Cambridge Assessment Admissions Testing

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION

2023

TMUA 2023 TEST Solution Book

- All Topics

ThrivingScholars

PAPER 1

1 Given that

$$\int_0^1 (ax+b)\,\mathrm{d}x = 1$$

and

$$\int_0^1 x(ax+b)\,\mathrm{d}x = 1$$

find the value of $a+b$.

A –1

B 0

C 1

D 2

E 3

F 4

G 5

We calculate the integrals in terms of a and b:

$$\begin{aligned}\int_0^1 (ax+b)\,\mathrm{d}x &= \left[\tfrac{1}{2}ax^2 + bx\right]_0^1 \\ &= \tfrac{1}{2}a + b \\ &= 1 \\ \int_0^1 x(ax+b)\,\mathrm{d}x &= \int_0^1 (ax^2+bx)\,\mathrm{d}x \\ &= \left[\tfrac{1}{3}ax^3 + \tfrac{1}{2}bx^2\right]_0^1 \\ &= \tfrac{1}{3}a + \tfrac{1}{2}b \\ &= 1\end{aligned}$$

Multiplying the first equation by 2 and the second by 6 gives the simultaneous equations

$$\begin{aligned} a + 2b &= 2 \qquad (1) \\ 2a + 3b &= 6 \qquad (2)\end{aligned}$$

Then $2\times(1)-(2)$ gives $b=-2$, so $a=6$. Therefore $a+b=4$ and the correct answer is option F.

2 The graphs of $y = x^2 + 5x + 6$ and $y = mx - 3$, where m is a constant, are plotted on the same set of axes.

Given that the graphs do not meet, what is the complete range of possible values of m?

A $-1 < m < 11$

B $m < -1,\ m > 11$

C $-\sqrt{11} < m < \sqrt{11}$

D $m < -\sqrt{11},\ m > \sqrt{11}$

E $-11 < m < 1$

F $m < -11,\ m > 1$

As the graphs do not meet, the simultaneous equations $y = x^2 + 5x + 6$ and $y = mx - 3$ have no solutions. Therefore $x^2 + 5x + 6 = mx - 3$ has no solutions, so the quadratic

$$x^2 + (5 - m)x + 9 = 0$$

has no solutions and must therefore have a negative discriminant. This gives

$$(5 - m)^2 - 4 \times 1 \times 9 = 25 - 10m + m^2 - 36 = m^2 - 10m - 11 < 0.$$

Factorising this quadratic in m gives

$$(m - 11)(m + 1) < 0.$$

This quadratic has roots -1 and 11, and as the coefficient of m^2 in $m^2 - 10m - 11$ is positive, the quadratic in m is negative between the roots. So the complete range of possible values of m is $-1 < m < 11$, which is option A.

3 For any integer $n \geq 0$,

$$\int_n^{n+1} f(x)\,\mathrm{d}x = n+1$$

Evaluate

$$\int_0^3 f(x)\,\mathrm{d}x + \int_1^3 f(x)\,\mathrm{d}x + \int_2^3 f(x)\,\mathrm{d}x + \int_4^3 f(x)\,\mathrm{d}x + \int_5^3 f(x)\,\mathrm{d}x$$

A –2

B 0

C 1

D 4

E 18

F 27

We work out the values of each of the individual integrals by splitting them into unit intervals:

$$\begin{aligned}
\int_0^3 f(x)\,\mathrm{d}x &= \int_0^1 f(x)\,\mathrm{d}x + \int_1^2 f(x)\,\mathrm{d}x + \int_2^3 f(x)\,\mathrm{d}x \\
&= 1+2+3 \\
&= 6 \\
\int_1^3 f(x)\,\mathrm{d}x &= \int_1^2 f(x)\,\mathrm{d}x + \int_2^3 f(x)\,\mathrm{d}x \\
&= 2+3 \\
&= 5 \\
\int_2^3 f(x)\,\mathrm{d}x &= 3 \\
\int_4^3 f(x)\,\mathrm{d}x &= -\int_3^4 f(x)\,\mathrm{d}x \\
&= -4 \\
\int_5^3 f(x)\,\mathrm{d}x &= -\int_3^5 f(x)\,\mathrm{d}x \\
&= -\left(\int_3^4 f(x)\,\mathrm{d}x + \int_4^5 f(x)\,\mathrm{d}x\right) \\
&= -(4+5) \\
&= -9
\end{aligned}$$

where for the final two integrals, we have used the rule $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$.

Adding these gives the required sum:

$$6+5+3-4-9=1$$

which is option C.

4 Evaluate

$$\sum_{n=0}^{\infty} \frac{\sin\left(n\pi + \frac{\pi}{3}\right)}{2^n}$$

A 0

B $\frac{1}{3}$

C $\frac{\sqrt{3}}{3}$

D $\sqrt{3}$

E 3

To help us understand what is going on here, we will work out the first few terms.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\sin\left(n\pi + \frac{\pi}{3}\right)}{2^n} &= \frac{\sin(\frac{\pi}{3})}{2^0} + \frac{\sin(\pi + \frac{\pi}{3})}{2^1} + \frac{\sin(2\pi + \frac{\pi}{3})}{2^2} + \frac{\sin(3\pi + \frac{\pi}{3})}{2^3} + \cdots \\ &= \frac{\sqrt{3}/2}{1} + \frac{-\sqrt{3}/2}{2} + \frac{\sqrt{3}/2}{4} + \frac{-\sqrt{3}/2}{8} + \cdots \\ &= \frac{\sqrt{3}}{2}\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots\right)\end{aligned}$$

We see that the part in the brackets is a geometric series with first term $a = 1$ and common ratio $r = -\frac{1}{2}$, so its sum is

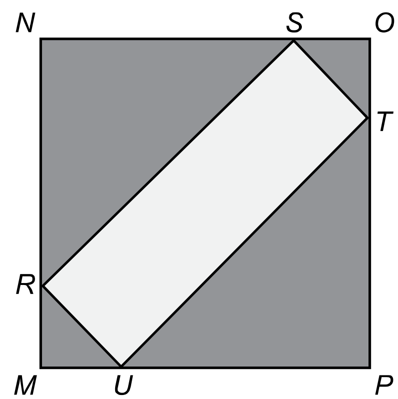
$$\frac{a}{1-r} = \frac{1}{1-(-\frac{1}{2})} = \frac{1}{3/2} = \frac{2}{3}$$

and hence the sum of the original series is

$$\frac{\sqrt{3}}{2} \times \frac{2}{3} = \frac{\sqrt{3}}{3}$$

which is option C.

5 The following shape has two lines of reflectional symmetry.



[diagram not to scale]

MNOP is a square of perimeter 40 cm.

The vertices of rectangle *RSTU* lie on the edge of square *MNOP*.

MR has length x cm.

What is the largest possible value of x such that *RSTU* has area 20 cm^2?

A $\sqrt{2}$

B $\sqrt{10}$

C $2\sqrt{15}$

D $10\sqrt{2}$

E $5+\sqrt{5}$

F $5+\sqrt{15}$

There are (at least) two ways to calculate the area of the rectangle $RSTU$: we could find the lengths of the sides RU and RS and then multiply them, or we could find the area of the square $MNOP$ and subtract the areas of the shaded triangles. We demonstrate both approaches.

We ignore the units, taking lengths to be in cm and areas to be in cm^2.

We first note that the symmetry of the shape means that $MU = MR$, $NR = NS$ and so on.

Also, as the perimeter of $MNOP$ is 40, the side length is 10, so $NR = 10 - x$.

Approach 1: Finding the lengths of the rectangle's sides

The triangle RMU is right-angled, with $MR = MU = x$. By Pythagoras's Theorem, this means that $RU^2 = x^2 + x^2 = 2x^2$, so $RU = x\sqrt{2}$.

Now $NR = 10 - x$, so by a similar argument, $RS = (10 - x)\sqrt{2}$.

Multiplying these gives the area as $A = x\sqrt{2} \times (10 - x)\sqrt{2} = 2x(10 - x)$.

Approach 2: Finding the area by subtraction

The area of $MNOP$ is $10^2 = 100$. The area of MRU and the area of OST are each $\frac{1}{2}x^2$, so the total area of these two triangles is x^2. Likewise, the area of NRS and the area of PTU are each $\frac{1}{2}(10 - x)^2$, so the total area of these two triangles is $(10 - x)^2$. Therefore the area of the rectangle $RSTU$ is

$$100 - x^2 - (10 - x)^2 = 100 - x^2 - (100 - 20x + x^2) = 20x - 2x^2 = 2x(10 - x).$$

We are told that $A = 20$, so we have to solve the equation $2x(10 - x) = 20$. Expanding and rearranging gives $2x^2 - 20x + 20 = 0$; dividing by 2 simplifies this to $x^2 - 10x + 10 = 0$. We can use the quadratic formula to find the possible values of x; they are

$$x = \frac{10 \pm \sqrt{(-10)^2 - 4 \times 1 \times 10}}{2} = \frac{10 \pm \sqrt{60}}{2} = 5 \pm \sqrt{15}.$$

The larger of these is $5 + \sqrt{15}$, and so the answer is option F.

6 In the simplified expansion of $(2+3x)^{12}$, how many of the terms have a coefficient that is divisible by 12?

A 0

B 2

C 5

D 10

E 11

F 12

G 13

A coefficient is divisible by 12 if and only if it is a multiple of $2^2 = 4$ and a multiple of 3. The coefficient of x^k (for $0 \leq k \leq 12$) is given by $\binom{12}{k}2^{12-k}3^k$.

This is certainly a multiple of 3 when $k \geq 1$.

This is certainly a multiple of $2^2 = 4$ when $k \leq 10$.

We therefore only need to check $k = 0$, $k = 11$ and $k = 12$.

When $k = 0$, the coefficient is $\binom{12}{0} \times 2^{12} = 2^{12}$, which is not a multiple of 3, and so not of 12 either.

When $k = 11$, the coefficient is $\binom{12}{11} \times 2 \times 3^{11}$. But $\binom{12}{11} = \binom{12}{1} = 12$, so this is a multiple of 12.

When $k = 12$, the coefficient is $\binom{12}{12} \times 3^{12} = 3^{12}$, which is not a multiple of 2, and so not of 12 either.

Therefore the coefficient is a multiple of 12 for $1 \leq k \leq 11$, so 11 terms have such a coefficient, and the answer is option E.

7 P(x) and Q(x) are defined as follows:

$$P(x) = 2^{x} + 4$$

$$Q(x) = 2^{(2x-2)} - 2^{(x+2)} + 16$$

Find the largest value of x such that P(x) and Q(x) are in the ratio 4 **:** 1, respectively.

A 5

B 12

C 32

D $\log_2 3$

E $\log_2 5$

F $\log_2 12$

G $\log_2 20$

Note that $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ being in the ratio $4:1$ means that $\mathrm{P}(x) = 4\mathrm{Q}(x)$, so

$$2^x + 4 = 4\big(2^{(2x-2)} - 2^{(x+2)} + 16\big).$$

Simplifying the powers, noting that $4 = 2^2$, gives

$$2^x + 4 = 2^{2x} - 2^{x+4} + 64.$$

This is a quadratic in 2^x, so let us write $u = 2^x$. The equation can then be written as

$$u + 4 = u^2 - 16u + 64.$$

This rearranges to $u^2 - 17u + 60 = 0$, which factorises as $(u-5)(u-12) = 0$, so $u = 5$ or $u = 12$. Therefore, as $u = 2^x$, we have $x = \log_2 5$ or $x = \log_2 12$, and the largest possible value of x is $\log_2 12$, which is option F.

8 A triangle *XYZ* is called *fun* if it has the following properties:

$$\text{angle } YXZ = 30°$$

$$XY = \sqrt{3}\,a$$

$$YZ = a$$

where a is a constant.

For a given value of a, there are two distinct *fun* triangles *S* and *T*, where the area of *S* is greater than the area of *T*.

Find the ratio

area of *S* **:** area of *T*

A 1 **:** 1

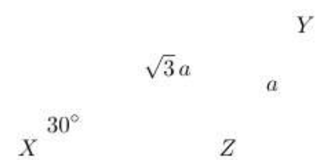
B 2 **:** 1

C 2 **:** $\sqrt{3}$

D $\sqrt{3}$ **:** 1

E 3 **:** 1

Let us sketch a fun triangle.



The area of XYZ is given by the '$\frac{1}{2}ab\sin C$' formula: it is $\frac{1}{2}XY.XZ\sin 30° = \frac{1}{4}\sqrt{3}\,a.XZ$. We must therefore find the possible values of XZ.

We can use the cosine rule to find XZ: we have

$$YZ^2 = XY^2 + XZ^2 - 2XY.XZ\cos 30°.$$

Filling in the known values, we obtain

$$a^2 = 3a^2 + XZ^2 - 3a.XZ$$

so XZ satisfies the quadratic equation

$$XZ^2 - 3a.XZ + 2a^2 = 0.$$

This factorises to give

$$(XZ - 2a)(XZ - a) = 0$$

so either $XZ = a$ or $XZ = 2a$.

It is useful to think about what the two resulting triangles look like and whether these values of XZ make sense.

We are asked to find the ratio of the areas of these possible fun triangles. Since the area is given by $\frac{1}{4}\sqrt{3}\,a.XZ$, the ratio of areas is given by the ratio of possible XZ values. Since triangle S has the larger area, it must have the larger of the XZ values, so the ratio is $2a : a$, which equals $2 : 1$. The correct answer is thus option B.

9 How many solutions are there to

$$(1+3\cos 3\theta)^2 = 4$$

in the interval $0° \le \theta \le 180°$?

A 1

B 2

C 3

D 4

E 5

F 6

We have $(1+3\cos 3\theta)^2 = 4$ if and only if $1+3\cos 3\theta = \pm 2$. We consider each possibility separately.

We have

$$\begin{aligned} & 1+3\cos 3\theta = 2 \\ \text{if and only if} \quad & 3\cos 3\theta = 1 \\ \text{if and only if} \quad & \cos 3\theta = \tfrac{1}{3}. \end{aligned}$$

Since $0° \le \theta \le 180°$, we have $0° \le 3\theta \le 540°$, and there are 3 values of 3θ which have $\cos 3\theta = \frac{1}{3}$ in this interval. (One is between $0°$ and $90°$, one is between $270°$ and $360°$, and one is between $360°$ and $450°$, by considering the graph of $y = \cos x$.)

Now considering the other possibility, we have

$$\begin{aligned} & 1+3\cos 3\theta = -2 \\ \text{if and only if} \quad & 3\cos 3\theta = -3 \\ \text{if and only if} \quad & \cos 3\theta = -1. \end{aligned}$$

Again, $0° \le 3\theta \le 540°$, but this time there are only 2 values of 3θ which satisfy the equation: $3\theta = 180°$ and $3\theta = 540°$.

Neither of these values of 3θ overlap with the values of 3θ found earlier, so in total there are $3+2=5$ values of 3θ in the interval $0° \le 3\theta \le 540°$, and hence 5 solutions to the original equation in the given interval. The correct answer is option E.

10 The trapezium rule with 4 strips is used to estimate the integral:

$$\int_{-2}^{2} \sqrt{4 - x^2}\, \mathrm{d}x$$

What is the positive difference between the estimate and the exact value of the integral?

A $2(\pi - 2 - 2\sqrt{3})$

B $2(\pi - 1 - \sqrt{3})$

C $2(2\pi - 1 - \sqrt{3})$

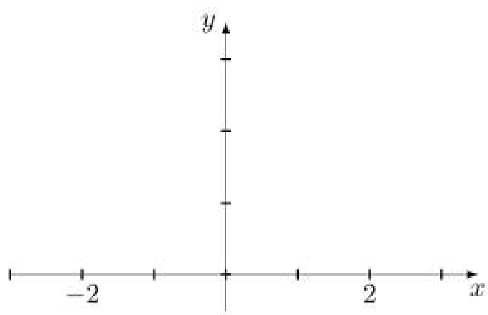
D $4(\pi - 1 - \sqrt{3})$

E $2\pi - 3\sqrt{3}$

F $4\pi - 6\sqrt{3}$

Though integrating functions like this is not on the syllabus, the question assumes that we can calculate the value of this integral. There is also a hint from the options offered that it may have something to do with circles, as they all involve π.

Let us therefore consider the graph $y = \sqrt{4 - x^2}$. Squaring this gives $y^2 = 4 - x^2$ or $x^2 + y^2 = 4$. This is the equation of a circle with centre at the origin and radius 2. However, as the graph involves taking the square root of this, the resulting graph is just the upper semicircle (on and above the x-axis).



The integral is from $x = -2$ to $x = 2$, which is the whole diameter, so the integral is the area of the semicircle, that is, $\frac{1}{2} \times \pi \times 2^2 = 2\pi$.

We now need to calculate the area as determined by the trapezium rule with 4 strips. We can draw a table of values:

x	-2	-1	0	1	2
$\sqrt{4 - x^2}$	0	$\sqrt{3}$	2	$\sqrt{3}$	0

The trapezium rule then approximates the area as

$$\tfrac{1}{2} \times 1 \times \left(0 + 2 \times \sqrt{3} + 2 \times 2 + 2 \times \sqrt{3} + 0\right) = \tfrac{1}{2} \times \left(4 + 4\sqrt{3}\right) = 2 + 2\sqrt{3}.$$

The positive difference (as the trapezium rule underestimates the area) is therefore

$$2\pi - \left(2 + 2\sqrt{3}\right) = 2\left(\pi - 1 - \sqrt{3}\right)$$

which is option B.

11 It is given that $f(x) = x^2 - 6x$

The curves $y = f(kx)$ and $y = f(x - c)$ have the same minimum point, where $k > 0$ and $c > 0$

Which of the following is a correct expression for k in terms of c?

A $k = \frac{3-c}{3}$

B $k = \frac{3}{c+3}$

C $k = \frac{c-6}{6}$

D $k = \frac{6}{6-c}$

E $k = \frac{c+9}{9}$

F $k = \frac{9}{c-9}$

Approach 1: Direct approach

We have, for the first graph,

$$y = f(kx) = (kx)^2 - 6(kx) = k^2x^2 - 6kx.$$

We can find the minimum point using either calculus or completing the square; we will use calculus here. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2k^2x - 6k$$

so $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ exactly when $x = \frac{3}{k}$. At this point, $y = k^2(\frac{3}{k})^2 - 6k(\frac{3}{k}) = -9$.

For the second graph, we have

$$y = f(x - c) = (x - c)^2 - 6(x - c) = x^2 - 2cx + c^2 - 6x + 6c = x^2 - (6 + 2c)x + c^2 + 6c.$$

We use completing the square to find the minimum point: we have

$$y = (x - (3 + c))^2 - (3 + c)^2 + c^2 + 6c = (x - (3 + c))^2 - 9$$

so the minimum point is at $x = 3 + c$, $y = -9$.

Approach 2: Using graph transformations

We first find the minimum point of $y = x^2 - 6x$. We can complete the square to get $y = (x - 3)^2 - 9$, so the minimum point is at $x = 3$, $y = -9$.

The graph $y = f(kx)$ is obtained by stretching the original graph by a factor of $1/k$ in the x-direction, so the minimum of this graph is at $x = 3/k$, $y = -9$.

The graph $y = f(x - c)$ is obtainedd by translating the original graph by c in the x-direction, so the minimum of this graph is at $x = 3 + c$, $y = -9$.

For the two minima to be at the same place, we require $3/k = 3 + c$, so taking the reciprocal gives

$$\frac{k}{3} = \frac{1}{3 + c}.$$

Multiplying by 3 then gives

$$k = \frac{3}{3 + c},$$

which is option B.

12 How many solutions are there to the equation

$$\frac{2^{\tan^2 x}}{4^{\sin^2 x}} = 1$$

in the range $0 \le x \le 2\pi$?

A 2

B 3

C 4

D 5

E 6

F 7

G 8

Let us start by multiplying both sides by $4^{\sin^2 x}$ to remove the fraction:

$$2^{\tan^2 x} = 4^{\sin^2 x}.$$

We note that $4^a = (2^2)^a = 2^{2a}$, so we can rewrite this equation as

$$2^{\tan^2 x} = 2^{2\sin^2 x},$$

and therefore the equation can be simplified to

$$\tan^2 x = 2\sin^2 x.$$

Next, we know that $\tan x = \frac{\sin x}{\cos x}$, so we can rewrite this as

$$\frac{\sin^2 x}{\cos^2 x} = 2\sin^2 x.$$

Multiplying by $\cos^2 x$ to eliminate fractions once more gives

$$\sin^2 x = 2\sin^2 x \cos^2 x,$$

which we can rearrange and factorise to give

$$\sin^2 x(1 - 2\cos^2 x) = 0.$$

Therefore either $\sin x = 0$ or $\cos^2 x = \frac{1}{2}$. The first has solutions $x = 0$, π, 2π. The second gives $\cos x = \pm\frac{1}{\sqrt{2}}$, so has 4 solutions in the range $0 \le x \le 2\pi$, none of which are 0, π or 2π. (We could list them explicitly: they are $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$.)

In total, there are 7 solutions, which is option F.

13 Point P lies on the circle with equation $(x - 2)^2 + (y - 1)^2 = 16$

Point Q lies on the circle with equation $(x - 4)^2 + (y + 5)^2 = 16$

What is the maximum possible length of PQ?

A 10

B 14

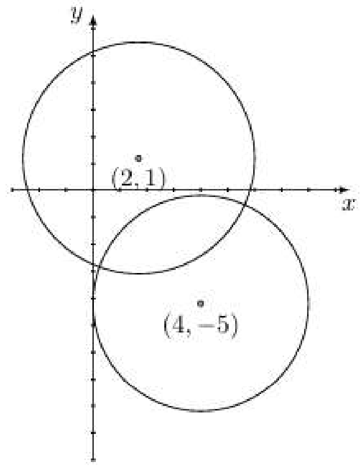
C 16

D $2\sqrt{34}$

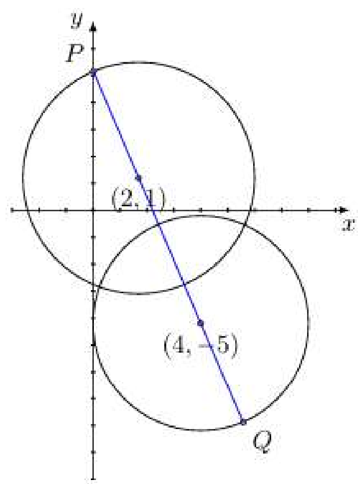
E $10\sqrt{2}$

F $8 + 2\sqrt{10}$

G $16 + 2\sqrt{6}$



It is clear that the furthest apart that P and Q could be is if they are on the line that passes through the two circle centres as shown below:



The distance between P and Q is then the sum of the radius of the first circle, the distance between the centres and the radius of the second circle.

The distance between the centres is $\sqrt{(4-2)^2 + (-5-1)^2} = \sqrt{40} = 2\sqrt{10}$ and each radius is 4, so the maximum distance is $8 + 2\sqrt{10}$, which is option F.

14 The function

$$f(x) = \frac{2}{3}x^3 + 2mx^2 + n, \quad m > 0$$

has three distinct real roots.

What is the complete range of possible values of n, in terms of m?

A $-\frac{8}{3}m^3 < n < 0$

B $-\frac{4}{3}m^3 < n < 0$

C $0 < n < \frac{3}{2}m^2$

D $0 < n < \frac{40}{3}m^3$

E $n < -\frac{8}{3}m^3$

F $n < \frac{3}{2}m^2$

G $n > -\frac{4}{3}m^3$

H $n > \frac{40}{3}m^3$

The function will have three distinct real roots if and only if the graph $y = f(x)$ has two stationary points, one above the x-axis and one below. We first find the stationary points. We have

$$f'(x) = 2x^2 + 4mx = 2x(x + 2m)$$

so $f'(x) = 0$ when $x = 0$ or $x = -2m$.

When $x = 0$, $f(x) = n$, and when $x = -2m$,

$$f(x) = \frac{2}{3}(-8m^3) + 2m(4m^2) + n = n + \frac{8}{3}m^3.$$

Since one of the stationary points has $f(x) = n$, we cannot have $n = 0$.

Since $m > 0$, it follows that if $n > 0$, then $f(-2m) > 0$ and both turning points would be above the x-axis. We must therefore have $n < 0$ and $n + \frac{8}{3}m^3 > 0$. The second inequality rearranges to $-\frac{8}{3}m^3 < n$, so we require $-\frac{8}{3}m^3 < n < 0$, which is option A.

15 The difference between the maximum and minimum values of the function $f(x) = a^{\cos x}$, where $a > 0$ and x is real, is 3.

Find the sum of the possible values of a.

A 0

B $\frac{3}{2}$

C $\frac{5}{2}$

D 3

E $\sqrt{10}$

F $\sqrt{13}$

The minimum and maximum values occur when $\cos x = \pm 1$ (though which corresponds to the maximum and which to the minimum depends on whether $a < 1$ or $a > 1$). Therefore the minimum and maximum values are a and $1/a$ in some order.

Assuming $a > 1$, so that $a > 1/a$, we require $a - \frac{1}{a} = 3$. Multiplying by a and rearranging gives $a^2 - 3a - 1 = 0$, which has roots $a = \frac{3 \pm \sqrt{13}}{2}$. The minus sign would give $a < 0$, so the root we want is $a = \frac{3+\sqrt{13}}{2}$, and this clearly satisfies $a > 1$.

Now assuming $a < 1$, so that $1/a > a$, we require $\frac{1}{a} - a = 3$. Multiplying by a and rearranging gives $a^2 + 3a - 1 = 0$, which has roots $a = \frac{-3 \pm \sqrt{13}}{2}$. The minus sign would give $a < 0$, so the root we want is $a = \frac{-3+\sqrt{13}}{2}$ (and as $3 < \sqrt{13} < 4$, this lies between 0 and $\frac{1}{2}$, hence $a < 1$).

The sum of these two values is $\frac{3+\sqrt{13}}{2} + \frac{-3+\sqrt{13}}{2} = \sqrt{13}$, which is option F.

16 A right-angled triangle has vertices at (2, 3), (9, –1) and (5, k).

Find the sum of all the possible values of k.

A –8

B –6

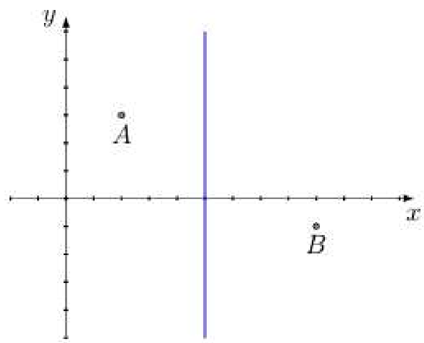
C 0.25

D 2

E 2.25

F 8.25

G 10.25



We could have C at about $(5,5)$ and have a right angle at A. We could have C a little lower and have a right angle at C. We could have C at about $(5,-5)$ and have a right angle at C, or we could have C a little lower and have a right angle at B. We work through these four possibilities.

- Right angle at A. The gradient of the line AB is $\frac{-1-3}{9-2} = -\frac{4}{7}$, so the gradient of the line AC is $\frac{7}{4}$.

  The equation of the line AC is thus $y - 3 = \frac{7}{4}(x-2)$. This intersects the line $x = 5$ when $y - 3 = \frac{7}{4}(5-2)$, giving $y = 3 + \frac{21}{4} = \frac{33}{4}$. So $k = \frac{33}{4}$ gives a right-angled triangle.

- Right angle at B. As in the first case, the gradient of the line BC is $\frac{7}{4}$.

  The equation of the line BC is thus $y + 1 = \frac{7}{4}(x-9)$. This intersects the line $x = 5$ when $y + 1 = \frac{7}{4}(5-9)$, giving $y = -1 - 7 = -8$. So $k = -8$ gives a right-angled triangle.

- Right angle at C. We could determine the possible values of k either by observing that the lines AC and BC must be perpendicular, or by using Pythagoras's Theorem to solve $AC^2 + BC^2 = AC^2$. Let us use the perpendicular approach here. We have

  $$\text{gradient of } AC = \frac{k-3}{5-2} = \frac{k-3}{3}$$
  $$\text{gradient of } BC = \frac{k+1}{5-9} = -\frac{k+1}{4}$$

  These must multiply to -1, so

  $$\frac{k-3}{3} \cdot \frac{k+1}{4} = 1$$

  which simplifies to $(k-3)(k+1) = 12$. We can solve this as a quadratic as usual: it rearranges to $k^2 - 2k - 15 = 0$ giving $k = 5$ and $k = -3$.

Adding these possible values of k gives a total of $\frac{33}{4} - 8 + 5 - 3 = \frac{9}{4}$, which is option E.

17 A circle C_n is defined by

$$x^2 + y^2 = 2n(x + y)$$

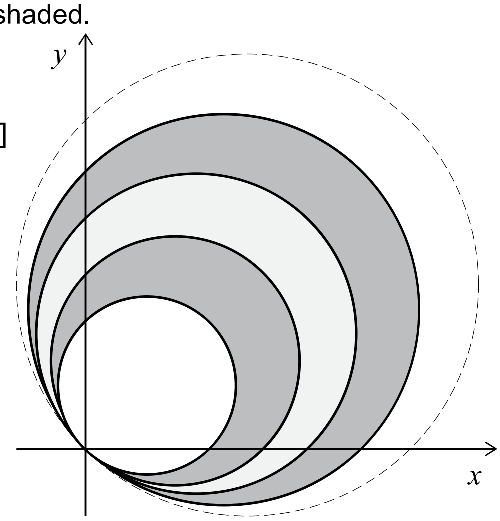
where n is a positive integer.

C_1 and C_2 are drawn and the area between them is shaded.

Next, C_3 and C_4 are drawn and the area between them is shaded.

This is shown in the diagram.

[diagram not to scale]



This process continues until 100 circles have been drawn.

What is the total shaded area?

A 100π

B 500π

C 2500π

D 5050π

E 10100π

F 40400π

We can prove this. Firstly, the circle C_n passes through the origin as $(x, y) = (0, 0)$ satisfies the equation of the circle. Next, we can solve the equation of the circle C_n simultaneously with $y = -x$. Substituting $y = -x$ into the equation of the circle gives

$$x^2 + x^2 = 2n(x + (-x)) = 0$$

so $2x^2 = 0$, which has a repeated root $x = 0$. Therefore this line is tangent to the circle at the origin.

Next, we need to find the area of C_n. We rearrange and complete the square to do this: we have

$$\begin{aligned} x^2 - 2nx + y^2 - 2ny &= 0 \\ \text{so} \quad (x-n)^2 - n^2 + (y-n)^2 - n^2 &= 0 \\ \text{so} \quad (x-n)^2 + (y-n)^2 &= 2n^2 \end{aligned}$$

This shows that the centre of C_n is at (n, n) and it has squared radius $2n^2$. Its area is therefore $2\pi n^2$.

We deduce that the shaded area between C_{n+1} and C_n is

$$2\pi(n+1)^2 - 2\pi n^2 = 2\pi(2n+1).$$

Summing this for $n = 1, 3, 5, \dots, 99$ (as we want the difference between circles C_1 and C_2, and between C_3 and C_4, and so on), the total shaded area is

$$2\pi(3 + 7 + 11 + \cdots + 199)$$

The sum here is an arithmetic series; we have first term $a = 3$, last term $l = 199$, and $n = 50$ terms (as there are 50 pairs of circles), so the total area is

$$2\pi \times \tfrac{1}{2}n(a + l) = \pi \times 50 \times 202 = 10100\pi$$

which is option E.

18 You are given that

$$S = 4 + \frac{8k}{7} + \frac{16k^2}{49} + \frac{32k^3}{343} + \cdots + 4\left(\frac{2k}{7}\right)^n + \cdots$$

The value for k is chosen as an integer in the range $-5 \le k \le 5$

All possible values for k are equally likely to be chosen.

What is the probability that the value of S is a finite number greater than 3?

A $\frac{1}{11}$

B $\frac{1}{10}$

C $\frac{3}{11}$

D $\frac{3}{10}$

E $\frac{5}{11}$

F $\frac{1}{2}$

G $\frac{7}{11}$

$\frac{7}{}$

The sum is an infinite geometric series with first term $a = 4$ and common ratio $r = \frac{2k}{7}$. For this series to have a finite sum, we require $|r| < 1$, so $-7 < 2k < 7$, that is $-\frac{7}{2} < k < \frac{7}{2}$. Thus we can only consider $k = -3, -2, \ldots, 2, 3$ out of the 11 possible values of k. (Note that there are 11 values: $k = 0$ is a possibility!)

In these cases, the sum is given by

$$S = \frac{a}{1-r} = \frac{4}{1 - 2k/7} = \frac{28}{7-2k}.$$

We need to solve the inequality $S > 3$, so

$$\frac{28}{7-2k} > 3.$$

Since $7 - 2k > 0$ for all $k < \frac{7}{2}$, which are the only values of k of interest to us, we can multiply by $7 - 2k$ to get

$$28 > 3(7 - 2k).$$

Expanding and rearranging gives $6k > -7$, so $k > -\frac{7}{6}$. Therefore the values of k that satisfy the conditions are $-1, 0, 1, 2, 3$, a total of 5 values out of the 11 possible values. Therefore the required probability is $\frac{5}{11}$, which is option E.

19 The solution to the differential equation

$$\frac{dy}{dx} = |-6x| \quad \text{for all } x$$

is $y = f(x) + c$, where c is a constant.

Which one of the following is a correct expression for $f(x)$?

A $-\frac{6x}{|x|}$

B $\frac{6x}{|x|}$

C $-3x|x|$

D $3x|x|$

E $-3x^2$

F $3x^2$

G $-x^3$

H x^3

When $x \geq 0$, the equation becomes $\frac{dy}{dx} = 6x$, so $y = 3x^2 + c$.

When $x \leq 0$, the equation becomes $\frac{dy}{dx} = -6x$, so $y = -3x^2 + \hat{c}$. (We use a different constant, $\hat{c}$, as we do not yet know whether it has the same value as the constant for the $x \geq 0$ region.)

Since the value of y has to be the same in both expressions when $x = 0$, we must have $\hat{c} = c$, so we have

$$y = \begin{cases} 3x^2 + c & \text{when } x \geq 0 \\ -3x^2 + c & \text{when } x < 0 \end{cases}$$

giving

$$f(x) = \begin{cases} 3x^2 & \text{when } x \geq 0 \\ -3x^2 & \text{when } x < 0 \end{cases}$$

The options offered require us to collapse this into a single expression. We can take out a common factor of $3x$ from both terms, to give

$$f(x) = \begin{cases} 3x(x) & \text{when } x \geq 0 \\ 3x(-x) & \text{when } x < 0 \end{cases}$$

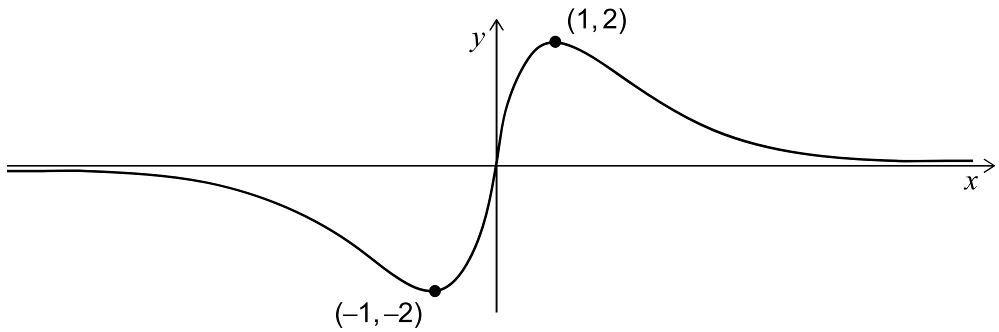
and this is $3x$ times $|x|$, since

$$|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

So we can write $f(x) = 3x\,|x|$, which is option D.

20 The diagram shows the graph of $y = f(x)$

The function f attains its maximum value of 2 at $x = 1$, and its minimum value of -2 at $x = -1$



Find the difference between the maximum and minimum values of $(f(x))^2 - f(x)$

A 2

B $\frac{9}{4}$

C 4

D $\frac{17}{4}$

E 6

F $\frac{25}{4}$

G 8

H $\frac{33}{4}$

Let us first consider the function $y = g(x)$, where $g(x) = x^2 - x$; we will then substitute $f(x)$ into this expression once we understand the behaviour of $y = g(x)$.

$g(x)$ is a quadratic and we can complete the square: $g(x) = (x - \frac{1}{2})^2 - \frac{1}{4}$. So $y = g(x)$ has a minimum value of $y = -\frac{1}{4}$ at $x = \frac{1}{2}$.

When we now substitute the values of $f(x)$ into this completed-square form for $g(x)$, obtaining $\left(f(x)\right)^2 - f(x)$. We can see from the graph that there is some value of x with $f(x) = \frac{1}{2}$, so this expression has minimum value of $-\frac{1}{4}$.

For the maximum value, we see that the value of $g(x)$ increases symmetrically either side of $x = \frac{1}{2}$. So the maximum value of $\left(f(x)\right)^2 - f(x)$ will either occur when $f(x) = 2$ or when $f(x) = -2$. Since -2 is further from $\frac{1}{2}$ than 2 is, $f(x) = -2$ will give the maximum value, which is $(-2)^2 - (-2) = 6$. (And if we hadn't made that observation, we could have calculated $2^2 - 2 = 2$ and seen that it is smaller than 6.)

Therefore the difference between the maximum and minimum values is $6 - (-\frac{1}{4}) = \frac{25}{4}$, which is option F.

PAPER 2

1 Given that

$$\frac{1}{\sqrt{x}-6} - \frac{1}{\sqrt{x}+6} = \frac{3}{11}$$

what is the value of x?

A $2\sqrt{15}$

B $4\sqrt{5}$

C $5\sqrt{2}$

D $\sqrt{58}$

E 50

F 58

G 60

H 80

We first multiply both sides of the equation by $11(\sqrt{x}-6)(\sqrt{x}+6)$ to eliminate all of the fractions, giving

$$11(\sqrt{x}+6) - 11(\sqrt{x}-6) = 3(\sqrt{x}-6)(\sqrt{x}+6).$$

We can now expand all of the brackets to give

$$11\sqrt{x} + 66 - 11\sqrt{x} + 66 = 3x - 108,$$

which simplifies to

$$240 = 3x.$$

Dividing by 3 gives $x = 80$, which is option H.

2 Evaluate

$$\int_9^{16}\left(\frac{1}{\sqrt{x}}+\sqrt{x}\right)^2 \mathrm{d}x - \int_9^{16}\left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)^2 \mathrm{d}x$$

A 0

B 2

C 4

D 7

E 14

F 28

G 75

H 175

We first expand the brackets to obtain

$$\int_9^{16} \frac{1}{x} + 2 + x \, \mathrm{d}x - \int_9^{16} \frac{1}{x} - 2 + x \, \mathrm{d}x.$$

Since these integrals have the same limits, we can combine them into a single integral to get

$$\int_9^{16} \left(\frac{1}{x} + 2 + x\right) - \left(\frac{1}{x} - 2 + x\right) = \int_9^{16} 4 \, \mathrm{d}x = 4 \times (16 - 9) = 28,$$

which is option F.

Note that integration of $\frac{1}{x}$ is not on the specification for this test; it is, of course, possible to answer this question by calculating both integrals explicitly and then subtracting, but it is much more work than the solution just presented.

3 Consider the claim:

For all positive real numbers x and y,

$$\sqrt{x^y} = x^{\sqrt{y}}$$

Which of the following is/are a **counterexample** to the claim?

I $x = 1,\ y = 16$

II $x = 2,\ y = 8$

III $x = 3,\ y = 4$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

We substitute the values into the two sides of the equation:

I $\sqrt{1^{16}} = \sqrt{1} = 1$ and $1^{\sqrt{16}} = 1^4 = 1$, so this is not a counterexample.

II $\sqrt{2^8} = 2^4 = 16$ and $2^{\sqrt{8}} \neq 16$, so this is a counterexample.

III $\sqrt{3^4} = 3^2 = 9$ and $3^{\sqrt{4}} = 3^2 = 9$, so this is not a counterexample.

Therefore the answer is option C, II only.

4 A student attempts to answer the following question.

What is the largest number of consecutive odd integers that are all prime?

The student's attempt is as follows:

I There are two consecutive odd integers that are prime (for example: 17, 19).

II Any three consecutive odd integers can be written in the form $n - 2$, n, $n + 2$ for some n.

III If n is one more than a multiple of 3, then $n + 2$ is a multiple of 3.

IV If n is two more than a multiple of 3, then $n - 2$ is a multiple of 3.

V The only other possibility is that n is a multiple of 3.

VI In each case, one of the integers is a multiple of 3, so not prime.

VII Therefore the largest number of consecutive odd integers that are all prime is two.

Which of the following best describes this attempt?

A It is completely correct.

B It is incorrect, and the first error is on line I.

C It is incorrect, and the first error is on line II.

D It is incorrect, and the first error is on line III.

E It is incorrect, and the first error is on line IV.

F It is incorrect, and the first error is on line V.

G It is incorrect, and the first error is on line VI.

H It is incorrect, and the first error is on line VII.

The argument looks very convincing and there is no obvious error. However, writing down the first few odd integers and underlining those which are prime shows that there is a problem:

$$1, \underline{3}, \underline{5}, \underline{7}, 9, \underline{11}, \underline{13}, 15, \underline{17}, \underline{19}, \ldots$$

There are three consecutive odd numbers that are all prime right at the start (3, 5, 7), so the student's final answer in line VII is incorrect. We must therefore find the first error in the argument.

I This line is correct: 17 and 19 are consecutive odd integers that are prime.

II This is correct: consecutive odd integers differ by 2, and we can call the middle one n.

III If $n = 3k + 1$, then $n + 2 = 3k + 3 = 3(k + 1)$, which is a multiple of 3, so this line is correct.

IV If $n = 3k + 2$, then $n - 2 = 3k$, which is a multiple of 3, so this line is correct.

V All integers are either multiples of 3, one more than a multiple of 3 or two more than a multiple of 3; a number which is three more than a multiple of 3 is itself a multiple of 3. Similarly, a number which is four more than a multiple of 3 is one more than the next multiple of 3 (in algebraic notation: $3k + 4 = 3(k + 1) + 1$), and so on. So this statement is correct.

VI In each case, one of $n + 2$, $n - 2$ and n is a multiple of 3. But our example of 3, 5, 7 shows that the last part of this statement is false: 3 is a multiple of 3, but *is* prime.

So the first error is on line VI, which is option G.

5 Consider the two statements

R: k is an integer multiple of π

S: $\int_0^k \sin 2x \, dx = 0$

Which of the following statements is true?

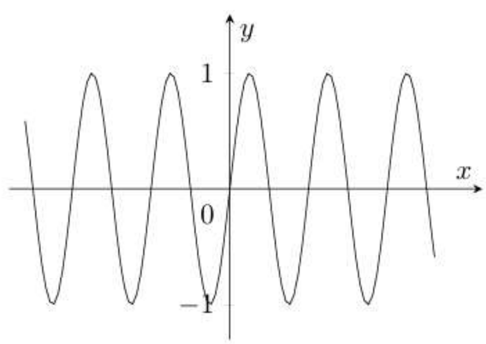
A R is **necessary and sufficient** for S.

B R is **necessary** but **not sufficient** for S.

C R is **sufficient** but **not necessary** for S.

D R is **not necessary** and **not sufficient** for S.

The graph of $y = \sin 2x$ is periodic with period π: it is the graph of $y = \sin x$ stretched by a factor of $\frac{1}{2}$ in the x-direction:



So if R is true, we see from the graph that $\int_0^k \sin 2x \, dx = 0$ as the areas above and below the x-axis cancel. Thus R is sufficient for S.

Conversely, if $\int_0^k \sin 2x \, dx = 0$, the areas above and below the x-axis must cancel. We can see from the graph that the only way to achieve this is if k is a multiple of π, so that we have an exact number of periods of $y = \sin 2x$. (The integral increases as k increases from 0 to $\frac{\pi}{2}$, and then decreases back to 0 as k increases from $\frac{\pi}{2}$ to π, and the same happens in every period.) Therefore S implies R, or R is necessary for S.

It follows that the correct answer is option A: R is necessary and sufficient for S.

6 Consider the following equation where a is a real number and $a > 1$:

$$(*) \quad a^x = x$$

Which of the following equations must have the same number of real solutions as $(*)$?

I $\log_a x = x$

II $a^{2x} = x^2$

III $a^{2x} = 2x$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

I If we take the logarithm to base a of the equation $(*)$, we get $x = \log_a x$.

This suggests that this equation must have the same number of real solutions as $(*)$, but we must be careful and check that we are allowed to take the logarithm of both sides of $(*)$ without adding or losing solutions. We can only take the logarithm of a positive number. The left hand side of $(*)$ is always positive, and if $x \leq 0$, then this value of x does not give a solution to $(*)$. So the solutions to $(*)$ (if there are any) all have $x > 0$. So we can take the logarithm of both sides without losing any solutions. And by exponentiating the resulting equation, we get back to $(*)$, so we do not introduce any new solutions. Therefore this equation must have the same number of real solutions as $(*)$.

II If we square both sides of $(*)$, we obtain $a^{2x} = x^2$. Squaring can introduce new solutions, though; does it on this occasion? If we have a solution to $a^{2x} = x^2$, we can deduce that $a^x = \pm x$. We always have $a^x > 0$, so it might be that we get a new solution with $x < 0$. Let us try a numerically simple example to see if this does happen. As $x < 0$, we have $0 < a^x < 1$, so we need $-1 < x < 0$. Taking $x = -\frac{1}{2}$, $(*)$ becomes $a^{-1/2} = \frac{1}{2}$; we then get $a = 4$. So in this case, $4^{2x} = x^2$ has an extra solution, $x = -\frac{1}{2}$, which is not a solution to $4^x = x$. Therefore this equation may not have the same number of real solutions as $(*)$.

Another way to think about it is graphically. The graphs of $y = a^x$ and $y = a^{2x}$ are both exponential graphs with $y < 1$ for all $x < 0$. The graph of $y = x$ is negative for all $x < 0$ so any solutions to $(*)$ have $x > 0$. On the other hand, $y = x^2$ is positive for all $x \neq 0$ and it reaches 1 at $x = -1$. Therefore there will always be a solution to $a^{2x} = x^2$ with $-1 < x < 0$.

III If we replace $2x$ with u in this equation, we get $a^u = u$, so there are the same number of real solutions for u as $(*)$ has. Every (real) value of u corresponds to a unique value of x, namely $x = u/2$, so there must be the same number of real solutions to $a^{2x} = 2x$ as $(*)$.

Thus the answer is option F, I and III only.

7 The graph of the line $ax + by = c$ is drawn, where a, b and c are real non-zero constants.

Which one of the following is a **necessary** but **not sufficient** condition for the line to have a positive gradient **and** a positive y-intercept?

A $\frac{c}{b} > 0$ and $\frac{a}{b} < 0$

B $\frac{c}{b} < 0$ and $\frac{a}{b} > 0$

C $a > b > c$

D $a < b < c$

E a and c have opposite signs

F a and c have the same sign

We can rearrange the equation to give $by = -ax + c$, so

$$y = -\frac{a}{b}x + \frac{c}{b}.$$

Therefore the gradient is $-\frac{a}{b}$ and the y-intercept is $\frac{c}{b}$.

We go through the conditions to determine whether they are necessary and/or sufficient for a positive gradient and positive y-intercept.

A $\frac{c}{b} > 0$ is equivalent to the y-intercept being positive. Next, $\frac{a}{b} < 0$ is equivalent to $-\frac{a}{b} > 0$, which is equivalent to the gradient being positive. So this is a necessary and sufficient condition.

B $\frac{c}{b} < 0$ is equivalent to the y-intercept being negative, so this is neither necessary nor sufficient.

C If we take $a > 0 > c > b$, then $-\frac{a}{b} > 0$ and $\frac{c}{b} > 0$, so the gradient and y-intercept are both positive. Therefore $a > b > c$ is not necessary.

D If we take $a < 0 < c < b$, then $-\frac{a}{b} > 0$ and $\frac{c}{b} > 0$, so the gradient and y-intercept are both positive. Therefore $a < b < c$ is not necessary.

E If $-\frac{a}{b} > 0$ and $\frac{c}{b} > 0$, then $\frac{a}{b} < 0 < \frac{c}{b}$, so a and c must have opposite signs. Therefore this condition is necessary. But if a and c have opposite signs, we know nothing about the sign of b, so we might or might not have $-\frac{a}{b} > 0$. So this condition is not sufficient.

F As in E, if the gradient and y-intercept are both positive, a and c must have opposite signs, so this condition is not necessary.

The correct option is therefore E.

8 A student draws a triangle that is acute-angled or obtuse-angled but **not** right-angled.

The student counts the number of straight lines that divide the triangle into two triangles, **at least one** of which is right-angled.

Which of the following statements is/are true?

I The student can draw a triangle for which there is exactly 1 such straight line.

II The student can draw a triangle for which there are exactly 2 such straight lines.

III The student can draw a triangle for which there are exactly 3 such straight lines.

A none of them

B I only

C II only

D III only

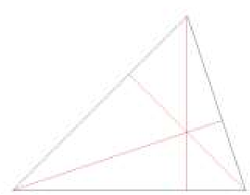
E I and II only

F I and III only

G II and III only

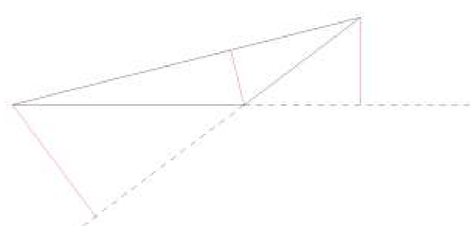
H I, II and III

Let us start with a triangle and see how many such straight lines it has. If we take an acute-angled triangle, then it has three such lines (shown in red), which are the altitudes of the triangle, going from each vertex perpendicular to the opposite side:

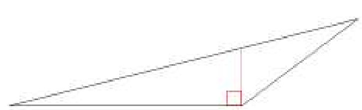


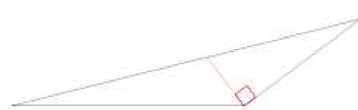
Therefore statement III is true.

But can the student draw a triangle for which there are fewer than 3 such straight lines? There is always such a straight line (altitude) from every vertex in an acute-angled triangle, so the only hope for fewer than 3 such straight lines is to consider an obtuse-angled triangle. There is only one usable altitude in this case, and that is from the obtuse angle; the other two altitudes land outside the triangle:



So it may appear that there is only one such line in this case. However, we are not restricted to altitudes! Instead, we can split the obtuse angle into a right angle and an acute angle in two different ways, giving another two lines:





Neither of these lines can meet the side opposite the obtuse angle perpendicularly, as if they did, we would have a triangle with two right angles, so they are always distinct from the altitude.

So in this case, we also get three such staight lines.

Hence the correct answer is option D, III only.

9 Consider the following statement about a pentagon P:

> ($*$) **If** at least one of the interior angles in P is 108°, **then** all the interior angles in P form an arithmetic sequence.

Which of the following is/are true?

- I The statement ($*$)
- II The contrapositive of ($*$)
- III The converse of ($*$)

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

We first note that the interior angles in a pentagon sum to $3 \times 180° = 540°$ (since a pentagon can be cut up into three triangles). Therefore the mean interior angle is $540°/5 = 108°$.

I The angles could be 108°, 108°, 108°, 107° and 109°: start with a regular pentagon and slightly rotate one of the sides. These do not form an arithmetic sequence, so statement ($*$) is false.

II The contrapositive of a statement always has the same truth value as the statement itself, so this is also false.

III The converse of ($*$) states: **if** all the interior angles in P form an arithmetic sequence, **then** at least one of the interior angles in P is 108°. If the interior angles in P form an arithmetic sequence with common difference d, then the angles can be written as $c-2d$, $c-d$, c, $c+d$, $c+2d$. (One could also write them as a, $a+d$, $\ldots$, $a+4d$, but this approach is easier.) Then the sum of the interior angles is $5c = 540°$, so $c = 108°$. Since one of the interior angles is c, the converse of ($*$) is true.

Thus the answer is option D, III only.

10 Here is an attempt to solve the inequality $x^4 - 2x^2 - 3 < 0$ by completing the square:

$x^4 - 2x^2 - 3 < 0$

I if and only if $x^4 - 2x^2 + 1 < 4$

II if and only if $(x^2 - 1)^2 < 4$

III if and only if $-2 < x^2 - 1 < 2$

IV if and only if $x^2 - 1 < 2$

V if and only if $x^2 < 3$

VI if and only if $-\sqrt{3} < x < \sqrt{3}$

Which of the following statements is true?

A The argument is completely correct.

B The first error occurs in line I.

C The first error occurs in line II.

D The first error occurs in line III.

E The first error occurs in line IV.

F The first error occurs in line V.

G The first error occurs in line VI.

Substituting $x = 0$ gives $x^4 - 2x^2 - 3 = -3 < 0$, so the final conclusion on line VI is at least plausible. We will check the argument line-by-line.

I This is obtained by adding 4 to both sides, and this is reversible. This line is correct.

II This is an algebraic rearrangement: $(x^2 - 1)^2 = x^4 - 2x^2 + 1$ is an identity, so this line is correct.

III We have $u^2 < 4$ if and only if $-2 < u < 2$. Taking $u = x^2 - 1$, we see that this line is equivalent to line II, so this step is correct.

IV This looks suspicious, as one of the inequalities has been dropped. The statement $x^2 - 1 < 2$ certainly follows from $-2 < x^2 - 1 < 2$, but the claim is that they follow from each other. Can we deduce $-2 < x^2 - 1 < 2$ from $x^2 - 1 < 2$?

Perhaps surprisingly, it turns out that we can. The function $x^2 - 1$ has a minimum value of -1, so we always have $x^2 - 1 \geq -1$, and therefore $-2 < x^2 - 1$ is true for all x. Hence the statement in line III does follow from $x^2 - 1 < 2$, and this line is correct.

V This is obtained by adding 1 to both sides, which is reversible, so this line is correct.

VI Similarly to our observation in line III, $x^2 < 3$ is true if and only if $-\sqrt{3} < x < \sqrt{3}$, so this line is correct.

Thus the entire argument is correct, and the correct option is A.

11 In this question, k is a positive integer.

Consider the following theorem:

If 2^k+1 is a prime, **then** k is a power of 2. $(*)$

Which of the following statements, taken individually, is/are equivalent to $(*)$?

I **If** k is a power of 2, **then** 2^k+1 is prime.

II 2^k+1 is **not** prime **only if** k is **not** a power of 2.

III A **sufficient** condition for k to be a power of 2 is that 2^k+1 is prime.

	Statement I is equivalent to $(*)$	Statement II is equivalent to $(*)$	Statement III is equivalent to $(*)$
A	Yes	Yes	Yes
B	Yes	Yes	No
C	Yes	No	Yes
D	Yes	No	No
E	No	Yes	Yes
F	No	Yes	No
G	No	No	Yes
H	No	No	No

I This is the converse of $(*)$, so is not equivalent to $(*)$.

II We can rewrite this as an 'if... then' statement, noting that '**A** only if **B**' means the same as 'if **A**, then **B**'. So this becomes '**if** 2^k+1 is **not** prime, **then** k is **not** a power of 2'. This is the contrapositive of '**if** k is a power of 2, **then** 2^k+1 is prime', which is the statement in I, namely the converse of $(*)$. So this statement is equivalent to the converse of $(*)$ but not equivalent to $(*)$ itself. (The contrapositive of the converse, which is the same as the converse of the contrapositive, is called the *inverse* of the original statement.)

III '**A** is a sufficient condition for **B**' is the same as 'if **A**, then **B**', so this statement can be rewritten as '**if** 2^k+1 is prime, **then** k is a power of 2'. (Note the subtlety of language here: the condition **A** is '2^k+1 is prime' and **B** is 'k is a power of 2', but they are written in the order 'a sufficient condition for **B** is **A**'.) This is equivalent to $(*)$ (and is in fact just a different way of saying the same thing).

The correct option is G.

> ***Commentary:*** *It turns out that statement I is false. It was conjectured by Pierre de Fermat in the 17th century, who noted that $2^{2^n}+1$ is prime for $n=1$, 2, 3 and 4. Such primes are called Fermat primes. The smallest counterexample to statement I is $2^{2^5}+1=641\times 6\,700\,417$, which was discovered by Leonhard Euler in the 18th century. It is not known whether $2^{2^n}+1$ is prime for any $n>4$.*

12 In this question, p is a real constant.

The equation $\sin x \cos^2 x = p^2 \sin x$ has n distinct solutions in the range $0 \leq x \leq 2\pi$

Which of the following statements is/are true?

I $n = 3$ is **sufficient** for $p > 1$

II $n = 7$ **only if** $-1 < p < 1$

A none of them

B I only

C II only

D I and II

We can rewrite the two statements in if/then language as

I if $n = 3$, then $p > 1$

II if $n = 7$, then $-1 < p < 1$

We will determine whether these statements are true or false using this if/then formulation.

The equation can be rearranged as $\sin x(\cos^2 x - p^2) = 0$, so this is true if and only if $\sin x = 0$ or $\cos x = \pm p$. The first of these has solutions $x = 0, \pi, 2\pi$, while the solutions to the second depend on the value of p:

- If $p = 0$, then it has solutions $x = \frac{\pi}{2}, \frac{3\pi}{2}$.
- If $-1 < p < 0$ or $0 < p < 1$, then it has four solutions, one with $0 < x < \frac{\pi}{2}$, one with $\frac{\pi}{2} < x < \pi$, one with $\pi < x < \frac{3\pi}{2}$ and one with $\frac{3\pi}{2} < x < 2\pi$; these are all different from the solutions to $\sin x = 0$.
- If $p = \pm 1$, then it has three solutions, $x = 0, \pi, 2\pi$, which are exactly the same as the solutions of $\sin x = 0$.
- If $p < -1$ or $p > 1$, then it has no solutions.

Therefore the original equation has:

- $n = 5$ distinct solutions if (and only if) $p = 0$
- $n = 7$ distinct solutions if (and only if) $-1 < p < 0$ or $0 < p < 1$
- $n = 3$ distinct solutions if (and only if) $p = \pm 1$ or $p < -1$ or $p > 1$

We can now return to the statements I and II in our rewritten form:

I This is false: we could have $p = 1$ or $p \leq -1$

II This is true, even though $p = 0$ is included here: if $-1 < p < 0$ or $0 < p < 1$, then certainly $-1 < p < 1$ is true

The correct option is therefore C, II only.

13 Let x be a real number.

Which **one** of the following statements is a **sufficient** condition for **exactly** three of the other four statements?

A $x \geq 0$

B $x = 1$

C $x = 0$ **or** $x = 1$

D $x \geq 0$ **or** $x \leq 1$

E $x \geq 0$ **and** $x \leq 1$

We work through them sequentially:

A If $x \geq 0$, then B may be false, C may be false, D may be false and E may be false

B If $x = 1$, then A is true, C is true, D is true and E is true

C If $x = 0$ or $x = 1$, then A is true, B may be false, D is true and E is true

D If $x \geq 0$ or $x \leq 1$, then A may be false (for example if $x = -1$), B may be false, C may be false and E may be false

E If $x \geq 0$ and $x \leq 1$, then A is true, B may be false, C may be false and D is true

The correct option is therefore C, which is sufficient for exactly three of the other four statements.

14 Three lines are given by the equations:

$$ax + by + c = 0$$
$$bx + cy + a = 0$$
$$cx + ay + b = 0$$

where a, b and c are non-zero real numbers.

Which one of the following is correct?

A **If** two of the lines are parallel, **then** all three are parallel.

B **If** two of the lines are parallel, **then** the third is perpendicular to the other two.

C **If** two of the lines are parallel, **then** the third is parallel to $y = x$.

D **If** two of the lines are parallel, **then** the third is perpendicular to $y = x$.

E **If** two of the lines are perpendicular, **then** all three meet at a point.

F **If** two of the lines are perpendicular, **then** the third is parallel to $y = x$.

G **If** two of the lines are perpendicular, **then** the third is perpendicular to $y = x$.

A If the first two lines are parallel, then $\frac{a}{b} = \frac{b}{c}$, but this does not imply that $\frac{c}{a}$ is equal to these. For example we could have $a = 1$, $b = 2$, $c = 4$ giving $\frac{a}{b} = \frac{b}{c} = \frac{1}{2}$ but $\frac{c}{a} = 4$.

B The same example shows that the third line need not be perpendicular to the other two. In fact, in this case, since $\frac{a}{c} = \left(-\frac{a}{b}\right)\left(-\frac{b}{c}\right) = \left(-\frac{a}{b}\right)^2 > 0$, the third line will always have a negative gradient (namely $-\frac{c}{a}$), even if the first two also have negative gradients.

C The same example shows that this is not necessarily the case; in fact, as the third line always has a negative gradient, it cannot be parallel to $y = x$.

D Again, the same example shows that the third line might not have gradient -1.

E If the first two lines are perpendicular, we have $\left(-\frac{a}{b}\right)\left(-\frac{b}{c}\right) = \frac{a}{c} = -1$, so $-\frac{c}{a} = 1$. It is not obvious why the lines should meet at a point, so let us return to this after we have looked at the remaining options.

F This is true: the calculation in E shows that the third line has gradient 1.

G This is false: the third line has gradient 1, not -1.

So the correct option is F.

We still need to eliminate option E. Let us consider an example with $a = 1$, $b = 2$; we therefore need $c = -1$ as $-\frac{c}{a} = 1$. Thus the three lines are

$$y = -\frac{1}{2}x - \frac{-1}{2}$$
$$y = -\frac{2}{-1}x - \frac{1}{-1}$$
$$y = -\frac{-1}{1}x - \frac{2}{1}$$

which simplifies to

$$y = -\tfrac{1}{2}x + \tfrac{1}{2}$$
$$y = 2x + 1$$
$$y = x - 2$$

The second and third lines meet at $(-3, -5)$, but the first line passes through $(-3, 2)$, so these three lines do not all meet at a point.

15 The base 10 number 0.03841 has the value

$$0 \times 10^{-1} + 3 \times 10^{-2} + 8 \times 10^{-3} + 4 \times 10^{-4} + 1 \times 10^{-5} = 0.03841$$

Similarly, the base 2 number 0.01101 has the value

$$0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} + 0 \times 2^{-4} + 1 \times 2^{-5} = \frac{13}{32}$$

What is the value of the recurring base 2 number $0.\dot{0}01\dot{1} = 0.001100110011\ldots$?

A $\frac{1}{3}$

B $\frac{1}{5}$

C $\frac{1}{15}$

D $\frac{2}{15}$

E $\frac{4}{15}$

F $\frac{3}{16}$

G $\frac{5}{16}$

H $\frac{6}{31}$

We have, on dropping zero terms and rearranging:

$$\begin{aligned} 0.\dot{0}01\dot{1} &= 0.001100110011\ldots \\ &= 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} + 1 \times 2^{-4} + 0 \times 2^{-5} + 0 \times 2^{-6} + 1 \times 2^{-7} + 1 \times 2^{-8} + \cdots \\ &= 2^{-3} + 2^{-4} + 2^{-7} + 2^{-8} + 2^{-11} + 2^{-12} + \cdots \\ &= (2^{-3} + 2^{-7} + 2^{-11} + \cdots) + (2^{-4} + 2^{-8} + 2^{-12} + \cdots) \end{aligned}$$

(Technical note: we are allowed to rearrange this infinite series as all of the terms are positive.)

We can now calculate the sums of these infinite geometric series to get

$$\begin{aligned} 0.\dot{0}01\dot{1} &= \frac{2^{-3}}{1 - 2^{-4}} + \frac{2^{-4}}{1 - 2^{-4}} \\ &= \frac{2}{16 - 1} + \frac{1}{16 - 1} \\ &= \frac{3}{15} \\ &= \frac{1}{5} \end{aligned}$$

16 A sequence is defined by:

$$u_1 = a$$
$$u_2 = b$$
$$u_{n+2} = u_n + u_{n+1} \quad \text{for } n \geq 1$$

where a and b are positive integers. The highest common factor of a and b is 7.

Which of the following statements **must** be true?

I u_{2023} is a multiple of 7

II If u_1 is not a factor of u_2, then u_1 is not a factor of u_n for any $n > 1$

III The highest common factor of u_1 and u_5 is 7

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

I Since the highest common factor of a and b is 7, they are both multiples of 7. So u_1 and u_2 are multiples of 7, hence $u_3 = u_1 + u_2$ is a multiple of 7. Then $u_4 = u_2 + u_3$ is a multiple of 7, and so on. So every u_n is a multiple of 7 and this statement is true.

II We have $u_3 = u_1 + u_2 = u_1(1 + u_2/u_1)$. As u_1 is not a factor of u_2, the expression in brackets is not an integer, so u_1 is not a factor of u_3.

Next, we have $u_4 = u_2 + u_3 = u_1 + 2u_2 = u_1(1 + 2u_2/u_1)$. It is feasible that u_1 is a factor of u_4: it will be if $2u_2/u_1$ is an integer. We could take $u_1 = 14$ and $u_2 = 7$ to achieve this. (Explicitly, in this case we have $u_3 = 21$, $u_4 = 28$, and 14 is a factor of 28.)

So this statement may be false.

III Following on from our calculations in II, we have $u_5 = u_3 + u_4 = (u_1 + u_2) + (u_1 + 2u_2) = 2u_1 + 3u_2$. The highest common factor of u_1 and u_5 is therefore the same as the highest common factor of u_1 and $3u_2$, and it is possible that this is 21 if u_1 is a multiple of 21 but u_2 is not. Let us check this explicitly: take $u_1 = 21$ and $u_2 = 7$, so the highest common factor of u_1 and u_2 is 7, as required. Then $u_5 = 2 \times 21 + 3 \times 7 = 3 \times 21$, and so the highest common factor of $u_1 = 21$ and $u_5 = 3 \times 21$ is 21. Therefore this statement may be false.

The correct answer is therefore option B, I only.

17 The ceiling of x, written $\lceil x \rceil$, is defined to be the value of x rounded up to the nearest integer.

For example: $\lceil \pi \rceil = 4$, $\lceil 2.1 \rceil = 3$, $\lceil 8 \rceil = 8$

What is the value of the following integral?

$$\int_0^{99} 2^{\lceil x \rceil}\,\mathrm{d}x$$

A 2^{99}

B $2^{99} - 1$

C $2^{99} - 2$

D 2^{100}

E $2^{100} - 1$

F $2^{100} - 2$

The value of $2^{\lceil x \rceil}$ changes at every integer. We have

$$2^{\lceil x \rceil} = \begin{cases} 2^0 & \text{when } x = 0 \\ 2^1 & \text{when } 0 < x \leq 1 \\ 2^2 & \text{when } 1 < x \leq 2 \\ 2^3 & \text{when } 2 < x \leq 3 \\ \dots & \\ 2^{99} & \text{when } 98 < x \leq 99 \end{cases}$$

We therefore break the integral up into unit intervals. The value of $2^{\lceil x \rceil}$ at the endpoints does not influence the value of the integral, so we have

$$\begin{aligned} \int_0^{99} 2^{\lceil x \rceil}\,\mathrm{d}x &= \int_0^1 2^{\lceil x \rceil}\,\mathrm{d}x + \int_1^2 2^{\lceil x \rceil}\,\mathrm{d}x + \int_2^3 2^{\lceil x \rceil}\,\mathrm{d}x + \cdots + \int_{98}^{99} 2^{\lceil x \rceil}\,\mathrm{d}x \\ &= \int_0^1 2^1\,\mathrm{d}x + \int_1^2 2^2\,\mathrm{d}x + \int_2^3 2^3\,\mathrm{d}x + \cdots + \int_{98}^{99} 2^{99}\,\mathrm{d}x \\ &= 2^1 + 2^2 + 2^3 + \cdots + 2^{99} \end{aligned}$$

as the integral of a constant is just the constant times the interval width. We now have a geometric series with first term $a = 2$, common ratio $r = 2$ and $n = 99$ terms, so the sum is

$$\frac{a(r^n - 1)}{r - 1} = \frac{2^1(2^{99} - 1)}{2 - 1} = 2^{100} - 2$$

which is option F.

18 The equation $x^4 + bx^2 + c = 0$ has four distinct real roots **if and only if** which of the following conditions is satisfied?

A $b^2 > 4c$

B $b^2 < 4c$

C $c > 0$ **and** $b > 2\sqrt{c}$

D $c > 0$ **and** $b < -2\sqrt{c}$

E $c < 0$ **and** $b < 0$

F $c < 0$ **and** $b > 0$

Writing $u = x^2$, the equation becomes $u^2 + bu + c = 0$. This has two distinct real roots u_1 and u_2 if and only if $b^2 - 4c > 0$, or $b^2 > 4c$, so this is a necessary condition.

The roots of the original equation are then given by $x = \pm\sqrt{u_1}$ and $x = \pm\sqrt{u_2}$. So for the original equation to have four distinct real roots, we require u_1 and u_2 to be distinct *positive* real numbers. We have

$$u_1, u_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

so the smaller of the two is $\frac{1}{2}\left(-b - \sqrt{b^2 - 4c}\right)$. If $b > 0$, then this will be negative, so we require $b < 0$. Supposing $b < 0$, we also require $-b > \sqrt{b^2 - 4c}$, which becomes on squaring (as both sides are positive) $b^2 > b^2 - 4c$, so $c > 0$.

Putting these all together, the original equation will have four distinct real roots if and only if all of the following hold:

- $b^2 > 4c$
- $b < 0$
- $c > 0$

As $c > 0$, we can deduce that $b^2 > 4c$ if and only if $b > 2\sqrt{c}$ or $b < -2\sqrt{c}$. Since we require $b < 0$, the first is not possible, so the pair of conditions $b < 0$ and $b^2 > 4c$ is equivalent to the single condition $b < -2\sqrt{c}$.

Therefore a necessary and sufficient condition for the original equation to have four distinct real roots is $c > 0$ and $b < -2\sqrt{c}$, which is option D.

19 In this question, $f(x)$ is a non-constant polynomial, and $g(x) = xf'(x)$

$f(x) = 0$ for exactly M real values of x.

$g(x) = 0$ for exactly N real values of x.

Which of the following statements is/are true?

I It is possible that $M < N$

II It is possible that $M = N$

III It is possible that $M > N$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Since $f(x) = 0$ for exactly M real values of x, $f(x)$ has at least $M - 1$ stationary points: there must be at least one stationary point between every adjacent pair of values of x for which $f(x) = 0$. But there may also be more stationary points between these pairs of values or before the first or after the last value.

Now $g(x) = xf'(x)$ is zero whenever $f(x)$ has a stationary point. So there are at least $M - 1$ values of x for which $g(x) = 0$, but there will be more if $f(x)$ has more than $M - 1$ stationary points. In addition, $g(x) = 0$ when $x = 0$. If $x = 0$ is a stationary point of $f(x)$, then $g(x) = 0$ at every stationary point of $f(x)$ and nowhere else. If $x = 0$ is not a stationary point of $f(x)$, then $g(x) = 0$ will be true for one further value of x.

Summarising:

- the number of stationary points of $f(x)$ is at least $M - 1$
- N equals the number of stationary points of $f(x)$ or one more than the number of stationary points of $f(x)$.

Therefore N is at least $M - 1$, so $N \geq M - 1$, which rearranges to $M \leq N + 1$.

Now considering the options:

I $M < N$ is possible if $f(x)$ has extra stationary points between zeros. For example, $f(x) = x^2 + 1$ has $M = 0$ but $N = 1$. As a second example, let $f(x)$ be a cubic with both turning points above the x-axis; this has $M = 1$ and $N = 2$ or $N = 3$.

II $M = N$ is possible if the number of stationary points of $f(x)$ is $M - 1$ and $x = 0$ is not a stationary point of $f(x)$. For example, $f(x) = (x-1)^2 - 1$ has $M = 2$ (at $x = 0$ and $x = 2$) and one stationary point at $x = 1$. So $g(x) = 0$ at $x = 0$ and $x = 1$, giving $N = 2$.

III $M > N$ is possible if the number of stationary points of $f(x)$ is $M - 1$ and $x = 0$ is a stationary point of $f(x)$. For example, $f(x) = x^2 - 1$ has $M = 2$ (at $x = \pm 1$) and one stationary point at $x = 0$. So $g(x) = 0$ at $x = 0$ only, giving $N = 1$.

The correct answer is option H, I, II and III.

20 Let f be a polynomial with real coefficients.

The integral $I_{p,q}$ where $p < q$ is defined by

$$I_{p,q} = \int_p^q (f(x))^2 - (f(|x|))^2 \, dx$$

Which of the following statements must be true?

1 $I_{p,q} = 0$ **only if** $0 < p$

2 $f'(x) < 0$ for all x **only if** $I_{p,q} < 0$ for all $p < q < 0$

3 $I_{p,q} > 0$ **only if** $p < 0$

A none of them

B 1 only

C 2 only

D 3 only

E 1 and 2 only

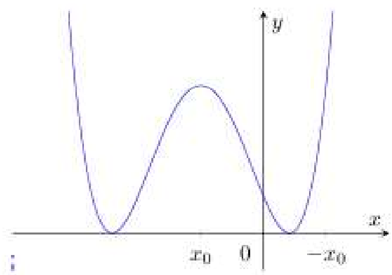
F 1 and 3 only

G 2 and 3 only

H 1, 2 and 3

1 We present a complicated approach first.

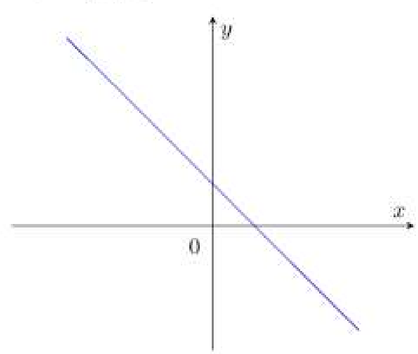
There seems to be no obvious reason why this must be true; one could imagine a polynomial for which the values of the integrand are positive for some negative values of x and negative for others, so the positive parts and negative parts cancel out making the integral zero even though $p < 0$. Here is one example: consider a quartic $f(x)$ that looks roughly like this:



For negative values of x close to $x = 0$, $\left(f(x)\right)^2 > \left(f(|x|)\right)^2$. But once $x < x_0$, we have $\left(f(x)\right)^2 < \left(f(|x|)\right)^2$ as $f(|x|) > f(x)$. We can therefore take $q = 0$ and find some $p < x_0$ for which $I_{p,q} = 0$.

A much simpler approach is that we can take $p = 0$, $q > 0$ and then $I_{p,q} = 0$, so this statement is not true. Were the question to have said $0 \leq p$ instead of $0 < p$, though, we would not have been able to use this simpler approach.

2 We could take $f(x) = 1 - x$, with $f'(x) = -1 < 0$ for all x.



Then $\left(f(x)\right)^2 > \left(f(|x|)\right)^2$ for all $x < 0$, so $I_{p,q} > 0$ for all $p < q < 0$.

3 We observed at the start of this question that the integrand is zero for all $x \geq 0$. So if $p \geq 0$, then $I_{p,q} = \int_p^q 0 \, dx = 0$. The contrapositive of this statement is: if $I_{p,q} \neq 0$, then $p < 0$. The given statement ($I_{p,q} > 0$ only if $p < 0$, which is the same as saying: if $I_{p,q} > 0$, then $p < 0$) is a special case of the more general result and is therefore true.

The correct answer is therefore option D, 3 only.