n+3} + 3^{2n-1}$ is divisible by 5 for all positive integers *n*.

Solution: When $n = 1$, $f(1) = 2^{2+3} + 3^{2-1} = 32 + 3 = 35 = \mathbf{5} \times 7$, and so the result is true for $n = 1$.

Assume that the result is true for $n = k$

$\Rightarrow$ $f(k) = 2^{2k+3} + 3^{2k-1}$ is divisible by 5

We could consider either (it does not matter which)

$f(k+1) - 2^2 \times f(k)$, which would eliminate the 2^{2k+3} term **I**

or $f(k+1) - 3^2 \times f(k)$, which would eliminate the 3^{2k-1} term **II**

I $\Rightarrow$ $f(k+1) - 2^2 \times f(k) = 2^{2(k+1)+3} + 3^{2(k+1)-1} - 2^2 \times (2^{2k+3} + 3^{2k-1})$

$= 2^{2k+5} + 3^{2k+1} - 2^{2k+5} - 2^2 \times 3^{2k-1}$

$\Rightarrow$ $f(k+1) - 4 \times f(k) = 9 \times 3^{2k-1} - 4 \times 3^{2k-1} = 5 \times 3^{2k-1}$

$\Rightarrow$ $f(k+1) = 4 \times \boldsymbol{f(k)} - \mathbf{5} \times 3^{2k-1}$

Since $\boldsymbol{f(k)}$ is divisible by **5** (induction assumption), and $\mathbf{5} \times 3^{2k-1}$ is divisible by **5**, then $f(k+1)$ must be divisible by **5**.

$\Rightarrow$ $f(n) = 2^{2n+3} + 3^{2n-1}$ is divisible by 5 for all positive integers, *n*, by mathematical induction.

Powers of matrices

Example: If $M = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$, prove that $M^n = \begin{pmatrix} 2^n & 1-2^n \\ 0 & 1 \end{pmatrix}$ for all positive integers *n*.

Solution: When $n = 1$, $M^1 = \begin{pmatrix} 2^1 & 1-2^1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = M$

$\Rightarrow$ the formula is true for $n = 1$.

Assume the formula is true for $n = k$ $\Rightarrow$ $M^k = \begin{pmatrix} 2^k & 1-2^k \\ 0 & 1 \end{pmatrix}$.

$$M^{k+1} = MM^k = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2^k & 1-2^k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 \times 2^k & 2 - 2 \times 2^k - 1 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow$ $M^{k+1} = \begin{pmatrix} 2^{k+1} & 1-2^{k+1} \\ 0 & 1 \end{pmatrix}$ $\Rightarrow$ The formula is true for $n = k + 1$

$\Rightarrow$ $M^n = \begin{pmatrix} 2^n & 1-2^n \\ 0 & 1 \end{pmatrix}$ is true for all positive integers, *n*, by mathematical induction.

7 Appendix

Complex roots of a real polynomial equation

Preliminary results:

I $(z_1 + z_2 + z_3 + z_4 + \ldots + z_n)^* = z_1^* + z_2^* + z_3^* + z_4^* + \ldots + z_n^*$,

by repeated application of $(z + w)^* = z^* + w^*$

II $(z^n)^* = (z^*)^n$

$(zw)^* = z^*w^*$

$\Rightarrow (z^n)^* = (z^{n-1}z)^* = (z^{n-1})^*(z)^* = (z^{n-2}z)^*(z)^* = (z^{n-2})^*(z)^*(z)^* \ldots = (z^*)^n$

Theorem: **If** $z = a + bi$ is a root of $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \alpha_{n-2} z^{n-2} + \ldots + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$,
and if all the α_i are **real**,
then the conjugate, $z^* = a - bi$ is also a root.

Proof: If $z = a + bi$ is a root of the equation $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots + \alpha_1 z + \alpha_0 = 0$

then $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$

$\Rightarrow (\alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots + \alpha_2 z^2 + \alpha_1 z + \alpha_0)^* = 0$ since $0^* = 0$

$\Rightarrow (\alpha_n z^n)^* + (\alpha_{n-1} z^{n-1})^* + \ldots + (\alpha_2 z^2)^* + (\alpha_1 z)^* + (\alpha_0)^* = 0$ using **I**

$\Rightarrow \alpha_n^*(z^n)^* + \alpha_{n-1}^*(z^{n-1})^* + \ldots + \alpha_2^*(z^2)^* + \alpha_1^*(z)^* + \alpha_0^* = 0$ since $(zw)^* = z^*w^*$

$\Rightarrow \alpha_n(z^n)^* + \alpha_{n-1}(z^{n-1})^* + \ldots + \alpha_2(z^2)^* + \alpha_1(z)^* + \alpha_0 = 0$ α_i **real** $\Rightarrow \alpha_i^* = \alpha_i$

$\Rightarrow \alpha_n(z^*)^n + \alpha_{n-1}(z^*)^{n-1} + \ldots + \alpha_2(z^*)^2 + \alpha_1(z^*) + \alpha_0 = 0$ using **II**

$\Rightarrow z^* = a - bi$ is also a root of the equation.

Formal definition of a linear transformation

A linear transformation $\boldsymbol{T}$ has the following properties:

(i) $\boldsymbol{T}\begin{pmatrix} kx \\ ky \end{pmatrix} = k\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix}$

(ii) $\boldsymbol{T}\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = \boldsymbol{T}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \boldsymbol{T}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

It can be shown that **any** matrix transformation is a linear transformation, and that **any** linear transformation can be represented by a matrix.

Derivative of x^n, for any integer

We can use proof by induction to show that $\frac{d}{dx}(x^n) = nx^{n-1}$, for any integer *n*.

1) We know that the derivative of x^0 is 0 which equals $0x^{-1}$,

since $x^0 = 1$, and the derivative of 1 is 0

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = 0$.

2) We know that the derivative of x^1 is 1 which equals $1 \times x^{1-1}$

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = 1$

Assume that the result is true for $n = k$

$\Rightarrow \quad \frac{d}{dx}(x^k) = kx^{k-1}$

$\Rightarrow \quad \frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \times x^k) = x \times \frac{d}{dx}(x^k) + 1 \times x^k$ product rule

$\Rightarrow \quad \frac{d}{dx}(x^{k+1}) = x \times kx^{k-1} + x^k = kx^k + x^k = (k+1)x^k$

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = k + 1$

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for all positive integers, *n*, by mathematical induction.

3) We know that the derivative of x^{-1} is $-x^{-2}$ which equals $-1 \times x^{-1-1}$

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = -1$

Assume that the result is true for $n = k$

$\Rightarrow \quad \frac{d}{dx}(x^k) = kx^{k-1}$

$\Rightarrow \quad \frac{d}{dx}(x^{k-1}) = \frac{d}{dx}\left(\frac{x^k}{x}\right) = \frac{x \times \frac{d}{dx}(x^k) - x^k \times 1}{x^2}$ quotient rule

$\Rightarrow \quad \frac{d}{dx}(x^{k+1}) = \frac{x \times kx^{k-1} - x^k}{x^2} = \frac{(k-1)x^k}{x^2} = (k-1)x^{k-2} = (k-1)x^{(k-1)-1}$

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = k - 1$

We are going backwards (**from** $n = k$ **to** $n = k - 1$), and, since we started from $n = -1$,

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for all negative integers, *n*, by mathematical induction.

Putting 1), 2) and 3), we have proved that

$\frac{d}{dx}(x^n) = nx^{n-1}$, for **any** integer *n*.

8 Index

Pure

Further Mathematics 2

Revision Notes

Further Pure 2

1 Inequalities

Algebraic solutions

Remember that if you multiply both sides of an inequality by a negative number, you must turn the inequality sign round: $2x > 3 \Rightarrow -2x < -3$.

A difficulty occurs when multiplying both sides by, for example, $(x - 2)$; this expression is sometimes positive ($x > 2$), sometimes negative ($x < 2$) and sometimes zero ($x = 2$). In this case we multiply both sides by $(x - 2)^2$, which is always positive (provided that $x \neq 2$).

Example 1: Solve the inequality $2x + 3 < \frac{x^2}{x-2}$, $x \neq 2$

Solution: Multiply both sides by $(x - 2)^2$ — we can do this since $(x - 2) \neq 0$

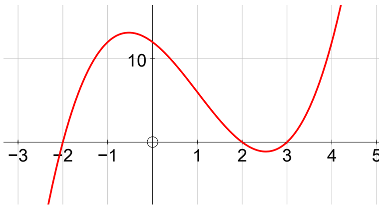
$\Rightarrow$ $(2x + 3)(x - 2)^2 < x^2(x - 2)$ — DO **NOT** MULTIPLY OUT

$\Rightarrow$ $(2x + 3)(x - 2)^2 - x^2(x - 2) < 0$

$\Rightarrow$ $(x - 2)(2x^2 - x - 6 - x^2) < 0$

$\Rightarrow$ $(x - 2)(x - 3)(x + 2) < 0$

$\Rightarrow$ $x < -2$, or $2 < x < 3$



Note – care is needed when the inequality is $\leq$ or $\geq$.

Example 2: Solve the inequality $\frac{x}{x+1} \geq \frac{2}{x+3}$, $x \neq -1$, $x \neq -3$

Solution: Multiply both sides by $(x + 1)^2(x + 3)^2$ — which cannot be zero

$\Rightarrow$ $x(x + 1)(x + 3)^2 \geq 2(x + 3)(x + 1)^2$ — DO **NOT** MULTIPLY OUT

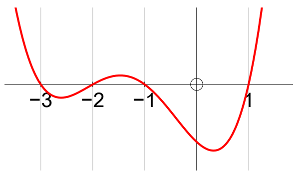
$\Rightarrow$ $x(x + 1)(x + 3)^2 - 2(x + 3)(x + 1)^2 \geq 0$

$\Rightarrow$ $(x + 1)(x + 3)(x^2 + 3x - 2x - 2) \geq 0$

$\Rightarrow$ $(x + 1)(x + 3)(x + 2)(x - 1) \geq 0$

from sketch it looks as though the solution is

$x \leq -3$ or $-2 \leq x \leq -1$ or $x \geq 1$



BUT since $x \neq -1$, $x \neq -3$,

the solution is $x < -3$ or $-2 \leq x < -1$ or $x \geq 1$

Graphical solutions

Example 1: On the same diagram sketch the graphs of $y = \frac{2x}{x+3}$ and $y = x - 2$.

Use your sketch to solve the inequality $\frac{2x}{x+3} \geq x - 2$

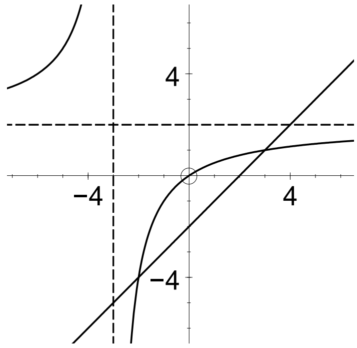
Solution: First find the points of intersection of the two graphs

$\Rightarrow \quad \frac{2x}{x+3} = x - 2$

$\Rightarrow \quad 2x = x^2 + x - 6$

$\Rightarrow \quad 0 = (x-3)(x+2)$

$\Rightarrow \quad x = -2 \text{ or } 3$



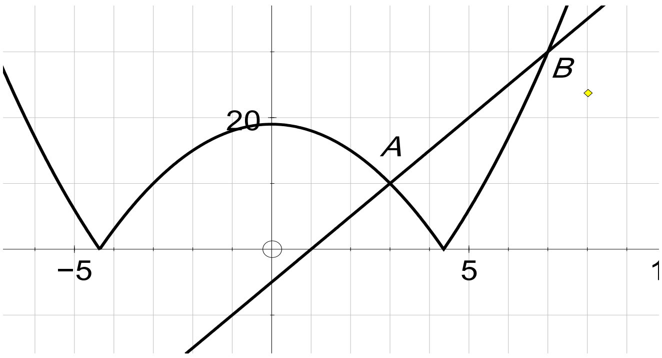
From the sketch we see that

$x < -3$ or $-2 \leq x \leq 3$. Note that $x \neq -3$

For inequalities involving |2*x* – 5| etc., it is often essential to sketch the graphs first.

Example 2: Solve the inequality $|x^2 - 19| < 5(x - 1)$.

Solution: It is essential to sketch the curves first in order to see which solutions are needed.



To find the point A, we need to solve

$-(x^2 - 19) = 5x - 5 \quad \Rightarrow \quad x^2 + 5x - 24 = 0$

$\Rightarrow \quad (x+8)(x-3) = 0 \quad \Rightarrow \quad x = -8 \text{ or } 3$

From the sketch $x \neq -8 \quad \Rightarrow \quad x = 3$

To find the point B, we need to solve

$$+(x^2 - 19) = 5x - 5 \quad \Rightarrow \quad x^2 - 5x - 14 = 0$$

$$\Rightarrow \quad (x-7)(x+2) = 0 \quad \Rightarrow \quad x = -2 \text{ or } 7$$

From the sketch $x \neq -2 \quad \Rightarrow \quad x = 7$

and the solution of $|x^2 - 19| < 5(x-1)$ is $3 < x < 7$

2 Series – Method of Differences

The trick here is to write each line out in full and see what cancels when you add.

Do not be tempted to work each term out – you will lose the pattern which lets you cancel when adding.

Example 1*:* Write $\frac{1}{r(r+1)}$ in partial fractions, and then use the method of differences to find the sum $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n(n+1)}$.

Solution: $\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$

put $r = 1 \Rightarrow \frac{1}{1\times 2} = \frac{1}{1} - \frac{1}{2}$

put $r = 2 \Rightarrow \frac{1}{2\times 3} = \frac{1}{2} - \frac{1}{3}$

put $r = 3 \Rightarrow \frac{1}{3\times 4} = \frac{1}{3} - \frac{1}{4}$

etc.

put $r = n \Rightarrow \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

adding $\Rightarrow \sum_1^n \frac{1}{r(r+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

Example 2: Write $\frac{2}{r(r+1)(r+2)}$ in partial fractions, and then use the method of differences to find the sum $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \cdots + \frac{1}{n(n+1)(n+2)}$.

Solution:
$$\frac{2}{r(r+1)(r+2)} = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}$$

put $r = 1$ $\Rightarrow$ $\frac{2}{1\times2\times3} = \frac{1}{1} - \frac{2}{2} + \frac{1}{3}$

put $r = 2$ $\Rightarrow$ $\frac{2}{2\times3\times4} = \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$

put $r = 3$ $\Rightarrow$ $\frac{2}{3\times4\times5} = \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$

put $r = 4$ $\Rightarrow$ $\frac{2}{4\times5\times6} = \frac{1}{4} - \frac{2}{5} + \frac{1}{6}$

⋮

etc.

⋮

put $r = n-1$ $\Rightarrow$ $\frac{2}{(n-1)n(n+1)} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$

put $r = n$ $\Rightarrow$ $\frac{2}{n(n+1)(n+2)} = \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$

---

adding $\Rightarrow$ $\sum_1^n \frac{2}{r(r+1)(r+2)} = \frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2}$

$$= \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

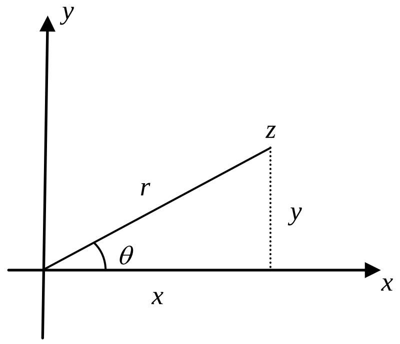
$$= \frac{n^2+3n+2-2n-4+2n+2}{2(n+1)(n+2)}$$

$\Rightarrow$ $\sum_1^n \frac{2}{r(r+1)(r+2)} = \frac{n^2+3n}{2(n+1)(n+2)}$

$\Rightarrow$ $\sum_1^n \frac{1}{r(r+1)(r+2)} = \frac{n^2+3n}{4(n+1)(n+2)}$

3 Complex Numbers

Modulus and Argument



The modulus of $z = x + iy$ is the length of z

$\Rightarrow \quad r = |z| = \sqrt{x^2 + y^2}$

and the argument of z is the angle made by z with the positive x-axis, between $-\pi$ and π.

N.B. $\arg z$ is **not always** equal to $\tan^{-1}\left(\frac{y}{x}\right)$

Properties

$z = r\cos\theta + i\,r\sin\theta$

$|zw| = |z|\,|w|$, and $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

$\arg(zw) = \arg z + \arg w$, and $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$

Euler's Relation $e^{i\theta}$

$z = e^{i\theta} = \cos\theta + i\sin\theta$

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

Example: Express $5e^{\left(\frac{i3\pi}{4}\right)}$ in the form $x + iy$.

Solution: $5e^{\left(\frac{i3\pi}{4}\right)} = 5\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

$= \frac{-5\sqrt{2}}{2} + i\frac{5\sqrt{2}}{2}$

Multiplying and dividing in mod-arg form

$re^{i\theta} \times se^{i\phi} = rs\,e^{i(\theta+\phi)}$

$\equiv \quad (r\cos\theta + i\,r\sin\theta) \times (s\cos\phi + i\,s\sin\phi) = rs\cos(\theta+\phi) + i\,rs\sin(\theta+\phi)$

and

$re^{i\theta} \div se^{i\phi} = \frac{r}{s}\,e^{i(\theta-\phi)}$

$\equiv (r\cos\theta + i\,r\sin\theta) \div (s\cos\phi + i\,s\sin\phi) = \frac{r}{s}\cos(\theta-\phi) + i\,\frac{r}{s}\sin(\theta-\phi)$

De Moivre's Theorem

$$\left(re^{i\theta}\right)^n \;=\; r^n e^{in\theta} \quad \equiv \quad (r\cos\theta + i\,r\sin\theta)^n = (r^n\cos n\theta + i\,r^n\sin n\theta)$$

Applications of De Moivre's Theorem

Example: Express $\sin 5\theta$ in terms of $\sin\theta$ only.

Solution: From De Moivre's Theorem we know that

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

$$= \cos^5\theta + 5i\cos^4\theta\,\sin\theta + 10i^2\cos^3\theta\,\sin^2\theta + 10i^3\cos^2\theta\,\sin^3\theta + 5i^4\cos\theta\,\sin^4\theta + i^5\sin^5\theta$$

Equating complex parts

$$\Rightarrow \quad \sin 5\theta = 5\cos^4\theta\,\sin\theta - 10\cos^2\theta\,\sin^3\theta + \sin^5\theta$$

$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$

$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

$z^n + \frac{1}{z^n} = 2\cos n\theta$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$

$$z = \cos\theta + i\sin\theta$$

$$\Rightarrow \quad z^n = (\cos\theta + i\sin\theta)^n = (\cos n\theta + i\,\sin n\theta)$$

and $$\frac{1}{z^n} = (\cos\theta - i\sin\theta)^n = (\cos n\theta - i\,\sin n\theta)$$

from which we can show that

$$\left(z + \frac{1}{z}\right) = 2\cos\theta \quad \text{and} \quad \left(z - \frac{1}{z}\right) = 2i\sin\theta$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Example: Express $\sin^5\theta$ in terms of $\sin 5\theta$, $\sin 3\theta$ and $\sin\theta$.

Solution: Here we are dealing with $\sin\theta$, so we use

$$(2i\sin\theta)^5 = \left(z - \frac{1}{z}\right)^5$$

$$\Rightarrow \quad 32i\sin^5\theta = z^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) - 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) - \left(\frac{1}{z^5}\right)$$

$$\Rightarrow \quad 32i\sin^5\theta = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$\Rightarrow \quad 32i\sin^5\theta = 2i\sin 5\theta - 5\times 2i\sin 3\theta + 10\times 2i\sin\theta$$

$$\Rightarrow \quad \sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

n^{th} roots of a complex number

The technique is the same for finding n^{th} roots of any complex number.

Example: Find the 4^{th} roots of $4 + 4i$, and show the roots on an Argand Diagram.

Solution: We need to solve the equation $z^4 = 4 + 4i$

1. Let $z = r\cos\theta + i\,r\sin\theta$

   $\Rightarrow \quad z^4 = r^4(\cos 4\theta + i\sin 4\theta)$

2. $|4 + 4i| = \sqrt{4^2 + 4^2} = \sqrt{32}$ and $\arg(4 + 4i) = \frac{\pi}{4}$

   $\Rightarrow \quad 4 + 4i = \sqrt{32}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

3. Then $z^4 = 4 + 4i$

   becomes $r^4(\cos 4\theta + i\sin 4\theta) = \sqrt{32}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

   $= \sqrt{32}\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)$ adding 2π

   $= \sqrt{32}\left(\cos\frac{17\pi}{4} + i\sin\frac{17\pi}{4}\right)$ adding 2π

   $= \sqrt{32}\left(\cos\frac{25\pi}{4} + i\sin\frac{25\pi}{4}\right)$ adding 2π

4. $\Rightarrow \quad r^4 = \sqrt{32}$

   and $\quad 4\theta = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}, \frac{25\pi}{4}$

   $\Rightarrow \quad r = \sqrt[8]{32} = 1.5422$

   and $\quad \theta = \frac{\pi}{16}, \frac{9\pi}{16}, \frac{17\pi}{16}, \frac{25\pi}{16}$

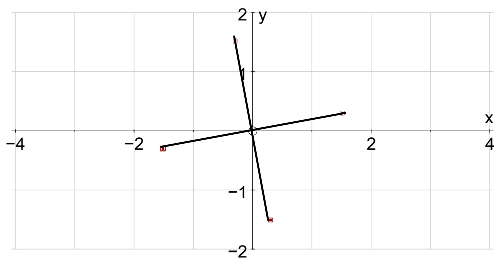
5. $\Rightarrow$ roots are

   $\sqrt[8]{32}\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right) = 1.513 + 0.301\,i$

   $\sqrt[8]{32}\left(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right) = -0.301 + 1.513\,i$

   $\sqrt[8]{32}\left(\cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16}\right) = -1.513 - 0.301\,i$

   $\sqrt[8]{32}\left(\cos\frac{25\pi}{16} + i\sin\frac{25\pi}{16}\right) = 0.301 - 1.513\,i$



Notice that the roots are symmetrically placed around the origin, and the angle between roots is $\frac{2\pi}{4} = \frac{\pi}{2}$ The angle between the n^{th} roots will always be $\frac{2\pi}{n}$.

For sixth roots the angle between roots will be $\frac{2\pi}{6} = \frac{\pi}{3}$, and so on.

Roots of polynomial equations with real coefficients

1. **Any** polynomial equation with real coefficients,
   $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots a_2x^2 + a_1x + a_0 = 0$,   …….. (I)
   where all a_i are real, has a complex solution
2. $\Rightarrow$ **any** complex n^{th} degree polynomial can be factorised into n linear factors over the complex numbers
3. If $z = a + ib$ is a root of (I), then its conjugate, $a - ib$ is also a root.
4. By pairing factors with conjugate pairs we can say that any polynomial with real coefficients can be factorised into a combination of linear and quadratic factors over the real numbers.

Example: Given that $3 - 2i$ is a root of $z^3 - 5z^2 + 7z + 13 = 0$

(a) Factorise over the real numbers

(b) Find all three real roots

Solution:

(a) $3 - 2i$ is a root $\Rightarrow$ $3 + 2i$ is also a root

$\Rightarrow$ $(z - (3 - 2i))(z - (3 + 2i)) = (z^2 - 6z + 13)$ is a factor

$\Rightarrow$ $z^3 - 5z^2 + 7z + 13 = (z^2 - 6z + 13)(z + 1)$ by inspection

(b) $\Rightarrow$ roots are $z = 3 - 2i,\ 3 + 2i$ and -1

Loci on an Argand Diagram

Two basic ideas

1. $|z - w|$ is the distance from w to z.
2. $\arg(z - (1 + i))$ is the angle made by the line joining $(1+i)$ to z, with the x-axis.

Example 1:

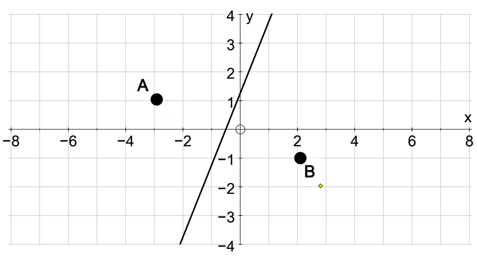
$|z - 2 - i| = 3$ is a circle with centre $(2 + i)$ and radius 3

Example 2:

$|z + 3 - i| = |z - 2 + i|$

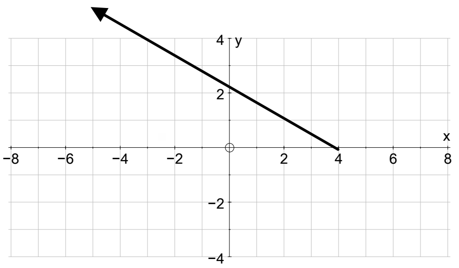
$\Leftrightarrow$ $|z - (-3 + i)| = |z - (2 - i)|$

is the locus of all points which are equidistant from the points $A\ (-3, 1)$ and $B\ (2, -1)$, and so is the perpendicular bisector of AB.



Example 3:

arg $(z - 4) = \frac{5\pi}{6}$ is a half line, from (4, 0), making an angle of $\frac{5\pi}{6}$ with the x-axis.



Example 4:

$|z - 3| = 2|z + 2i|$ is a circle (Apollonius's circle).

To find its equation, put $z = x + iy$

$\Rightarrow \quad |(x-3) + iy| = 2|x + i(y+2)|$ square both sides

$\Rightarrow \quad (x-3)^2 + y^2 = 4\left(x^2 + (y+2)^2\right)$ leading to

$\Rightarrow \quad 3x^2 + 6x + 3y^2 + 16y + 7 = 0$

$\Rightarrow \quad (x+1)^2 + \left(y + \frac{8}{3}\right)^2 = \frac{52}{9}$

which is a circle with centre $\left(-1, \frac{-8}{3}\right)$, and radius $\frac{2\sqrt{13}}{3}$.

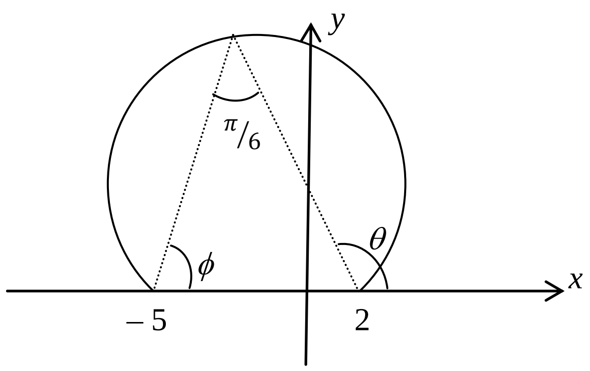
Example 5:

$\arg\left(\frac{z-2}{z+5}\right) = \frac{\pi}{6}$

$\Rightarrow \quad \arg(z-2) - \arg(z+5) = \frac{\pi}{6}$

$\Rightarrow \quad \theta - \phi = \frac{\pi}{6}$

which gives the arc of the circle as shown.

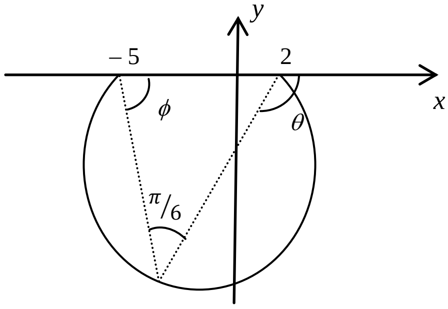


N.B.

The corresponding arc below the x-axis would have equation

$$\arg\left(\frac{z-2}{z+5}\right) = -\frac{\pi}{6}$$

as $\theta - \phi$ would be negative in this picture.



Transformations of the Complex Plane

Always start from the z-plane and transform to the w-plane, $z = x + iy$ and $w = u + iv$.

Example 1: Find the image of the circle $|z - 5| = 3$ under the transformation $w = \frac{1}{z-2}$.

Solution: *First* rearrange to find z

$$w = \frac{1}{z-2} \Rightarrow z - 2 = \frac{1}{w} \qquad \Rightarrow \qquad z = \frac{1}{w} + 2$$

Second substitute in equation of circle

$$\Rightarrow \quad \left|\frac{1}{w} + 2 - 5\right| = 3 \qquad \Rightarrow \qquad \left|\frac{1-3w}{w}\right| = 3$$

$$\Rightarrow \quad |1 - 3w| = 3|w| \qquad \Rightarrow \qquad 3\left|\frac{1}{3} - w\right| = 3|w|$$

$$\Rightarrow \quad \left|w - \frac{1}{3}\right| = |w|$$

which is the equation of the perpendicular bisector of the line joining 0 to $\frac{1}{3}$,

$\Rightarrow$ the image is the line $u = \frac{1}{6}$

Always consider the 'modulus technique' (above) first;

if this does not work then use the *u + iv* method shown below.

Example 2: Show that the image of the line $x + 4y = 4$ under the transformation $w = \frac{1}{z-3}$ is a circle, and find its centre and radius.

Solution: *First* rearrange to find $z \Rightarrow z = \frac{1}{w} + 3$

The 'modulus technique' is not suitable here.

$z = x + iy$ and $w = u + iv$

$$\Rightarrow \quad z = \frac{1}{w} + 3 = \frac{1}{u+iv} + 3 = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} + 3$$

$$\Rightarrow \quad x + iy = \frac{u-iv}{u^2+v^2} + 3$$

Equating real and imaginary parts $x = \frac{u}{u^2+v^2} + 3$ and $y = \frac{-v}{u^2+v^2}$

$$\Rightarrow \quad x + 4y = 4 \quad \text{becomes} \quad \frac{u}{u^2+v^2} + 3 - \frac{4v}{u^2+v^2} = 4$$

$$\Rightarrow \quad u^2 - u + v^2 + 4v = 0$$

$$\Rightarrow \quad \left(u - \frac{1}{2}\right)^2 + (v + 2)^2 = \frac{17}{4}$$

which is a circle with centre $\left(\frac{1}{2}, -2\right)$ and radius $\frac{\sqrt{17}}{2}$.

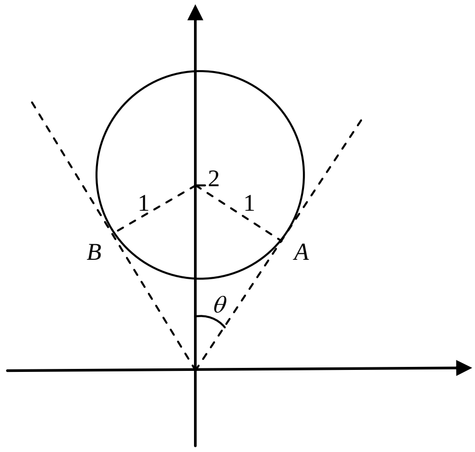
There are many more examples in the book, but these are the two important techniques.

Loci and geometry

It is always important to think of diagrams.

Example: z lies on the circle $|z - 2i| = 1$.
Find the greatest and least values of $\arg z$.

Solution: Draw a picture!



The greatest and least values of $\arg z$ will occur at B and A.

Trigonometry tells us that

$\theta = \frac{\pi}{6}$

and so greatest and least values of

$\arg z$ are $\frac{2\pi}{3}$ and $\frac{\pi}{3}$

4 First Order Differential Equations

Separating the variables, families of curves

Example: Find the general solution of

$\frac{dy}{dx} = \frac{y}{x(x+1)}$, for $x > 0$,

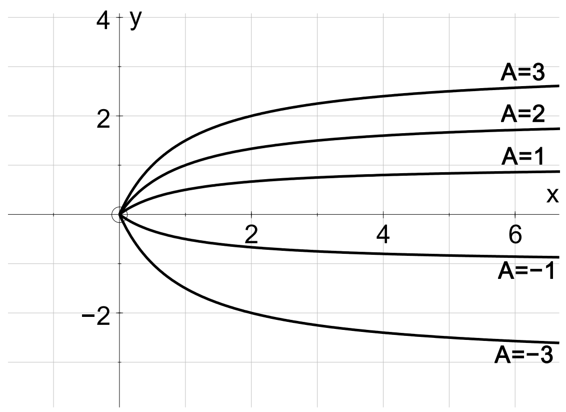
and sketch the family of solution curves.

Solution: $\frac{dy}{dx} = \frac{y}{x(x+1)} \quad \Rightarrow \quad \int \frac{1}{y}\, dy = \int \frac{1}{x(x+1)}\, dx = \int \frac{1}{x} - \frac{1}{x+1}\, dx$

$\Rightarrow \quad \ln y = \ln x - \ln(x+1) + \ln A$

$\Rightarrow \quad y = \frac{Ax}{x+1} = \frac{A(x+1-1)}{x+1} = A\left(1 - \frac{1}{x+1}\right)$

Thus for varying values of A and for $x > 0$, we have



Exact Equations

In an exact the L.H.S. is an exact derivative (really a preparation for Integrating Factors).

Example: Solve $\sin x \frac{dy}{dx} + y\cos x = 3x^2$

Solution: Notice that the L.H.S. is an exact derivative

$\sin x \frac{dy}{dx} + y\cos x = \frac{d}{dx}(y \sin x)$

$\Rightarrow \quad \frac{d}{dx}(y\sin x) = 3x^2$

$\Rightarrow \quad y\sin x = \int 3x^2\, dx = x^3 + c$

$\Rightarrow \quad y = \frac{x^3 + c}{\sin x}$

Integrating Factors

$\frac{dy}{dx} + Py = Q$ where P and Q are functions of x only.

In this case, multiply both sides by an Integrating Factor, $R = e^{\int P\,dx}$.
The L.H.S. will now be an exact derivative, $\frac{d}{dx}(Ry)$.
Proceed as in the above example.

Example: Solve $x\frac{dy}{dx} + 2y = 1$

Solution: First divide through by x

$\Rightarrow \quad \frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x}$ now in the correct form

Integrating Factor, I.F., is $R = e^{\int P\,dx} = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$

$\Rightarrow \quad x^2\frac{dy}{dx} + 2xy = x$ multiplying by x^2

$\Rightarrow \quad \frac{d}{dx}(x^2y) = x$, check that it is an exact derivative

$\Rightarrow \quad x^2 y = \int x\,dx = \frac{x^2}{2} + c$

$\Rightarrow \quad y = \frac{1}{2} + \frac{c}{x^2}$

Using substitutions

Example 1*:* Use the substitution $y = vx$ (where v is a function of x) to solve the equation

$$\frac{dy}{dx} = \frac{3yx^2 + y^3}{x^3 + xy^2}.$$

Solution: $y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3yx^2+y^3}{x^3+xy^2} \quad \Rightarrow \quad v + x\frac{dv}{dx} = \frac{3(vx)x^2 + (vx)^3}{x^3 + x(vx)^2} = \frac{3v + v^3}{1+v^2}$

and we can now separate the variables

$\Rightarrow \quad x\frac{dv}{dx} = \frac{3v+v^3}{1+v^2} - v = \frac{3v + v^3 - v - v^3}{1+v^2} = \frac{2v}{1+v^2}$

$\Rightarrow \quad \frac{1+v^2}{2v}\frac{dv}{dx} = \frac{1}{x}$

$\Rightarrow \quad \int \frac{1}{2v} + \frac{v}{2}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow \quad \frac{1}{2}\ln v + \frac{v^2}{4} = \ln x + c$

But $v = \frac{y}{x}$, $\quad \Rightarrow \quad \frac{1}{2}\ln\frac{y}{x} + \frac{y^2}{4x^2} = \ln x + c$

$\Rightarrow \quad 2x^2 \ln y + y^2 = 6x^2 \ln x + c'x^2$ c' is new arbitrary constant

and I would not like to find y!!!

Example 2: Use the substitution $y = \frac{1}{z}$ to solve the differential equation $\frac{dy}{dx} = y^2 + y\cot x$.

Solution: $y = \frac{1}{z} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-1}{z^2}\frac{dz}{dx}$

$\Rightarrow \quad \frac{-1}{z^2}\frac{dz}{dx} = \frac{1}{z^2} + \frac{1}{z}\cot x$

$\Rightarrow \quad \frac{dz}{dx} + z\cot x = -1$

Integrating factor is $R = e^{\int \cot x \ dx} = e^{\ln(\sin x)} = \sin x$

$\Rightarrow \quad \sin x\frac{dz}{dx} + z\cos x = -\sin x$

$\Rightarrow \quad \frac{d}{dx}(z\sin x) = -\sin x$ check that it is an exact derivative

$\Rightarrow \quad z\sin x = \cos x + c$

$\Rightarrow \quad z = \frac{\cos x + c}{\sin x}$ but $z = \frac{1}{y}$

$\Rightarrow \quad y = \frac{\sin x}{\cos x + c}$

Example 3: Use the substitution $z = x + y$ to solve the differential equation $\frac{dy}{dx} = \cos(x + y)$

Solution: $z = x + y \quad \Rightarrow \quad \frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \quad \frac{dz}{dx} = 1 + \cos z$

$\Rightarrow \quad \int \frac{1}{1+\cos z}\, dz = \int dx$ separating the variables

$\Rightarrow \quad \int \frac{1}{2}\sec^2\left(\frac{z}{2}\right) dz = x + c$ $1 + \cos z = 1 + 2\cos^2\left(\frac{z}{2}\right) - 1 = 2\cos^2\left(\frac{z}{2}\right)$

$\Rightarrow \quad \tan\left(\frac{z}{2}\right) = x + c$

But $z = x + y \quad \Rightarrow \quad \tan\left(\frac{x+y}{2}\right) = x + c$

5 Second Order Differential Equations

Linear with constant coefficients

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ where a, b and c are constants.

(1) when *f*(*x*) = 0

First write down the Auxiliary Equation, A.E

A.E. $am^2 + bm + c = 0$

and solve to find the roots $m = \alpha$ or β

(i) If α and β are both real numbers, and if $\alpha \neq \beta$
then the Complimentary Function, C.F., is
$y = A\,e^{\alpha x} + Be^{\beta x}$, where A and B are arbitrary constants of integration

(ii) If α and β are both real numbers, and if $\alpha = \beta$
then the Complimentary Function, C.F., is
$y = (A + Bx)\,e^{\alpha x}$, where A and B are arbitrary constants of integration

(iii) If α and β are both complex numbers, and if $\alpha = a + ib,\ \beta = a - ib$
then the Complimentary Function, C.F.,
$y = e^{\alpha x}(A \sin bx + B \cos bx)$,
where A and B are arbitrary constants of integration

Example 1*:* Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$

Solution: A.E. is $m^2 + 2m - 3 = 0$

$\Rightarrow$ $(m-1)(m+3) = 0$

$\Rightarrow$ $m = 1$ or -3

$\Rightarrow$ $y = Ae^{x} + Be^{-3x}$ when $f(x) = 0$, the C.F. is the solution

Example 2*:* Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

Solution: A.E. is $m^2 + 6m + 9 = 0$

$\Rightarrow$ $(m+3)^2 = 0$

$\Rightarrow$ $m = -3$ (and -3) repeated root

$\Rightarrow$ $y = (A + Bx)e^{-3x}$ when $f(x) = 0$, the C.F. is the solution

Example 3: Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Solution: A.E. is $m^2 + 4m + 13 = 0$

$\Rightarrow$ $(m + 2)^2 - (3i)^2 = 0$

$\Rightarrow$ $(m + 2 + 3i)(m + 2 - 3i) = 0$

$\Rightarrow$ $m = -2 - 3i$ or $-2 + 3i$

$\Rightarrow$ $y = e^{-2x}(A \sin 3x + B \cos 3x)$ when $f(x) = 0$, the C.F. is the solution

(2) when $f(x) \neq 0$, Particular Integrals

First proceed as in (1) to find the Complimentary Function, then use the rules below to find a Particular Integral, P.I.

Second the General Solution, G.S. , is found by adding the C.F. and the P.I.

$\Rightarrow$ **G.S. = C.F. + P.I.**

Note that it does not matter what P.I. you use, so you might as well find the easiest, which is what these rules do.

(1) $\boldsymbol{f(x) = e^{kx}}$.

Try $y = Ae^{kx}$
unless e^{kx} appears in the C.F., in which case try $y = Cxe^{kx}$
unless xe^{kx} appears in the C.F., in which case try $y = Cx^2e^{kx}$.

(2) $\boldsymbol{f(x) = \sin kx}$ **or** $\boldsymbol{f(x) = \cos kx}$

Try $y = C \sin kx + D \cos kx$
unless $\sin kx$ or $\cos kx$ appear in the C.F., in which case
try $y = x(C \sin kx + D \cos kx)$

(3) $\boldsymbol{f(x)}$ **= a polynomial of degree** $\boldsymbol{n}$**.**

Try $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$
unless a number, on its own, appears in the C.F., in which case
try $f(x) = x(a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0)$

(4) In general
to find a P.I., try something like $f(x)$, unless this appears in the C.F. (or if there is a problem), then try something like $x f(x)$.

Example 1: Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 2x$

Solution: A.E. is $m^2 + 6m + 5 = 0$

$\Rightarrow$ $(m + 5)(m + 1) = 0$ $\Rightarrow$ $m = -5$ or -1

$\Rightarrow$ C.F. is $y = Ae^{-5x} + Be^{-x}$

For the P.I., try $y = Cx + D$

$\Rightarrow$ $\frac{dy}{dx} = C$ and $\frac{d^2y}{dx^2} = 0$

Substituting in the differential equation gives

$0 + 6C + 5(Cx + D) = 2x$

$\Rightarrow$ $5C = 2$ comparing coefficients of x

$\Rightarrow$ $C = \frac{2}{5}$

and $6C + 5D = 0$ comparing constant terms

$\Rightarrow$ $D = \frac{-12}{25}$

$\Rightarrow$ P.I. is $y = \frac{2}{5}x - \frac{12}{25}$

$\Rightarrow$ G.S. is $y = Ae^{-5x} + Be^{-x} + \frac{2}{5}x - \frac{12}{25}$

Example 2: Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}$

Solution: A.E. is is $m^2 - 6m + 9 = 0$

$\Rightarrow$ $(m - 3)^2 = 0$

$\Rightarrow$ $m = 3$ repeated root

$\Rightarrow$ C.F. is $y = (Ax + B)e^{3x}$

In this case, both e^{3x} and xe^{3x} appear in the C.F.,
so for a P.I. we try $y = Cx^2e^{3x}$

$\Rightarrow$ $\frac{dy}{dx} = 2Cxe^{3x} + 3Cx^2e^{3x}$

and $\frac{d^2y}{dx^2} = 2Ce^{3x} + 6Cxe^{3x} + 6Cxe^{3x} + 9Cx^2e^{3x}$

Substituting in the differential equation gives

$2Ce^{3x} + 12Cxe^{3x} + 9Cx^2e^{3x} - 6(2Cxe^{3x} + 3Cx^2e^{3x}) + 9Cx^2e^{3x} = e^{3x}$

$\Rightarrow$ $2Ce^{3x} = e^{3x}$

$\Rightarrow$ $C = \frac{1}{2}$

$\Rightarrow$ P.I. is $y = \frac{1}{2}x^2e^{3x}$

$\Rightarrow$ G.S. is $y = (Ax + B)e^{3x} + \frac{1}{2}x^2e^{3x}$

Example 3: Solve $\frac{d^2x}{dt^2} - x = 4\cos 2t$
given that $x = 0$ and $\dot{x} = 1$ when $t = 0$.

Solution: A.E. is $m^2 - 1 = 0$

$\Rightarrow$ $m = \pm 1$

$\Rightarrow$ C.F. is $x = Ae^t + Be^{-t}$

For the P.I. try $x = C\sin 2t + D\cos 2t$

$\Rightarrow$ $\dot{x} = 2C\cos 2t - 2D\sin 2t$

and $\ddot{x} = -4C\sin 2t - 4D\cos 2t$

Substituting in the differential equation gives

$(-4C\sin 2t - 4D\cos 2t) - (C\sin 2t + D\cos 2t) = 4\cos 2t$

$\Rightarrow$ $-5C = 0$ comparing coefficients of $\sin 2t$

and $-5D = 4$ comparing coefficients of $\cos 2t$

$\Rightarrow$ $C = 0$ and $D = \frac{-5}{4}$

$\Rightarrow$ P.I. is $x = \frac{-5}{4}\cos 2t$

$\Rightarrow$ G.S. is $x = Ae^t + Be^{-t} - \frac{5}{4}\cos 2t$

$\Rightarrow$ $\dot{x} = Ae^t - Be^{-t} + \frac{5}{2}\sin 2t$

$x = 0$ and when $t = 0$ $\Rightarrow$ $0 = A + B - \frac{5}{4}$

and $\dot{x} = 1$ when $t = 0$ $\Rightarrow$ $1 = A - B$

$\Rightarrow$ $A = \frac{9}{8}$ and $B = \frac{1}{8}$

$\Rightarrow$ solution is $x = \frac{9}{8}e^t + \frac{1}{8}e^{-t} - \frac{5}{4}\cos 2t$

D.E.s of the form $ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = f(x)$

Substitute $x = e^u$

$\Rightarrow \quad \frac{dx}{du} = e^u = x$

and $\frac{dy}{du} = \frac{dx}{du} \times \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{du} = x\frac{dy}{dx}$ result I

But $\frac{d^2y}{du^2} = \frac{d\left(\frac{dy}{du}\right)}{du} = \frac{d\left(\frac{dy}{du}\right)}{dx} \times \frac{dx}{du}$ using the chain rule

$= \frac{d\left(x\frac{dy}{dx}\right)}{dx} \times \frac{dx}{du}$ using result I

$= \left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) \times \frac{dx}{du}$ product rule

$\Rightarrow \quad \frac{d^2y}{du^2} = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ since $\frac{dx}{du} = x$

$\Rightarrow \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ using result I

Thus we have $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ and $x\frac{dy}{dx} = \frac{dy}{du}$

substituting these in the original equation leads to a second order D.E. with constant coefficients.

Example: Solve the differential equation $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = -2x^2$.

Solution: Using the substitution $x = e^u$, and proceeding as above

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ and $x\frac{dy}{dx} = \frac{dy}{du}$

$\Rightarrow \quad \frac{d^2y}{du^2} - \frac{dy}{du} - 3\frac{dy}{du} + 3y = -2e^{2u}$

$\Rightarrow \quad \frac{d^2y}{du^2} - 4\frac{dy}{du} + 3y = -2e^{2u}$

$\Rightarrow$ A.E. is $m^2 - 4m + 3 = 0$

$\Rightarrow \quad (m-3)(m-1) = 0 \quad \Rightarrow \quad m = 3$ or 1

$\Rightarrow$ C.F. is $y = Ae^{3u} + Be^u$

For the P.I. try $y = Ce^{2u}$

$\Rightarrow \quad \frac{dy}{du} = 2Ce^{2u}$ and $\frac{d^2y}{du^2} = 4Ce^{2u}$

$\Rightarrow \quad 4Ce^{2u} - 8Ce^{2u} + 3Ce^{2u} = -2e^{2u}$

$\Rightarrow \quad C = 2$

$\Rightarrow$ G.S. is $y = Ae^{3u} + Be^u + 2e^{2u}$

But $x = e^u \quad \Rightarrow \quad$ G.S. is $y = Ax^3 + Bx + 2x^2$

6 Maclaurin and Taylor Series

1) Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

2) Taylor series

$$f(x+a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a) + \cdots + \frac{x^n}{n!}f^n(a) + \cdots$$

3) Taylor series – as a power series in $(x - a)$

replacing x by $(x - a)$ in 2) we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots$$

4) Solving differential equations using Taylor series

(a) If we are given the value of y when $x = 0$, then we use the Maclaurin series with

$f(0) = y_0$ — the value of y when $x = 0$

$f'(0) = \left(\frac{dy}{dx}\right)_0$ — the value of $\frac{dy}{dx}$ when $x = 0$

etc. to give

$$f(x) = y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \cdots + \frac{x^n}{n!}\left(\frac{d^ny}{dx^n}\right)_0 + \cdots$$

(b) If we are given the value of y when $x = a$, then we use the Taylor power series with

$f(a) = y_a$ — the value of y when $x = a$

$f'(a) = \left(\frac{dy}{dx}\right)_a$ — the value of $\frac{dy}{dx}$ when $x = a$

etc. to give

$$y = y_a + (x-a)\left(\frac{dy}{dx}\right)_a + \frac{(x-a)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_a + \frac{(x-a)^3}{3!}\left(\frac{d^3y}{dx^3}\right)_a + \cdots$$

NOTE THAT 4 (a) and 4 (b) are not in the formula book, but can easily be found using the results in 1) and 3).

Standard series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$ converges for all real x

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$$ converges for all real x

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!} + \cdots$$ converges for all real x

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$ converges for $-1 < x \le 1$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \cdots$$ converges for $-1 < x < 1$

Example 1: Find the Maclaurin series for $f(x) = \tan x$, up to and including the term in x^3

Solution: $f(x) = \tan x \quad \Rightarrow \quad f(0) = 0$

$\Rightarrow \quad f'(x) = \sec^2 x \quad \Rightarrow \quad f'(0) = 1$

$\Rightarrow \quad f''(x) = 2\sec^2 x \tan x \quad \Rightarrow \quad f''(0) = 0$

$\Rightarrow \quad f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \quad \Rightarrow \quad f^{iv}(0) = 2$

and $\quad f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \ldots$

$\Rightarrow \quad \tan x \cong 0 + x \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times 2$ up to the term in x^3

$\Rightarrow \quad \tan x \cong x + \frac{x^3}{3}$

Example 2: Using the Maclaurin series for e^x to find an expansion of e^{x+x^2}, up to and including the term in x^3.

Solution: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$\Rightarrow \quad e^{x+x^2} \cong 1 + (x + x^2) + \frac{(x+x^2)^2}{2!} + \frac{(x+x^2)^3}{3!}$ up to the term in x^3

$\cong 1 + x + x^2 + \frac{x^2 + 2x^3 + \cdots}{2!} + \frac{x^3 + \cdots}{3!}$ up to the term in x^3

$\Rightarrow \quad e^{x+x^2} \cong 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3$ up to the term in x^3

Example 3: Find a Taylor series for $\cot\left(x+\frac{\pi}{4}\right)$, up to and including the term in x^2.

Solution: $f(x) = \cot x$ and we are looking for

$$f\left(x+\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) + xf'\left(\frac{\pi}{4}\right) + \frac{x^2}{2!}f''\left(\frac{\pi}{4}\right)$$

$f(x) = \cot x \quad \Rightarrow \quad f\left(\frac{\pi}{4}\right) = 1$

$\Rightarrow \quad f'(x) = -\text{cosec}^2 x \quad \Rightarrow \quad f'\left(\frac{\pi}{4}\right) = -2$

$\Rightarrow \quad f''(x) = 2\text{cosec}^2 x \cot x \quad \Rightarrow \quad f''\left(\frac{\pi}{4}\right) = 4$

$\Rightarrow \quad \cot\left(x+\frac{\pi}{4}\right) \cong 1 - 2x + \frac{x^2}{2!} \times 4$ up to the term in x^2

$\Rightarrow \quad \cot\left(x+\frac{\pi}{4}\right) \cong 1 - 2x + 2x^2$ up to the term in x^2

Example 4: Use a Taylor series to solve the differential equation,

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0 \qquad \text{equation I}$$

up to and including the term in x^3, given that $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$.

In this case we shall use

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

$\Leftrightarrow \quad y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0.$

We already know that $y_0 = 1$ and $\left(\frac{dy}{dx}\right)_0 = 2$ values when $x = 0$

$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)_0 = \left(-\frac{1}{y}\left(\frac{dy}{dx}\right)^2 - 1\right)_0 = -5$ values when $x = 0$

Differentiating $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0$

$\Rightarrow \quad y\frac{d^3y}{dx^3} + \frac{dy}{dx}\times\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\times\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Substituting $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = 2$ and $\left(\frac{d^2y}{dx^2}\right)_0 = -5$ values when $x = 0$

$\Rightarrow \quad \left(\frac{d^3y}{dx^3}\right)_0 + 2 \times (^-5) + 2 \times 2 \times (^-5) + 2 = 0$

$\Rightarrow \quad \left(\frac{d^3y}{dx^3}\right)_0 = 28$

$\Rightarrow \quad$ solution is $y \cong 1 + 2x + \frac{x^2}{2!}\times(^-5) + \frac{x^3}{3!}\times 28$

$\Rightarrow \quad y \cong 1 + 2x - \frac{5}{2}x^2 + \frac{14}{3}x^3$

Series expansions of compound functions

Example: Find a polynomial expansion for

$$\frac{\cos 2x}{1-3x}, \qquad \text{up to and including the term in } x^3.$$

Solution: Using the standard series

$\cos 2x = 1 - \frac{(2x)^2}{2!} + \cdots$ up to and including the term in x^3

and $(1-3x)^{-1} = 1 + 3x + \frac{-1\times -2}{2!}(-3x)^2 + \frac{-1\times -2\times -3}{3!}(-3x)^3$

$= 1 + 3x + 9x^2 + 27x^3$ up to and including the term in x^3

$\Rightarrow \quad \frac{\cos 2x}{1-3x} = \left(1 - \frac{(2x)^2}{2!}\right)(1 + 3x + 9x^2 + 27x^3)$

$= 1 + 3x + 9x^2 + 27x^3 - 2x^2 - 6x^3$ up to and including the term in x^3

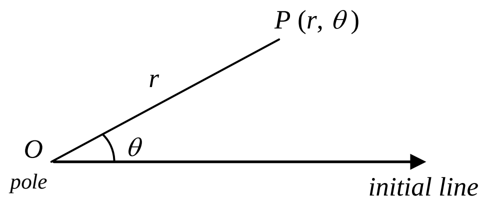
$\Rightarrow \quad \frac{\cos 2x}{1-3x} = 1 + 3x + 7x^2 + 21x^3$ up to and including the term in x^3

7 Polar Coordinates

The polar coordinates of P are (r, θ)

$r = OP$, the distance from the origin or *pole*,

and θ is the angle made anti-clockwise with the initial line.



In the Edexcel syllabus *r* is always taken as positive

(But in most books r can be negative, thus $\left(-4, \frac{\pi}{2}\right)$ is the same point as $\left(4, \frac{3\pi}{2}\right)$)

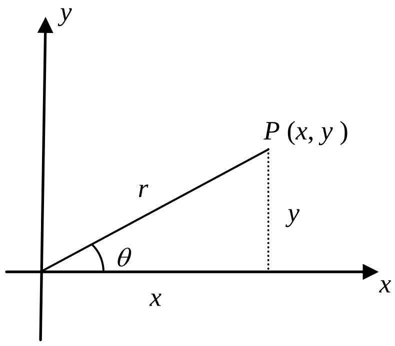
Polar and Cartesian coordinates

From the diagram

$r = \sqrt{x^2 + y^2}$

and $\tan\theta = \frac{y}{x}$ (use sketch to find θ).

$x = r\cos\theta$ and $y = r\sin\theta$.



Sketching curves

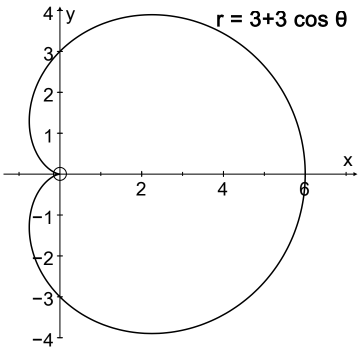
In practice, if you are asked to sketch a curve, it will probably be best to plot a few points. The important values of θ are those for which $r = 0$.

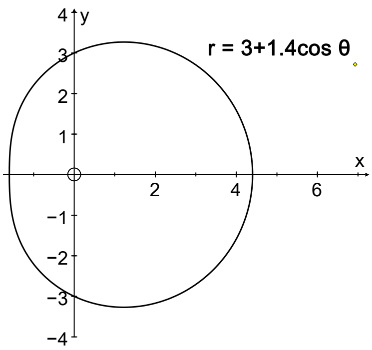
The sketches in these notes will show when *r* is negative by plotting a dotted line; these sections should be ignored as far as Edexcel A-level is concerned.

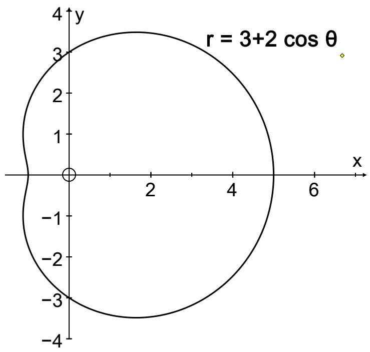
Some common curves

$\boldsymbol{r = a + b\cos\theta}$

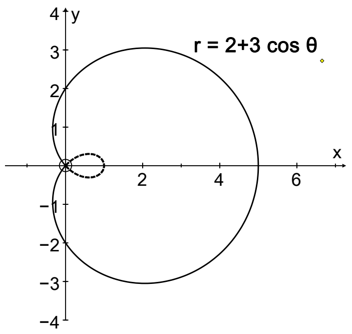
Cardiod	**Limacon without dimple**	**Limacon with a dimple**
$a = b$	$a \geq 2b$,	$b \leq a < 2b$

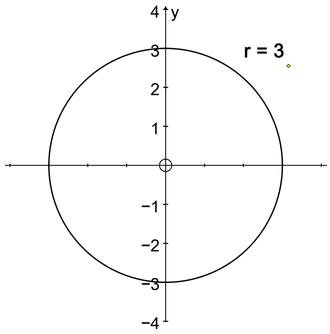


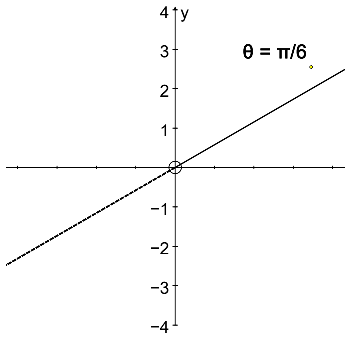




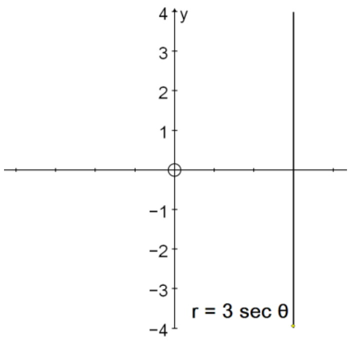
Limacon with a loop	**Circle**	**Line**
$a < b$		
r negative in the loop		*r* negative in bottom half

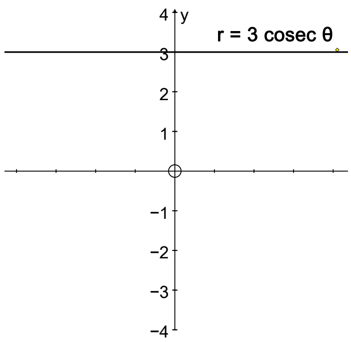


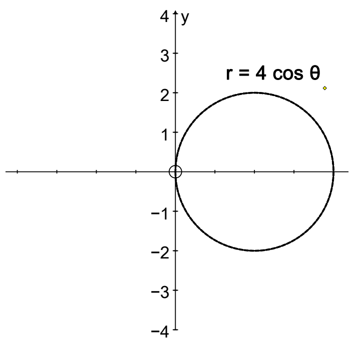




Line	**Line**	**Circle**

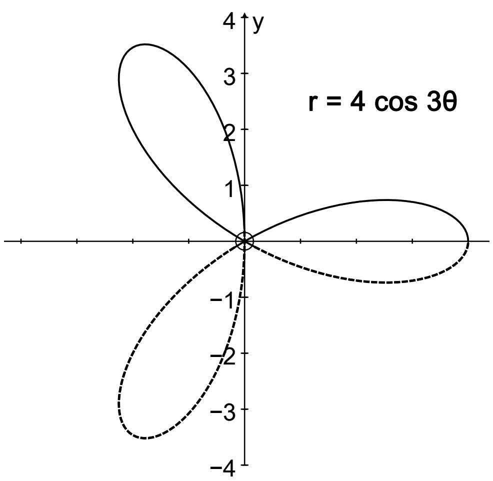






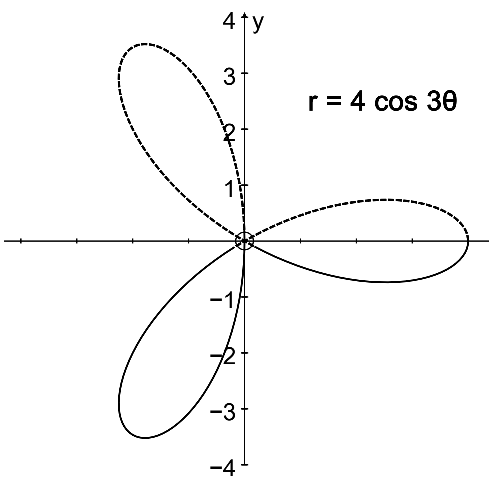
Rose Curves

$r = 4\cos 3\theta$
$0 \le \theta \le \pi$



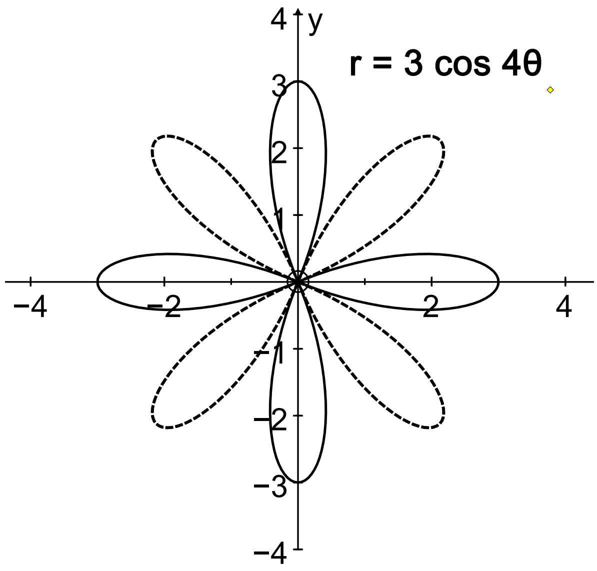
below *x*-axis, *r* negative

r = 4 cos 3θ
$\pi \le \theta \le 2\pi$



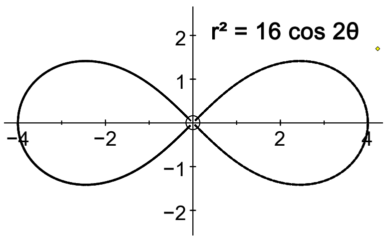
above *x*-axis, *r* negative

$r = 3 \cos 4\theta$

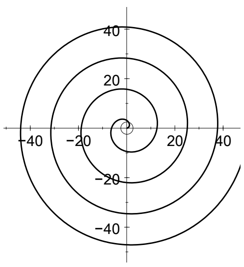


Thus the rose curve $r = a \cos\theta$ always has ***n* petals**, when **only the positive values of *r* are taken.**

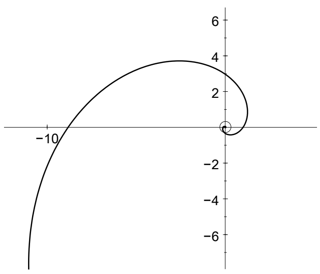
Leminiscate of Bernoulli



Spiral $r = 2\theta$

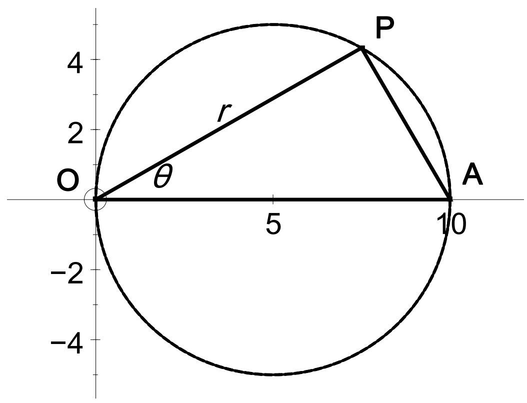


Spiral $r = e^{\theta}$



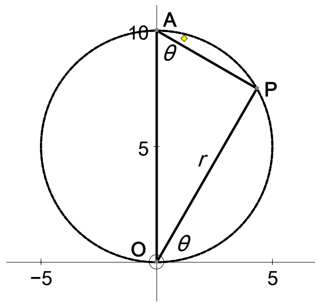
Circle $r = 10 \cos \theta$

Notice that in the circle on OA as diameter, the angle P is 90° (angle in a semi-circle) and trigonometry gives us that $r = 10 \cos \theta$.



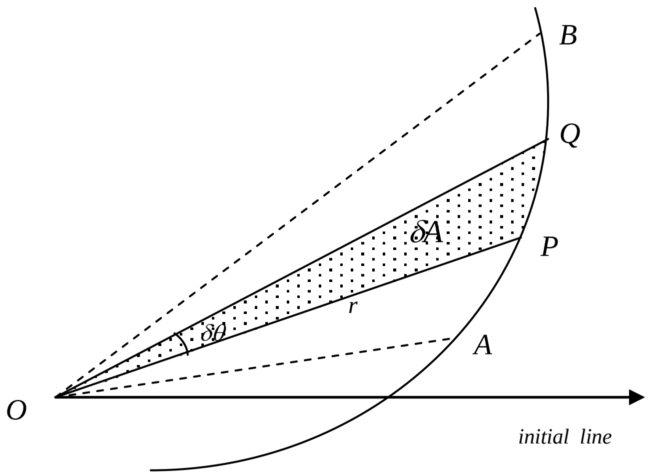
Circle $r = 10 \sin \theta$

In the same way $r = 10 \sin \theta$ gives a circle on the y-axis.



Areas using polar coordinates

Remember: area of a sector is $\frac{1}{2}r^2\theta$



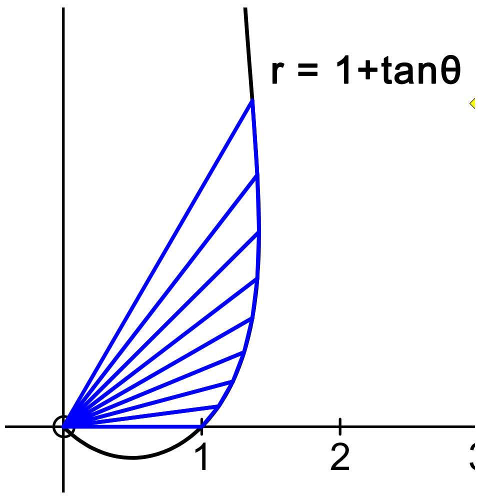
Area of $OPQ = \delta A \approx \frac{1}{2}r^2\delta\theta$

$\Rightarrow$ Area $OAB \approx \sum\left(\frac{1}{2}r^2\delta\theta\right)$

as $\delta\theta \to 0$

$\Rightarrow$ Area $OAB = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2 \, d\theta$

Example: Find the area between the curve $r = 1 + \tan \theta$ and the half lines $\theta = 0$ and $\theta = \frac{\pi}{3}$



Solution: Area $= \int_0^{\pi/3} \frac{1}{2}r^2 \, d\theta$

$= \int_0^{\pi/3} \frac{1}{2}(1 + \tan\theta)^2 \, d\theta$

$= \int_0^{\pi/3} \frac{1}{2}(1 + 2\tan\theta + \tan^2\theta) \, d\theta$

$= \int_0^{\pi/3} \frac{1}{2}(2\tan\theta + \sec^2\theta) \, d\theta$

$= \frac{1}{2}[2\ln(\sec\theta) + \tan\theta]_0^{\pi/3}$

$= \ln 2 + \frac{\sqrt{3}}{2}$

Tangents parallel and perpendicular to the initial line

$y = r\sin\theta$ and $x = r\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

1) Tangents will be parallel to the initial line ($\theta = 0$), or horizontal, when $\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{d\theta} = 0$

$\Rightarrow \quad \frac{d}{d\theta}(r\sin\theta) = 0$

2) Tangents will be perpendicular to the initial line ($\theta = 0$), or vertical, when $\frac{dy}{dx}$ is infinite

$\Rightarrow \quad \frac{dx}{d\theta} = 0$

$\Rightarrow \quad \frac{d}{d\theta}(r\cos\theta) = 0$

Note that if both $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} = 0$, then $\frac{dy}{dx}$ is not defined, and you should look at a sketch to help (or use l'Hôpital's rule).

Example: Find the coordinates of the points on $r = 1 + \cos\theta$ where the tangents are
(*a*) parallel to the initial line,
(*b*) perpendicular to the initial line.

Solution: $r = 1 + \cos\theta$ is shown in the diagram.

(*a*) Tangents parallel to $\theta = 0$ (horizontal)

$\Rightarrow \quad \frac{dy}{d\theta} = 0 \quad \Rightarrow \quad \frac{d}{d\theta}(r\sin\theta) = 0$

$\Rightarrow \quad \frac{d}{d\theta}((1+\cos\theta)\sin\theta) = 0 \quad \Rightarrow \quad \frac{d}{d\theta}(\sin\theta + \sin\theta\cos\theta) = 0$

$\Rightarrow \quad \cos\theta - \sin^2\theta + \cos^2\theta = 0 \quad \Rightarrow \quad 2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow \quad (2\cos\theta - 1)(\cos\theta + 1) = 0 \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \text{ or } -1$

$\Rightarrow \quad \theta = \pm\frac{\pi}{3} \text{ or } \pi$

(*b*) Tangents perpendicular to $\theta = 0$ (vertical)

$\Rightarrow \quad \frac{dx}{d\theta} = 0 \quad \Rightarrow \quad \frac{d}{d\theta}(r\cos\theta) = 0$

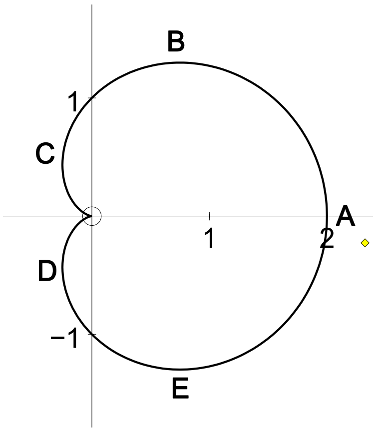
$\Rightarrow \quad \frac{d}{d\theta}((1+\cos\theta)\cos\theta) = 0 \quad \Rightarrow \quad \frac{d}{d\theta}(\cos\theta + \cos^2\theta) = 0$

$\Rightarrow \quad -\sin\theta - 2\cos\theta\sin\theta = 0 \quad \Rightarrow \quad \sin\theta\,(1 + 2\cos\theta) = 0$

$\Rightarrow \quad \cos\theta = -\frac{1}{2} \text{ or } \sin\theta = 0$

$\Rightarrow \quad \theta = \pm\frac{2\pi}{3} \text{ or } 0, \pi$

From the above we can see that



(*a*) the tangent is parallel to $\theta = 0$

at $B\ \left(\theta = \frac{\pi}{3}\right)$, and $E\ \left(\theta = -\frac{\pi}{3}\right)$,

also at $\theta = \pi$, the origin – see below

(*b*) the tangent is perpendicular to $\theta = 0$

at $A\ (\theta = 0)$, $C\ \left(\theta = \frac{2\pi}{3}\right)$ and $D\ \left(\theta = \frac{-2\pi}{3}\right)$

(*c*) we also have both $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} = 0$ when $\theta = \pi$!!!

From the graph it looks as if the tangent is parallel to $\theta = 0$ at the origin, $(\theta = \pi)$, and from l'Hôpital's rule it can be shown that this is true.

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