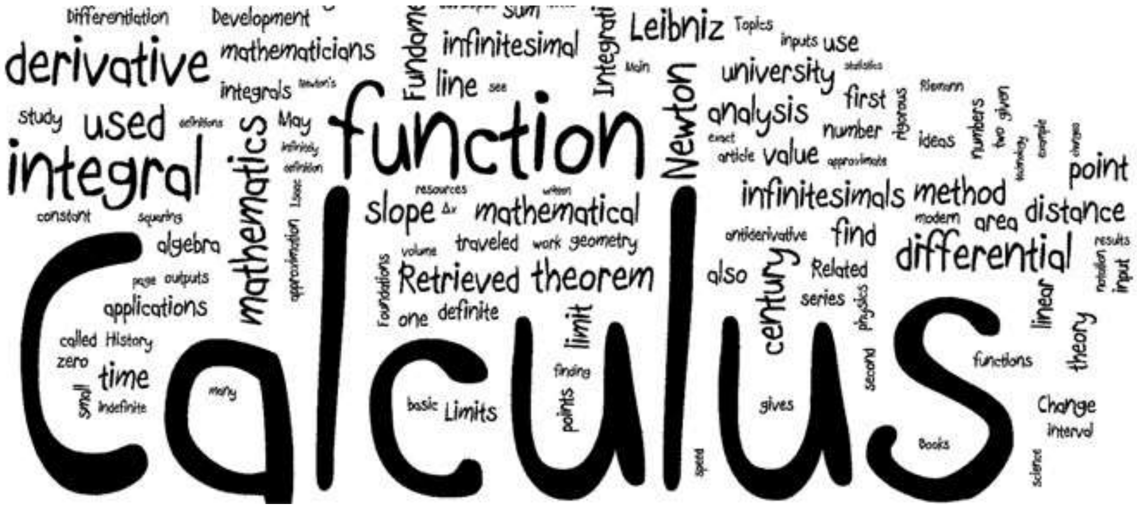


AP CALCULUS

Complete Notes

MathWithRajat

STUFF YOU MUST KNOW COLD . . .

Alternate Definition of the Derivative:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Basic Derivatives

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos\ x$$

$$\frac{d}{dx}(\cos x) = -\sin\ x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

Where u is a function of x,
and a is a constant.

Differentiation Rules

Chain Rule:

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx} \quad OR \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Product Rule:

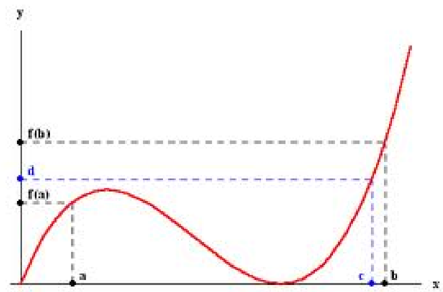
$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad OR\ u\,v' + v\,u'$$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad OR \quad \frac{v\,u' - u\,v'}{v^2}$$

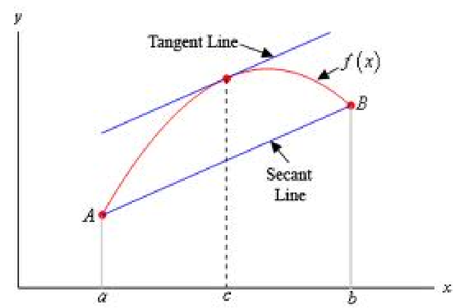
Intermediate Value Theorem

If the function $f(x)$ is continuous on $[a, b]$, and y is a number between $f(a)$ and $f(b)$, then there exists at least one number $x = c$ in the open interval (a, b) such that $f(c) = y$.



Mean Value Theorem

If the function $f(x)$ is continuous on $[a, b]$, **AND** *__the first derivative exists on the interval__* (a, b) then there is at least one number $x = c$ in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

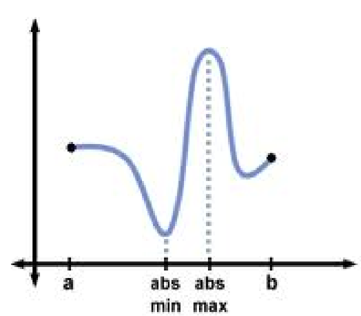


Rolle's Theorem

If the function $f(x)$ is continuous on $[a, b]$, **AND** *__the first derivative exists on the interval__* (a, b) **AND** $f(a) = f(b)$, then there is at least one number $x = c$ in (a, b) such that $f'(c) = 0$.

Extreme Value Theorem

If the function $f(x)$ is continuous on $[a, b]$, then the function is guaranteed to have an absolute maximum and an absolute minimum on the interval.



Derivative of an Inverse Function:

If f has an inverse function g then:

$$g'(x) = \frac{1}{f'(g(x))}$$

derivatives are reciprocal slopes

Implicit Differentiation

Remember that in implicit differentiation you will have a $\frac{dy}{dx}$ for each y in the original function or equation. Isolate the $\frac{dy}{dx}$. If you are taking the second derivative $\frac{d^2y}{dx^2}$, you will often substitute the expression you found for the first derivative somewhere in the process.

Average Rate of Change ARoC:

$$m_{sec} = \frac{f(b) - f(a)}{b - a}$$

Instantaneous Rate of Change IRoC:

$$m_{tan} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Curve Sketching And Analysis

$y = f(x)$ must be continuous at each:

Critical point: $\frac{dy}{dx} = 0$ or undefined

LOOK OUT FOR ENDPOINTS

Local minimum:

$\frac{dy}{dx}$ goes $(-, 0, +)$ or $(-, und, +)$ OR $\frac{d^2y}{dx^2} > 0$

Local maximum:

$\frac{dy}{dx}$ goes $(+, 0, -)$ or $(+, und, -)$ OR $\frac{d^2y}{dx^2} < 0$

Point of inflection: concavity changes

$\frac{d^2y}{dx^2}$ goes from $(+, 0, -)$, $(-, 0, +)$, $(+, und, -)$, OR $(-, und, +)$

First Derivative:

$f'(x) > 0$ function is increasing.

$f'(x) < 0$ function is decreasing.

$f'(x) = 0$ or DNE: Critical Values at x.

Relative Maximum: $f'(x) = 0$ or DNE and sign of $f'(x)$ changes from $+$ to $-$.

Relative Minimum: $f'(x) = 0$ or DNE and sign of $f'(x)$ changes from $-$ to $+$.

Absolute Max or Min:

MUST CHECK ENDPOINTS ALSO

The maximum value is a y-value.

Second Derivative:

$f''(x) > 0$ function is concave up.

$f''(x) < 0$ function is concave down.

$f'(x) = 0$ and sign of $f''(x)$ changes, then there is a point of inflection at x.

Relative Maximum: $f''(x) < 0$

Relative Minimum: $f''(x) > 0$

Write the equation of a tangent line at a point:

You need a slope (derivative) and a point.

$$y_2 - y_1 = m(x_2 - x_1)$$

Horizontal Asymptotes:

1. If the largest exponent in the numerator is < largest exponent in the denominator then $\lim_{x \to \pm\infty} f(x) = 0$.

2. If the largest exponent in the numerator is > the largest exponent in the denominator then $\lim_{x \to \pm\infty} f(x) = DNE$

3. If the largest exponent in the numerator is = to the largest exponent in the denominator then the quotient of the leading coefficients is the asymptote.

$\lim_{x \to \pm\infty} f(x) = \frac{a}{b}$

ONLY FOUR THINGS YOU CAN DO ON A CALCULATOR THAT NEEDS NO WORK SHOWN:

1. Graphing a function within an arbitrary view window.
2. Finding the zeros of a function.
3. Computing the derivative of a function numerically.
4. Computing the definite integral of a function numerically.

Distance, Velocity, and Acceleration

$x(t)$ = position function
$v(t)$ = velocity function
$a(t)$ =acceleration function

The derivative of position (*ft*) is velocity (*ft/sec*); the derivative of velocity (*ft/sec*) is acceleration (*ft/sec*^{2}).

The integral of acceleration (*ft/sec*^{2}) is velocity (*ft/sec*) ; the integral of velocity (*ft/sec*) is position (*ft*).

Speed is | velocity |

If acceleration and velocity have the ***same sign***, then the speed is *increasing*

If the acceleration and velocity have ***different signs***, then the speed is *decreasing*,

Displacement = $\int_{t_0}^{t_f} v(t)\, dt$

Distance $= \int_{initial\ time}^{final\ time} |\, v(t)|\ dt$

Average Velocity

$$= \frac{\text{final position} - \text{initial position}}{\text{total time}} = \frac{\Delta x}{\Delta t}$$

The Accumulation Function

$$F(x) = f(a) + \int_a^x f'(t)\, dt$$

The total amount, $F(x)$, at any time x , is the initial amount, $f\ (a)$, plus the amount of change between $t = a$ and $t = x$, given by the integral.

LOGARITHMS
Definition:

$$ln\ N = p\ \leftrightarrow e^p = N$$

$ln\ e = 1$
$\ln 1 = 0$
$\ln(MN) = \ln M + \ln N$
$\ln\left(\frac{M}{N}\right) = \ln M - \ln N$
$p\ \cdot \ln M = \ln M^p$

EXPONENTIAL GROWTH and DECAY:

When you see these words use: $y = Ce^{kt}$

"y is a differentiable function of t such that $y\ > 0$ and $y' =\ ky$ "

"the rate of change of y is proportional to y"

When solving a differential equation:

1. Separate variables first
2. Integrate
3. Add +C to one side
4. Use initial conditions to find "C"
5. Write the equation if the form of $y = f(x)$

"PLUS A CONSTANT"

The Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where $F'(x) =\ f(x)$

Corollary to FTC

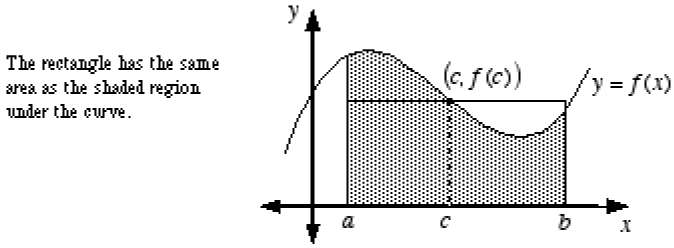
$$\frac{d}{dx}\int_a^{g(u)} f(t)dt = f\big(g(u)\big)\,\frac{du}{dx}$$

Mean Value Theorem for Integrals: The Average Value

If the function $f(x)$ is continuous on $[a, b]$ and the first derivative exists on the interval (a, b), then there exists a number $x = c$ on (a, b) such that

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{\int_a^b f(x)dx}{b-a}$$

This value $f(c)$ is the "average value" of the function on the interval $[a, b]$.



Riemann Sums

A Riemann Sum means a rectangular approximation. Approximation means that you ***DO NOT EVALUATE THE INTEGRAL***; you add up the areas of the rectangles.

Trapezoidal Rule

For uneven intervals, may need to calculate area of one trapezoid at a time and total.

$A_{Trap} = \frac{1}{2}h[b_1 + b_2]$

For even intervals:

$$\int_a^b f(x)\,dx = \frac{b-a}{2n}\left[\begin{matrix} y_0 + 2y_1 + 2y_2 + \ldots \\ +2y_{n-1} + y_n \end{matrix}\right]$$

Values of Trigonometric Functions for Common Angles

θ	$sin\ \theta$	$cos\ \theta$	$tan\ \theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	"∞"
π	0	-1	0

Must know both inverse trig and trig values:

EX. $tan\ \frac{\pi}{4} = 1$ and $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

ODD and EVEN:

$\sin(-x) = -\sin x$ (odd)

$\cos(-x) = \cos x$ (even)

Trigonometric Identities

Pythagorean Identities:

$sin^2\theta + cos^2\theta = 1$

The other two are easy to derive by dividing by $\sin^2\theta$ or $\cos^2\theta$.

$1 + \tan^2\theta = sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

Double Angle Formulas:

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

Power-Reducing Formulas:

$\cos^2 x = \frac{1}{2}(1 + \cos 2x\ )$

$\sin^2 x = \frac{1}{2}\ (1 - \cos 2x)$

Quotient Identities:

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\quad \cot\theta = \frac{\cos\theta}{\sin\theta}$

Reciprocal Identities:

$csc\ x = \frac{1}{sin\ x}$ or $\sin x\csc x = 1$

$sec\ x = \frac{1}{\cos x}$ or $\cos x\sec x = 1$

Basic Integrals

$$\int du = u + C$$

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \; n \neq -1$$

$$\int \frac{du}{u} = \ln| u | + C$$

$$\int e^u \, du = e^u + C$$

$$\int a^u \, du = \frac{a^u}{\ln a} + C$$

$$\int \sin u \, du = -\cos u + C$$

$$\int \cos u \, du = \sin u + C$$

$$\int \tan u \, du = -\ln|\cos u + C|$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u \, du = -\ln|\csc u + \cot u| + C$$

$$\int \sec^2 u \, due = \tan u + C$$

$$\int \csc^2 u \, du = -\cot u + C$$

$$\int \sec u \tan u \, du = \sec u + C$$

$$\int \csc u \cot u \, du = -\csc u + C$$

Area and Solids of Revolution:

NOTE: $(\boldsymbol{a}, \boldsymbol{b})$ **are** $\boldsymbol{x}$**-coordinates and** $(\boldsymbol{c}, \boldsymbol{d})$ **are** $\boldsymbol{y}$**-coordinates**

Area Between Two Curves:

Slices ⊥ to $\boldsymbol{x}$**-axis:** $A = \int_a^b [f(x) - g(x)] \, dx$

Slices ⊥ to $\boldsymbol{y}$**-axis:** $A = \int_c^d [f(y) - g(y)] \, dy$

Volume By Disk Method:

About $\boldsymbol{x}$**-axis:** $V = \pi \int_a^b [R(x)]^2 \, dx$

About $\boldsymbol{y}$**-axis:** $V = \pi \int_c^d [R(y)]^2 \, dy$

Volume By Washer Method:

About $\boldsymbol{x}$**-axis:** $V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) \, dx$

About $\boldsymbol{y}$**-axis:** $V = \pi \int_c^d ([R(y)]^2 - [r(y)]^2) \, dy$

Volume By Shell Method:

About $\boldsymbol{x}$**-axis:** $V = 2\pi \int_c^d y \, [R(y)] \, dy$

About $\boldsymbol{y}$**-axis:** $V = 2\pi \int_a^b x \, [R(x)] \, dx$

Not in the syllabus, may skip

General Equations for Known Cross Section where *base* is the distance between the two curves and *a* and *b* are the limits of integration.

SQUARES: $V = \int_a^b (base)^2 \, dx$

TRIANGLES

EQUILATERAL: $V = \frac{\sqrt{3}}{4} \int_a^b (base)^2 \, dx$

ISOSCELES RIGHT: $V = \frac{1}{4} \int_a^b (base)^2 \, dx$

RECTANGLES: $V = \int_a^b (base) \cdot h \, dx$

where h is the height of the rectangles.

SEMI-CIRCLES: $V = \frac{\pi}{2} \int_a^b (radius)^2 \, dx$

where radius is ½ distance between the two curves.

MORE DERIVATIVES:

$$\frac{d}{dx}\left[\sin^{-1}\frac{u}{a}\right] = \frac{1}{\sqrt{a^2-u^2}}\,\frac{du}{dx}$$

$$\frac{d}{dx}[\cos^{-1}x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\tan^{-1}\frac{u}{a}\right] = \frac{a}{a^2+u^2}\,\frac{du}{dx}$$

$$\frac{d}{dx}\,[\cot^{-1}x] = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left[\sec^{-1}\frac{u}{a}\right] = \frac{a}{|u|\sqrt{u^2-a^2}}\,\frac{du}{dx}$$

$$\frac{d}{dx}[\csc^{-1}x] = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\,(a^u) = a^u \ln a\,\frac{du}{dx}$$

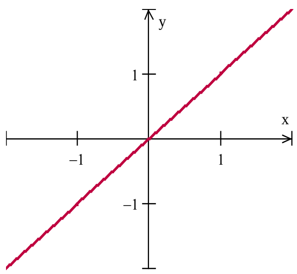
$$\frac{d}{dx}[\log_a x] = \frac{1}{x\ln a}$$

MORE INTEGRALS:

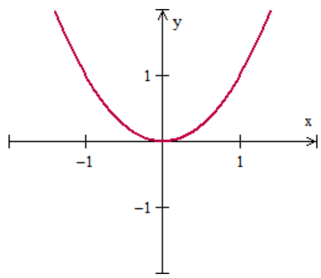
$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

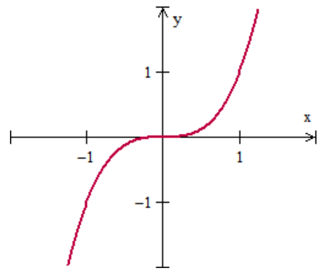
$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$$



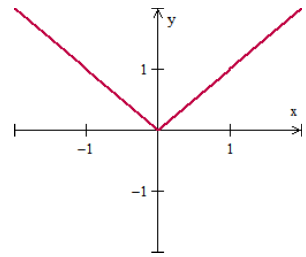
$y = x$



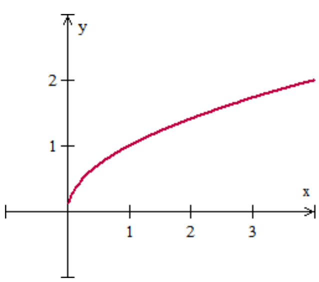
$y = x^2$



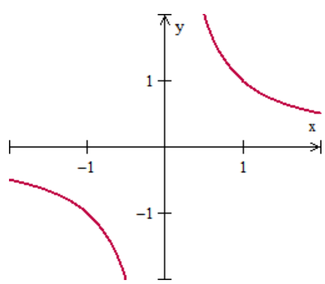
$y = x^3$



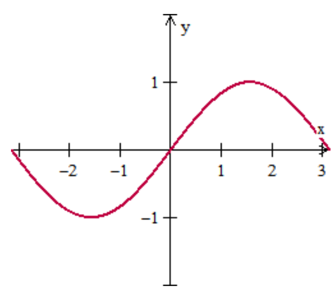
$y = |x|$



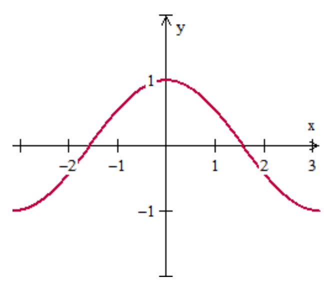
$y = \sqrt{x}$



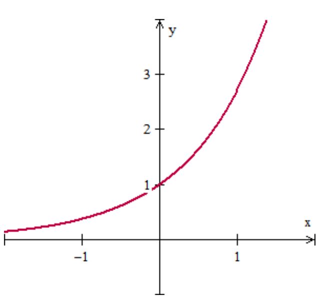
$y = \frac{1}{x}$



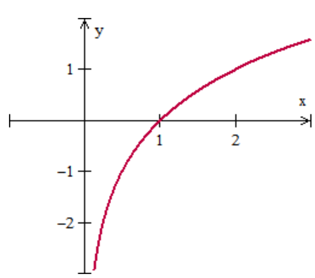
$y = \sin x$



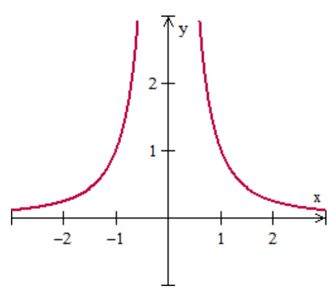
$y = \cos x$



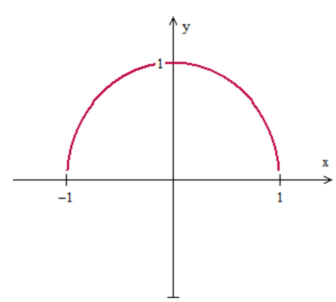
$y = e^x$



$y = \ln x$



$y = \frac{1}{x^2}$



$y = \sqrt{a^2 - x^2}$

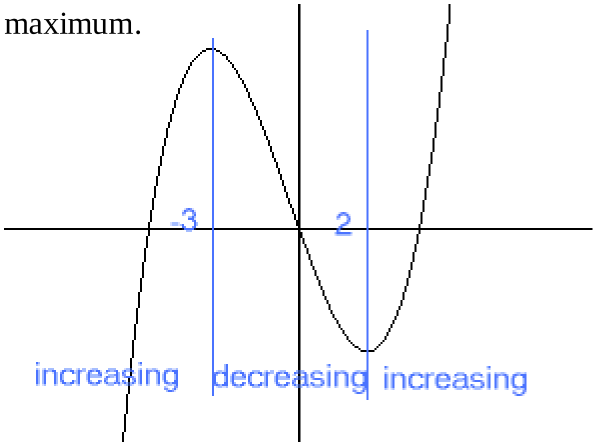
Extrema, Increasing/Decreasing Functions, the First Derivative Test and the Second Derivative Test

Finding Extrema on a Closed Interval [a,b]

1) Find the critical numbers of *f(x)*.
2) Evaluate *f(x)* at each critical number.
3) Evaluate *f(x)* at the endpoints.
4) The least value is a minimum. The greatest value is the maximum.

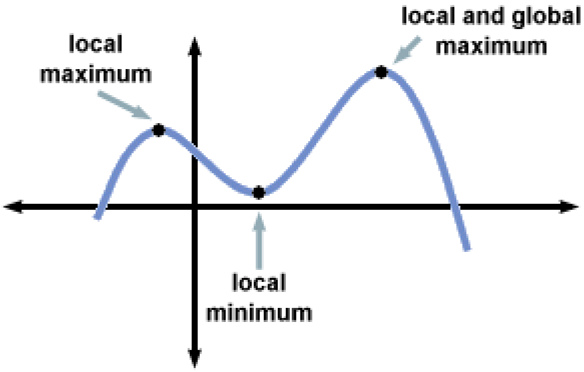
Determining if *f(x)* is Increasing or Decreasing on (a,b)

1) Find the critical numbers of *f(x)*.
2) Determine the intervals of *f(x)* to test.
3) Determine the sign of *f'(x)* at one value in the intervals.
4) If *f'(x)* > 0, then *f(x)* is increasing on the interval (a,b).
5) If *f'(x)* < 0, then *f(x)* is decreasing on the interval (a,b).
6) If *f'(x)* = 0, then *f(x)* is constant on (a,b).



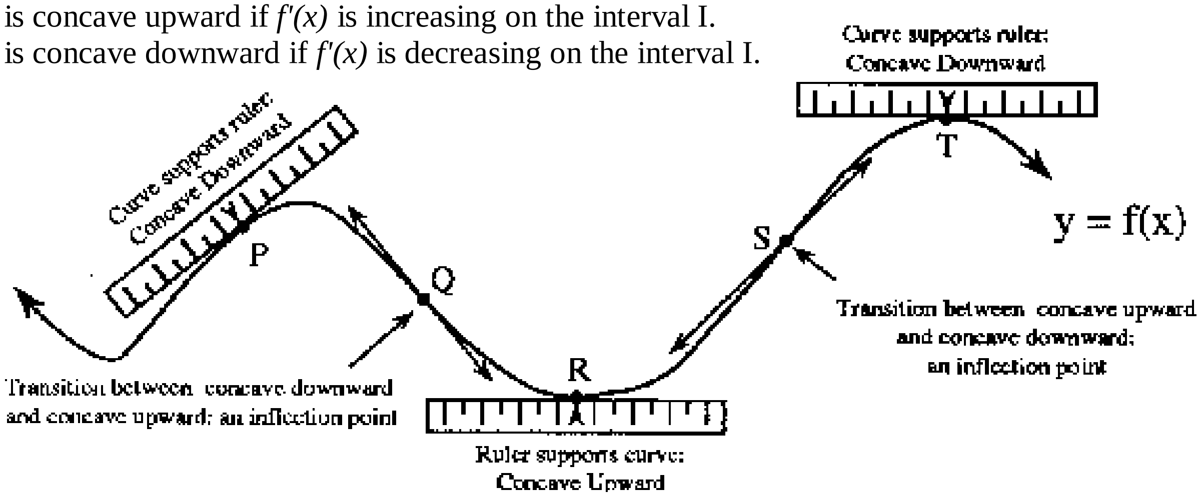
The First Derivative Test (c is a critical number of *f(x)*)

1) If *f'(x)* changes from negative to positive at c, then *f(c)* is a relative (local) minimum of *f(x)*
2) If *f'(x)* changes from positive to negative at c, then *f(c)* is a relative (local) maximum of *f(x)*.



Definition of Concavity

1) *f(x)* is concave upward if *f'(x)* is increasing on the interval I.
2) *f(x)* is concave downward if *f'(x)* is decreasing on the interval I.

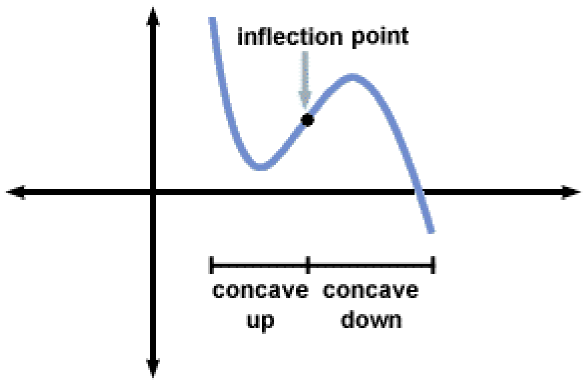


Determining if $f(x)$ is Concave Up or Down

1) Find $f''(x)$ and locate the points at which $f''(x) = 0$ or is undefined.
2) Use the points found in #1 to determine your test intervals.
3) Evaluate one test point from each of your intervals.
4) If $f''(x) > 0$, then $f(x)$ is concave up on the interval.
5) If $f''(x) < 0$, then $f(x)$ is concave down on the interval.

Points of Inflection

Points of inflection occur when the graph of $f(x)$ changes from concave up to concave down (or vice versa). Points of inflection only occur at values where $f''(x) = 0$ or is undefined. **NOTE: not all values of $f''(x) = 0$/undefined are points of inflection, therefore we must always check these points.**



Second Derivative Test (c is a critical number)

1) Find the critical numbers of f(x) {$f'(x) = 0$ or undefined}.
2) If $f''(c) > 0$, then $f(c)$ is a relative minimum because $f(c)$ is concave up.
3) If $f''(c) < 0$, then $f(c)$ is a relative maximum because $f(c)$ is concave down.
4) If $f''(c) = 0$, then the test fails. Use the first derivative test.

BC Only!

10.1: Parametric Equations

What we need to know about parametric equations (more or less):

1. How to write parametric equations given a Cartesian (x-y) equation:

   A graph of any function $y = f(x)$ can be parametrized by setting $x(t) = t$ and $y(t) = f(t)$.

2. How to eliminate *t* to obtain a Cartesian equation.

3. How to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for a parametric curve:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \textbf{and} \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

4. How to find the length of a smooth parametric curve:

   If a smooth curve $x = f(t)$, $y = g(t), a \le t \le b,$ is traced exactly once as t increases from *a* to *b*, then the arc length of the curve is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

5. How to find the highest/lowest and rightmost/leftmost points on a parametric curve and the points at which the tangent line to the curve is horizontal/vertical. For closed intervals, check endpoints as well.

   Highest/lowest: $\frac{dy}{dt} = 0, \frac{dx}{dt} \ne 0,$ and appropriate sign change or 2nd derivative test

   Rightmost/leftmost: $\frac{dx}{dt} = 0, \frac{dy}{dt} \ne 0,$ and appropriate sign change or 2nd derivative test

   Horizontal tangent: $\frac{dy}{dt} = 0, \frac{dx}{dt} \ne 0$ **Vertical tangent**: $\frac{dx}{dt} = 0, \frac{dy}{dt} \ne 0$

   At points where dx/dt and dy/dt are both 0, dy/dx becomes an indeterminate form; such points are called singular points. No general statement can be made about the behavior of the curve at such points; they must be analyzed case by case. (Anton, 9th ed.)

BC Only!

10.2: Vector Functions

Algebraically, an *n*-dimensional vector is an ordered set of *n* numbers. A two-dimensional vector, $\vec{v}$, is an ordered pair of real numbers $\langle x, y\rangle$ called components of the vector. We can add two vectors and multiply a vector by a real number (a scalar) as follows:

$$\vec{u}+\vec{v}=\langle x_1, y_1\rangle+\langle x_2, y_2\rangle=\langle x_1+x_2, y_1+y_2\rangle \quad \textbf{and} \qquad k\vec{v}=k\langle x, y\rangle=\langle kx, ky\rangle$$

Geometrically, a two-dimensional vector is a directed segment between two points on the plane:

If an arrow has an initial point $(x_{1,} y_1)$ and a terminal point $(x_{2,} y_2)$, it represents the vector $\langle x_2-x_1, y_2-y_1\rangle$.

Two vectors are deemed equal if they have the same length and direction:

The length or magnitude of a vector $\vec{u}=\langle x, y\rangle$ is given by $|\vec{u}|=\sqrt{x^2+y^2}$ and its direction θ is described by $\tan\theta=\dfrac{y}{x}$, where $0\le\theta<2\pi$.

A vector $\langle x, y\rangle$ can be drawn as a directed segment that goes from the origin to the point on the plane with coordinates *x* and *y*. Thus, there is a one-to-one correspondence between two-dimensional vectors and points in the plane. When a particle is moving in the plane, its position can be described as a vector $\langle x, y\rangle$. This vector connects the origin to the point at which the particle is located. The velocity of the particle is also a vector. It is often convenient to draw the velocity vector from the point of the current location.

A vector function is a function that produces vectors as outputs. When a particle moves on the *xy*-plane, the coordinates of its position can be given as parametric equations $x=f(t)$ and $y=g(t)$ for some interval $a\le t\le b$. The particle's position vector $\vec{r}(t)=\langle x(t), y(t)\rangle$ is a vector function of *t*. The coordinates of the position of the point at time *t* are the same as the components of $\vec{r}$ at time *t*. Therefore, **a vector function is essentially a different notation for a parametric function.** The functions $x(t)$ and $y(t)$ are called component or coordinate functions.

Here's what we need to know about vector functions:

If $\vec{r}(t)=\langle x(t), y(t)\rangle$ is the position vector of a particle moving along a smooth curve in the *xy*-plane then, at any time *t*,

1. **The particle's velocity vector $\vec{v}(t)$ is $\langle x'(t), y'(t)\rangle$; if drawn from the position point, it is tangent to the curve and points in the direction of increasing *t*.**

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2. **The particle's speed along the curve is the length of the velocity vector,** $|\vec{v}| = \sqrt{x'(t)^2 + y'(t)^2}$.

3. **The particle's acceleration vector** $\vec{a}(t)$ **is** $\langle x''(t), y''(t) \rangle$**, is the derivative of the velocity vector, and is the second derivative of the position vector.**

4. $\dfrac{\vec{v}}{|\vec{v}|}$, **a unit vector, is the direction of motion; note that** $\dfrac{\vec{v}}{|\vec{v}|} = \langle \cos\theta, \sin\theta \rangle$ **. The velocity is the product of the speed and direction:** $|\vec{v}|\left(\dfrac{\vec{v}}{|\vec{v}|}\right)$**.**

If $\vec{v}(t) = \langle x'(t), y'(t) \rangle$ **is the velocity vector of a particle moving along a smooth curve in the** ***xy*****-plane then**

5. **The displacement from** $t = a$ **to** $t = b$ **is given by the vector:**

$$\left\langle \int_a^b x'(t)\,dt, \int_a^b y'(t)\,dt \right\rangle$$

The preceding vector is added to the position at time $t = a$ **to get the position at time** $t = b$**.**

6. **The distance traveled from** $t = a$ **to** $t = b$ **is given by**

$$\int_a^b |\vec{v}(t)|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

Note that this is the same as the arc length for a parametric curve.

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10.3: Polar Function

The polar coordinates for a point *P* are $(r,\ \theta)$, where *r* represents the distance from the origin (pole) to the point *P* and θ is the measure of an angle from the positive *x*-axis (polar axis) to the ray joining the origin to point *P*. In working with polar forms of equations, it is sometimes necessary to convert the coordinates to Cartesian form:

> **The Cartesian coordinates (x, y) and the polar coordinates (r, θ) are related by the following equations:**
>
> $$x = r\cos\theta, \quad y = r\sin\theta, \quad x^2 + y^2 = r^2, \quad \tan\theta = \frac{y}{x}$$

Note that in the polar coordinate system every point has an infinite number of pairs of coordinates. For any r and θ, the pairs (r, θ) and $(r, \theta + 2\pi)$ describe the same point and the pairs (r, θ) and $(-r, \theta + \pi)$ describe the same point. In addition, a point *P* lies on the graph of a polar equation if *P* has any pair of polar coordinates that satisfy the equation.

Slope:

Now, let's consider curves of the form $r = f(\theta)$, where *f* is a differentiable function. The slope of the tangent line to a point on the polar curve is still found by $\dfrac{dy}{dx}$:

> **Let a curve *C* be given in polar coordinates by a function $r = f(\theta), \alpha \le \theta \le \beta$, where f and f' are continuous on (α, β) and not simultaneously zero. Then, for θ in (α, β), the slope of *C* at $r(\theta) = (f(\theta), \theta)$ is given by**
>
> $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Area:

The area enclosed by a polar curve is obtained, as in other cases, by transforming a Riemann sum into a definite integral. Whereas rectangles are used to find the area under the graph of a Cartesian function, here sectors of circles are used. The area of a sector with an angle $\Delta\theta$ and radius $r = f(\theta)$ is $\frac{1}{2}r^2\Delta\theta$ (which is obtained from $\pi r^2(\Delta\theta/2\pi)$). Therefore:

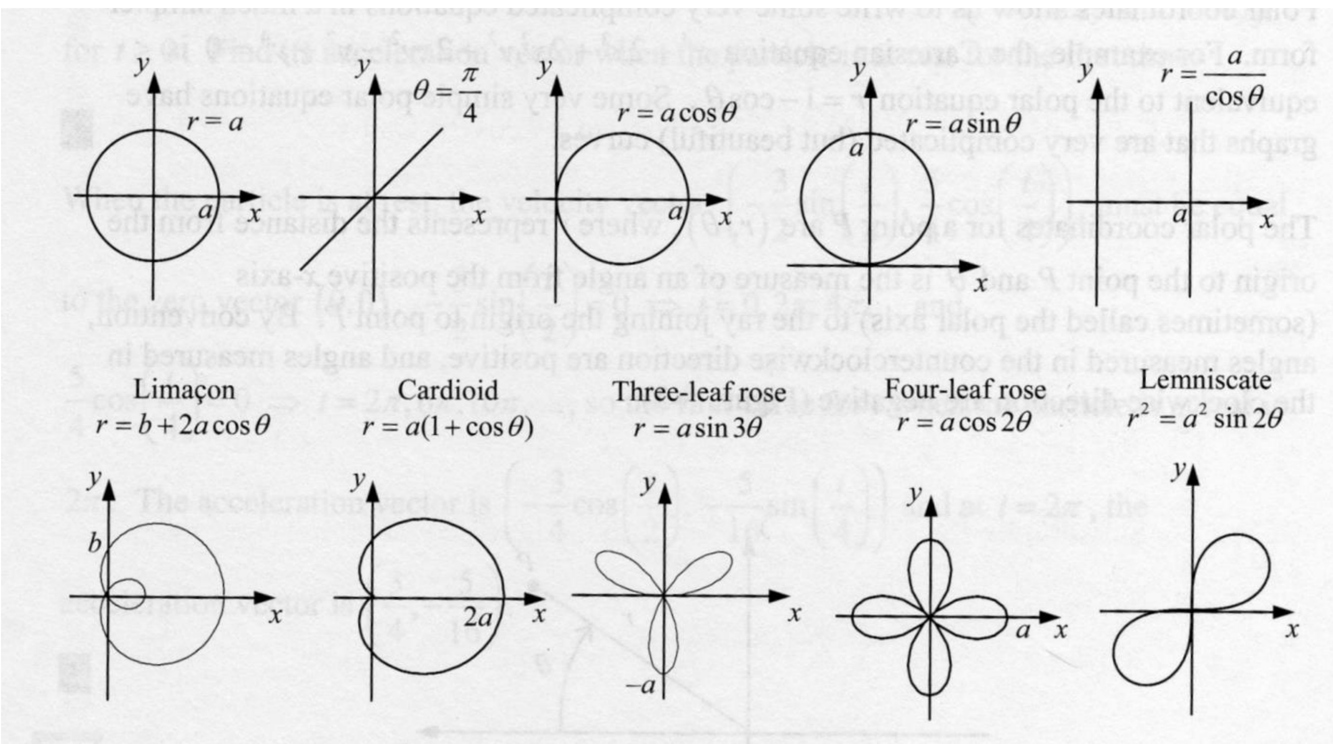
> **If *f* is a continuous function, and *R* is the region in the *xy*-plane bounded by the polar curve $r = f(\theta)$ and the rays $\theta = \alpha$ to $\theta = \beta$, then the**
>
> $$\textbf{Area of } R = \int_\alpha^\beta \frac{r^2}{2}\,d\theta.$$

Area between two polar curves:

The area of the region that lies between two polar curves $r_1 = f_1(\theta)$ **and** $r_2 = f_2(\theta)$ **from** $\theta = \alpha$ **to** $\theta = \beta$ **is given by**

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_1^{\,2} - r_2^{\,2}\right) d\theta .$$

You should be familiar with the calculator's polar graphing mode in case you need to draw a polar curve. You should also be familiar with the graphs of the most common polar equations (from *Be Prepared for the AP Calculus Exam* by Howell and Montgomery):



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Sequences: $\{a_n\}$

1. Listing the terms of a sequence
2. Finding a formula for the nth term of a sequence
3. Determining the convergence/divergence of a sequence

Series: $\sum_{k=1}^{\infty} a_k$

1. Geometric: $\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + \cdots, a \neq 0, r \neq 0$
    a. Converges if $|r| < 1$, diverges otherwise
    b. If it converges, the sum = $\dfrac{a}{1-r}$
2. Harmonic: $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (diverges)
3. P-series: $\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$
    a. Converges if $p > 1$
    b. Diverges if $0 < p \leq 1$
4. Alternating:
    a. Be able to approximate the sum of an alternating series
    b. Error of approximation is less than the next unused term
    c. Absolute convergence: $\sum_{k=1}^{\infty} a_k$ converges absolutely if $\sum_{k=1}^{\infty} |a_k|$ converges.
    d. Conditional convergence: If $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ does not
    e. If a series converges absolutely, then it converges (two for one.)

Tests for Convergence of Series

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1. Geometric series – see above
2. p-series test (including harmonic) – see above
3. Divergence/nth term test: If $\lim_{k\to\infty} a_k \neq 0 \Rightarrow \sum_{k=1}^{\infty} a_k$ diverges.
4. Integral test: $\int_c^{\infty} f(x)dx$ and $\sum_{k=1}^{\infty} a_k$ both converge or both diverge
5. Ratio test: If $\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ <1 then the series converges, if ρ >1 then it diverges, if ρ =1 then the test is inconclusive.
6. Root test: If $\rho = \lim_{k\to\infty} \left(a_k\right)^{\frac{1}{k}}$ < 1 then the series converges, if ρ >1 then it diverges, if ρ =1 then the test is inconclusive.
7. Limit comparison test: If $\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$ is finite and greater than 0, then both series converge or both diverge.
8. Comparison test: If $\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k$ term by term, then
   a. If $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ converges. (If the bigger converges, the smaller must.)
   b. If $\sum_{k=1}^{\infty} a_k$ diverges then $\sum_{k=1}^{\infty} b_k$ diverges. (If the smaller diverges, the bigger must.)
9. Alternating series test: an alternating series converges if
   a. $a_1 > a_2 > a_3 \cdots$, i.e. the *sequence* $\{a_n\}$ is decreasing (prove), and
   b. $\lim_{k\to\infty} a_k = 0$
10. Ratio test for absolute convergence: If $\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$ <1 then the series converges absolutely, if ρ >1 then it diverges, if ρ =1 then the test is inconclusive.

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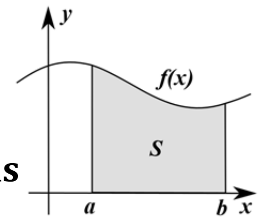
Summary of the convergence tests that may appear on the AP Calculus

Test Name	The series …	will converge if	Or will diverge if	Comments
n^{th} –term test	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	For divergence only; the converse is false.
Geometric	$\sum_{n=1}^{\infty} ar^{n-1}$	$-1 < r < 1$	$r \leq -1$ or $r \geq 1$	Sum = $\frac{a}{1-r}$
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$\lvert a_{n+1}\rvert < \lvert a_n\rvert$ and $\lim_{n\to\infty} a_n = 0$		Error bound $\lvert S_\infty - S_n\rvert < \lvert a_{n+1}\rvert$
Integral test	$\sum_{n=1}^{\infty} a_n$ and $a_n = f(n) \geq 0$	$\int_1^\infty f(x)\,dx$ converges	$\int_1^\infty f(x)\,dx$ diverges	f must be continuous, positive and decreasing.
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Direct comparison	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert} > 1$	If $\lim_{n\to\infty} \frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert} = 1$ the ratio test cannot be used.

Other useful convergence tests that may be used.

Test Name	The series …	will converge if	Or will diverge if	Comments
Limit Comparison	$\sum_{n=1}^{\infty} a_n$	$a_n > 0, b_n > 0$ $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$a_n > 0, b_n > 0$ $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Root Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{a_n} < 1$	$\lim_{n\to\infty} \sqrt[n]{a_n} > 1$	The test cannot be used if $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$

10 Things to know for the Free Response Questions



1. You will be given 6 Free Response questions. For two questions you are allowed to use the graphing calculator and for the remaining four there is no calculator allowed. Each Free Response Question is worth 9 points. Not all parts are weighted equally.
2. Always round to 4 decimal places. (AP only requires 3 but 4 will always get you points).
3. No simplification is needed; $e^0 - 4 + 6$ is okay! If you simply and you simplify wrong you will be awarded no points!
4. If you think it, write it. Never give a bald answer without any supporting work. If just the answer were okay then it would be a multiple-choice question, not free response.
5. Answer the question; don't say too much. If you say something correctly and then begin to say additional wrong information you will lose points.
6. Never erase. Graders are trained to ignore crossed out work.
7. Always bring the problem back to Calculus. Never use "it" or "the function" when justifying an answer. You must use the name of the function you are describing. Calculus always gives you the points. Pre-Calculus will sometimes give you the points.

   Ex. $f'(x)$ is positive (Calculus) vs.

   $f(x)$ is increasing (Pre-Calculus)
8. Don't use calculator syntax. If you use your calculator, describe it clearly in math terms, not in calculator terms.
9. Watch for linkage issues. Use arrows instead of equal signs.
10. Don't write $f(x) = 2(1.5) + 3$ when you mean $f(1.5) = 2(1.5) + 3$.

AP Calculus – Final Review Sheet

When you see the words ….	This is what you think of doing
1. Find the zeros	Set function = 0, factor or use quadratic equation if quadratic, graph to find zeros on calculator
2. Find equation of the line tangent to $f(x)$ on $[a,b]$	Take derivative - $f'(a)=m$ and use $y-y_1=m(x-x_1)$
3. Find equation of the line normal to $f(x)$ on $[a,b]$	Same as above but $m=\dfrac{-1}{f'(a)}$
4. Show that $f(x)$ is even	Show that $f(-x)=f(x)$ - symmetric to *y*-axis
5. Show that $f(x)$ is odd	Show that $f(-x)=-f(x)$ - symmetric to origin
6. Find the interval where $f(x)$ is increasing	Find $f'(x)$, set both numerator and denominator to zero to find critical points, make sign chart of $f'(x)$ and determine where it is positive.
7. Find interval where the slope of $f(x)$ is increasing	Find the derivative of $f'(x)=f''(x)$, set both numerator and denominator to zero to find critical points, make sign chart of $f''(x)$ and determine where it is positive.
8. Find the minimum value of a function	Make a sign chart of $f'(x)$, find all relative minimums and plug those values back into $f(x)$ and choose the smallest.
9. Find the minimum slope of a function	Make a sign chart of the derivative of $f'(x)=f''(x)$, find all relative minimums and plug those values back into $f'(x)$ and choose the smallest.
10. Find critical values	Express $f'(x)$ as a fraction and set both numerator and denominator equal to zero.
11. Find inflection points	Express $f''(x)$ as a fraction and set both numerator and denominator equal to zero. Make sign chart of $f''(x)$ to find where $f''(x)$ changes sign. (+ to – or – to +)
12. Show that $\lim\limits_{x\to a} f(x)$ exists	Show that $\lim\limits_{x\to a^-} f(x)=\lim\limits_{x\to a^+} f(x)$
13. Show that $f(x)$ is continuous	Show that 1) $\lim\limits_{x\to a} f(x)$ exists $\left(\lim\limits_{x\to a^-} f(x)=\lim\limits_{x\to a^+} f(x)\right)$ 2) $f(a)$ exists 3) $\lim\limits_{x\to a} f(x)=f(a)$
14. Find vertical asymptotes of $f(x)$	Do all factor/cancel of $f(x)$ and set denominator = 0
15. Find horizontal asymptotes of $f(x)$	Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$
16. Find the average rate of change of $f(x)$ on $[a,b]$	Find $\dfrac{f(b)-f(a)}{b-a}$
17. Find instantaneous rate of change of $f(x)$ at *a*	Find $f'(a)$

18. Find the average value of $f(x)$ on $[a,b]$	Find $\dfrac{\int_a^b f(x)\,dx}{b-a}$
19. Find the absolute maximum of $f(x)$ on $[a,b]$	Make a sign chart of $f'(x)$, find all relative maximums and plug those values back into $f(x)$ as well as finding $f(a)$ and $f(b)$ and choose the largest.
20. Show that a piecewise function is differentiable at the point a where the function rule splits	First, be sure that the function is continuous at $x=a$. Take the derivative of each piece and show that $\lim_{x\to a^-} f'(x) = \lim_{x\to a^+} f'(x)$
21. Given $s(t)$ (position function), find $v(t)$	Find $v(t)=s'(t)$
22. Given $v(t)$, find how far a particle travels on $[a,b]$	Find $\int_a^b \lvert v(t)\rvert dt$
23. Find the average velocity of a particle on $[a,b]$	Find $\dfrac{\int_a^b v(t)dt}{b-a} = \dfrac{s(b)-s(a)}{b-a}$
24. Given $v(t)$, determine if a particle is speeding up at $t=k$	Find $v(k)$ and $a(k)$. Multiply their signs. If both positive, the particle is speeding up, if different signs, then the particle is slowing down.
25. Given $v(t)$ and $s(0)$, find $s(t)$	$s(t)=\int v(t)dt + C$ Plug in $t = 0$ to find C
26. Show that Rolle's Theorem holds on $[a,b]$	Show that f is continuous and differentiable on the interval. If $f(a)=f(b)$, then find some c in (a,b) such that $f'(c)=0$.
27. Show that Mean Value Theorem holds on $[a,b]$	Show that f is continuous and differentiable on the interval. Then find some c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.
28. Find domain of $f(x)$	Assume domain is $(-\infty,\infty)$. Restrictable domains: denominators $\neq 0$, square roots of only non negative numbers, log or ln of only positive numbers.
29. Find range of $f(x)$ on $[a,b]$	Use max/min techniques to rind relative max/mins. Then examine $f(a), f(b)$
30. Find range of $f(x)$ on $(-\infty,\infty)$	Use max/min techniques to rind relative max/mins. Then examine $\lim_{x\to\pm\infty} f(x)$.
31. Find $f'(x)$ by definition	$f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ or $f'(x)=\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}$
32. Find derivative of inverse to $f(x)$ at $x=a$	Interchange x with y. Solve for $\dfrac{dy}{dx}$ implicitly (in terms of y). Plug your x value into the inverse relation and solve for y. Finally, plug that y into your $\dfrac{dy}{dx}$.

33. y is increasing proportionally to y	$\frac{dy}{dt} = ky$ translating to $y = Ce^{kt}$
34. Find the line $x = c$ that divides the area under $f(x)$ on $[a,b]$ to two equal areas	$\int_a^c f(x)dx = \int_c^b f(x)dx$
35. $\frac{d}{dx}\int_a^x f(t)\,dt =$	2nd FTC: Answer is $f(x)$
36. $\frac{d}{dx}\int_a^u f(t)dt$	2nd FTC: Answer is $f(u)\frac{du}{dx}$
37. The rate of change of population is …	$\frac{dP}{dt} = \ldots$
38. The line $y = mx + b$ is tangent to $f(x)$ at (x_1, y_1)	Two relationships are true. The two functions share the same slope ($m = f'(x)$) and share the same *y* value at x_1.
39. Find area using left Riemann sums	$A = base[x_0 + x_1 + x_2 + \ldots + x_{n-1}]$
40. Find area using right Riemann sums	$A = base[x_1 + x_2 + x_3 + \ldots + x_n]$
41. Find area using midpoint rectangles	Typically done with a table of values. Be sure to use only values that are given. If you are given 6 sets of points, you can only do 3 midpoint rectangles.
42. Find area using trapezoids	$A = \frac{base}{2}[x_0 + 2x_1 + 2x_2 + \ldots + 2x_{n-1} + x_n]$ This formula only works when the base is the same. If not, you have to do individual trapezoids.
43. Solve the differential equation …	Separate the variables – *x* on one side, *y* on the other. The *dx* and *dy* must all be upstairs.
44. Meaning of $\int_a^x f(t)\,dt$	The accumulation function – accumulated area under the function $f(x)$ starting at some constant *a* and ending at *x*.
45. Given a base, cross sections perpendicular to the *x*-axis are squares	The area between the curves typically is the base of your square. So the volume is $\int_a^b (base^2)dx$
46. Find where the tangent line to $f(x)$ is horizontal	Write $f'(x)$ as a fraction. Set the numerator equal to zero.
47. Find where the tangent line to $f(x)$ is vertical	Write $f'(x)$ as a fraction. Set the denominator equal to zero.
48. Find the minimum acceleration given $v(t)$	First find the acceleration $a(t) = v'(t)$. Then minimize the acceleration by examining $a'(t)$.
49. Approximate the value of $f(0.1)$ by using the tangent line to *f* at $x = 0$	Find the equation of the tangent line to *f* using $y - y_1 = m(x - x_1)$ where $m = f'(0)$ and the point is $(0, f(0))$. Then plug in 0.1 into this line being sure to use an approximate $(\approx)$ sign.

50. Given the value of $F(a)$ and the fact that the anti-derivative of *f* is *F*, find $F(b)$1	Usually, this problem contains an antiderivative yo cannot take. Utilize the fact that if $F(x)$is the antiderivative of *f*, then $\int_a^b F(x)dx = F(b)-F(a)$. So solve for $F(b)$ using the calculator to find the definite integral.
51. Find the derivative of $f(g(x))$	$f'(g(x))\cdot g'(x)$
52. Given $\int_a^b f(x)\,dx$, find $\int_a^b [f(x)+k]\,dx$	$\int_a^b [f(x)+k]\,dx = \int_a^b f(x)dx + \int_a^b kdx$
53. Given a picture of $f'(x)$, find where $f(x)$ is increasing	Make a sign chart of $f'(x)$ and determine where $f'(x)$ is positive.
54. Given $v(t)$ and $s(0)$, find the greatest distance from the origin of a particle on $[a,b]$	Generate a sign chart of $v(t)$ to find turning points. Then integrate $v(t)$ using $s(0)$ to find the constant to find $s(t)$. Finally, find *s*(all turning points) which will give you the distance from your starting point. Adjust for the origin.
55. Given a water tank with *g* gallons initially being filled at the rate of $F(t)$ gallons/min and emptied at the rate of $E(t)$ gallons/min on $[t_1,t_2]$, find a) the amount of water in the tank at *m* minutes	$g+\int_{t}^{t_2}(F(t)-E(t))dt$
56. b) the rate the water amount is changing at *m*	$\frac{d}{dt}\int_t^m (F(t)-E(t))dt = F(m)-E(m)$
57. c) the time when the water is at a minimum	$F(m)-E(m)$=0, testing the endpoints as well.
58. Given a chart of *x* and $f(x)$ on selected values between *a* and *b*, estimate $f'(c)$ where *c* is between *a* and b.	Straddle *c*, using a value *k* greater than *c* and a value *h* less than *c*. so $f'(c)\approx \frac{f(k)-f(h)}{k-h}$
59. Given $\frac{dy}{dx}$, draw a slope field	Use the given points and plug them into $\frac{dy}{dx}$, drawing little lines with the indicated slopes at the points.
60. Find the area between curves $f(x), g(x)$ on $[a,b]$	$A=\int_a^b [f(x)-g(x)]dx$, assuming that the *f* curve is above the *g* curve.
61. Find the volume if the area between $f(x), g(x)$ is rotated about the *x*-axis	$V=\pi\int_a^b \left[(f(x))^2-(g(x))^2\right]dx$ assuming that the *f* curve is above the *g* curve.

BC Problems

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62. Find $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}$ if $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty} g(x)=0$	Use L'Hopital's Rule.
63. Find $\int_0^\infty f(x)\ dx$	$\lim\limits_{h\to\infty}\int_0^h f(x)dx$
64. $\dfrac{dP}{dt}=\dfrac{k}{M}P(M-P)$ or $\dfrac{dP}{dt}=kP\left(1-\dfrac{P}{M}\right)$	Signals logistic growth. $\lim\limits_{t\to\infty}\dfrac{dP}{dt}=0\Rightarrow M=P$
65. Find $\int\dfrac{dx}{x^2+ax+b}$ where x^2+ax+b factors	Factor denominator and use Heaviside partial fraction technique.
66. The position vector of a particle moving in the plane is $r(t)=\langle x(t),y(t)\rangle$ a) Find the velocity.	$v(t)=\langle x'(t),y'(t)\rangle$
67. b) Find the acceleration.	$a(t)=\langle x''(t),y''(t)\rangle$
68. c) Find the speed.	$\lVert v(t)\rVert=\sqrt{[x'(t)]^2+[y'(t)]^2}$
69. a) Given the velocity vector $v(t)=\langle x(t),y(t)\rangle$ and position at time 0, find the position vector.	$s(t)=\int x(t)dt+\int y(t)dt+C$ Use $s(0)$ to find C, remembering it is a vector.
70. b) When does the particle stop?	$v(t)=0\to x(t)=0$ AND $y(t)=0$
71. c) Find the slope of the tangent line to the vector at t_1.	This is the acceleration vector at t_1.
72. Find the area inside the polar curve $r=f(\theta)$.	$A=\dfrac{1}{2}\int_{\theta_1}^{\theta_2}[f(\theta)]^2d\theta$
73. Find the slope of the tangent line to the polar curve $r=f(\theta)$.	$x=r\cos\theta, y=r\sin\theta\Rightarrow\dfrac{dy}{dx}=\dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$
74. Use Euler's method to approximate $f(1.2)$ given $\dfrac{dy}{dx}$, $(x_0,y_0)=(1,1)$, and $\Delta x=0.1$	$dy=\dfrac{dy}{dx}(\Delta x), y_{\text{new}}=y_{\text{old}}+dy$
75. Is the Euler's approximation an underestimate or an overestimate?	Look at sign of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in the interval. This gives you increasing.decreasing/concavity. Draw picture to ascertain

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	answer.
76. Find $\int x^n e^{ax} dx$ where *a*, *n* are integers	Integration by parts, $\int u\,dv = uv - \int v\,du + C$
77. Write a series for $x^n \cos x$ where *n* is an integer	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ Multiply each term by x^n
78. Write a series for $\ln(1+x)$ centered at $x = 0$.	Find Maclaurin polynomial: $P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$
79. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if…..	$p > 1$
80. If $f(x) = 2 + 6x + 18x^2 + 54x^3 + \ldots$, find $f\left(-\frac{1}{2}\right)$	Plug in and factor. This will be a geometric series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$
81. Find the interval of convergence of a series.	Use a test (usually the ratio) to find the interval and then test convergence at the endpoints.
82. Let S_4 be the sum of the first 4 terms of an alternating series for $f(x)$. Approximate $\lvert f(x) - S_4 \rvert$	This is the error for the 4th term of an alternating series which is simply the 5th term. It will be positive since you are looking for an absolute value.
83. Suppose $f^{(n)}(x) = \frac{(n+1)\ n!}{2^n}$. Write the first four terms and the general term of a series for $f(x)$ centered at $x = c$	You are being given a formula for the derivative of $f(x)$. $f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$
84. Given a Taylor series, find the Lagrange form of the remainder for the *n*th term where *n* is an integer at *x* = *c*.	You need to determine the largest value of the 5th derivative of *f* at some value of *z*. Usually you are told this. Then: $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$
85. $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	$f(x) = e^x$
86. $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots$	$f(x) = \sin x$
87. $f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots$	$f(x) = \cos x$
88. Find $\int (\sin x)^m (\cos x)^n dx$ where *m* and *n* are integers	If *m* is odd and positive, save a sine and convert everything else to cosine. The sine will be the *du*. If n is odd and positive, save a cosine and convert everything else to sine. The cosine will be the *du*. Otherwise use the fact that:

	$\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$
89. Given $x = f(t), y = g(t)$, find $\frac{dy}{dx}$	$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
90. Given $x = f(t), y = g(t)$, find $\frac{d^2y}{dx^2}$	$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$
91. Given $f(x)$, find arc length on $[a,b]$	$L = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$
92. $x = f(t), y = g(t)$, find arc length on $[t_1, t_2]$	$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
93. Find horizontal tangents to a polar curve $r = f(\theta)$	$x = r\cos\theta, y = r\sin\theta$ Find where $r\sin\theta = 0$ where $r\cos\theta \neq 0$
94. Find vertical tangents to a polar curve $r = f(\theta)$	$x = r\cos\theta, y = r\sin\theta$ Find where $r\cos\theta = 0$ where $r\sin\theta \neq 0$
95. Find the volume when the area between $y = f(x), x = 0, y = 0$ is rotated about the y-axis.	Shell method: $V = 2\pi \int_0^b \text{radius} \cdot \text{height}\, dx$ where b is the root.
96. Given a set of points, estimate the volume under the curve using Simpson's rule on $[a,b]$.	$A \approx \frac{b-a}{3n}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + ... + 4y_{n-1} + y_n\right]$
97. Find the dot product: $\langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle$	$\langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle = u_1v_1 + u_2v_2$
98. Multiply two vectors:	You get a scalar.

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