AMC 10
MOCK TEST 4
Solution Book

Arithmetic and Algebra

ThrivingScholars

1. The five integers 2, 5, 6, 9, 14 are arranged into a different order. In the new arrangement, the sum of the first three integers is equal to the sum of the last three integers.

What is the middle number in the new arrangement?

A 2 B 5 C 6 D 9 E 14

Solution **E**

Let the integers in the new arrangement be in the order p, q, r, s, t. Because the sum of the first three integers is the same as the sum of the last three,

$$p + q + r = r + s + t,$$

and hence

$$p + q = s + t.$$

We therefore see that the pair of integers p, q has the same sum as the pair s, t. Also, the middle number, r, is the one that is not included in either of these pairs.

It is straightforward to check that $2 + 9 = 5 + 6$ and that 2, 9 and 5, 6 are the only two pairs of the given integers with the same sum.

Therefore the middle integer in the new arrangement is 14, as this does not occur in either pair.

For investigation

- In how many different ways may the integers 2, 5, 6, 9, 14 be arranged into a different order so that the sum of the first three integers is equal to the sum of the last three integers?
- Suppose that the integers 3, 7, 8, 10, 12 are arranged into a different order so that the sum of the first three integers is equal to the sum of the last three. What is the middle number in the new arrangement?
- The integers 3, 6, 9, 12, 15 are to be arranged into a different order so that the sum of the first three integers is equal to the sum of the last three. How many different possibilities are there for the middle number in the new arrangement?
- Five different integers are to be arranged in order so that the sum of the first three integers is the same as the sum of the last three. What is the maximum number of possibilities for the middle number in the new arrangement?
- (a) What is the largest number of integers that may be chosen from the set of all positive integers from 1 to 10, inclusive, so that no two pairs of the chosen integers have the same total?

  (b) What is the largest number of integers that may be chosen from the set of all positive integers from 1 to 20, inclusive, so that no two pairs of the chosen integers have the same total?

2. The base of a triangle is increased by 20% and its height is decreased by 15%.

What happens to its area?

A It decreases by 3% B It remains the same C It decreases by 2%
D It increases by 2% E It increases by 5%

Solution **D**

Suppose the original triangle has base b and height h. The area of this triangle is X, where $X = \frac{1}{2}(b \times h)$.

When the base of the triangle is increased by 20%, its base becomes $b' = \frac{6}{5}b$. When its height is decreased by 15%, its height becomes $h' = \frac{17}{20}h$.

Therefore the area of the changed triangle is X', where $X' = \frac{1}{2}(b' \times h') = \frac{1}{2}(\frac{6}{5}b \times \frac{17}{20}h) = (\frac{6}{5} \times \frac{17}{20})(\frac{1}{2}(b \times h)) = \frac{102}{100}X$.

Therefore the effect of the changes is to increase the area of the triangle by 2%.

For investigation

- The base of a triangle is decreased by 15% and its height is increased by 20%. What happens to its area?
- The base of a triangle is increased by 20% and its height is decreased by 20%. What happens to its area?
- The base of a triangle is increased by 20%. By what percentage should its height be decreased to keep the area unchanged?

3. The positive integer k is a solution of the equation $(k \div 12) \div (15 \div k) = 20$.

What is the sum of the digits of k?

A 15 B 12 C 9 D 6 E 3

SOLUTION **D**

We have

$$\begin{aligned}(k \div 12) \div (15 \div k) &= \frac{k}{12} \div \frac{15}{k} \\ &= \frac{k}{12} \times \frac{k}{15} \\ &= \frac{k \times k}{12 \times 15} \\ &= \frac{k^2}{180}.\end{aligned}$$

It follows that

$$\begin{aligned}(k \div 12) \div (15 \div k) = 20 &\Leftrightarrow \frac{k^2}{180} = 20 \\ &\Leftrightarrow k^2 = 3600 \\ &\Leftrightarrow k = 60, \text{ as } k > 0.\end{aligned}$$

The sum of the digits of 60 is $6 + 0 = 6$.

FOR INVESTIGATION

Find the solutions of the following equations.

(a) $(x \div 5) \div (5 \div x) = 4$.

(b) $(x \div 2) \div ((x \div 10) \div (x \div 3)) = 15$.

4. Which of the following is equal to $2017 - \frac{1}{2017}$?

A $\frac{2017^2}{2016}$ B $\frac{2016}{2017}$ C $\frac{2018}{2017}$ D $\frac{4059}{2017}$

E $\frac{2018 \times 2016}{2017}$

SOLUTION **E**

Writing both 2017 and $\frac{1}{2017}$ over a common denominator, we have

$$2017 - \frac{1}{2017} = \frac{2017^2 - 1}{2017}.$$

Now,

$$2017^2 - 1 = 2017^2 - 1^2.$$

Hence, using the standard factorization of the difference of two squares, we have

$$2017 - \frac{1}{2017} = \frac{2017^2 - 1^2}{2017} = \frac{(2017 + 1)(2017 - 1)}{2017} = \frac{2018 \times 2016}{2017}.$$

5. Lucy is asked to choose p, q, r and s to be the numbers 1, 2, 3 and 4, in some order, so as to make the value of $\frac{p}{q} + \frac{r}{s}$ as small as possible.

What is the smallest value Lucy can achieve in this way?

A $\frac{7}{12}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D $\frac{5}{6}$ E $\frac{11}{12}$

Solution **D**

To make the value of $\frac{p}{q} + \frac{r}{s}$ as small as possible, q and s need to be as large as possible, and so have the values 3 and 4.

Therefore the expression with the smallest value that Lucy can achieve is either $\frac{1}{3} + \frac{2}{4}$ or $\frac{2}{3} + \frac{1}{4}$.

Now,

$$\begin{aligned} \frac{1}{3} + \frac{2}{4} &= \frac{1}{3} + \frac{1}{4} + \frac{1}{4} \\ &< \frac{1}{3} + \frac{1}{3} + \frac{1}{4} \\ &= \frac{2}{3} + \frac{1}{4}. \end{aligned}$$

It follows that the smallest value that Lucy can achieve is

$$\frac{1}{3} + \frac{2}{4} = \frac{4+6}{12} = \frac{10}{12} = \frac{5}{6}.$$

For investigation

- Lucy is asked to choose p, q, r, s, t and u to be the numbers 1, 2, 3, 4, 5 and 6, in some order, so as to make the value of $\frac{p}{q} + \frac{r}{s} + \frac{t}{u}$ as small as possible.

  What is the smallest value Lucy can achieve in this way?

- Explain why the statement in the first line of the above solution that

  "to make the value of $\frac{p}{q} + \frac{r}{s}$ as small as possible, q and s need to be as large as possible"

  is correct.

6. There are fewer than 30 students in the A-level mathematics class. One half of them play the piano, one quarter play hockey and one seventh are in the school play.

How many of the students play hockey?

A 3 B 4 C 5 D 6 E 7

SOLUTION **E**

Because one half of the students play the piano, the number of students is a multiple of 2.

Because one quarter of the students play hockey, the number of students is a multiple of 4.

Because one seventh of the students are in the school play, the number of students is a multiple of 7.

Therefore the number of students is a multiple of 2, 4 and 7. Hence the number of students is a multiple of 28.

Because there are fewer than 30 students in the class, it follows that there are 28 students in the class.

Therefore, because one quarter of the 28 students play hockey, the number of students who play hockey is 7.

7. A sequence begins 2023, 2022, 1, ... After the first two terms, each term is the positive difference between the previous two terms.

What is the value of the 25^{th} term?

A 2010 B 2009 C 2008 D 2007 E 2006

SOLUTION **D**

The sequence begins

$$2023,\ 2022,\ 1,\ 2021,\ 2020,\ 1,\ 2019,\ 2018,\ 1,\ \ldots.$$

From this it seems that, in general, for each non-negative integer k, the terms in positions $3k + 1$, $3k + 2$ and $3k + 3$ are $2023 - 2k$, $2023 - 2k - 1$ and 1. [In fact, this holds only provided $2023 - 2k - 1 \geq 0$, that is, only for $k \leq 1011$. See Problems 7.1 and 7.4.]

Now $25 = 3\times8+1$. Therefore, by putting $k = 8$, we deduce that the 25 th term is $2023-2\times8 = 2007$.

FOR INVESTIGATION

(a) What are the 3034 th, 3045 th and 3046 th terms of the sequence of this question?

(b) What is the 5000 th term of the sequence of this question?

A sequence begins

$$2023,\ 2021,\ 2, \ldots$$

After the first two terms, each term is the positive difference between the previous two terms.

What is the 25 th term of the sequence?

A sequence begins 2023, s, $2023 - s$, ... After the first two terms, each term is the positive difference between the previous two terms.

Which is the positive integer s for which the 25 th term of this sequence is 199?

We let u_n be the n th term of the sequence of this question.

If you have met the method of *Proof by Mathematical Induction,* use this method to prove that for each non-negative integer k,

$$u_{3k+1} = \begin{cases} 2023 - 2k, \text{ if } k \leq 1011, \\ 1, \text{ otherwise,} \end{cases}$$

$$u_{3k+2} = \begin{cases} 2023 - 2k - 1, \text{ if } k \leq 1011, \\ 0, \text{ otherwise,} \end{cases}$$

and

$$u_{3k+3} = 1.$$

8. In 2021, a first class postage stamp cost 85p and a second class postage stamp cost 66p. In order to spend an exact number of pounds and to buy at least one of each type, what is the smallest total number of stamps that should be purchased?

A 10 B 8 C 7 D 5 E 2

Solution **C**

The cost of r first class postage stamps at 85p each, and s second class postage stamps at 66p each is $(85r + 66s)$p.

This is an exact number of pounds provided that

$$85r + 66s = 100t \qquad (1)$$

for some positive integer t.

We therefore seek the solution of (1) in which r, s and t are positive integers with $r + s$ as small as possible.

Equation (1) may be rearranged as

$$85r = 100t - 66s. \qquad (2)$$

Since 100 and 66 are both divisible by 2, it follows from (2) that $85r$ is divisible by 2. Therefore r is divisible by 2. Hence $r \geq 2$.

Equation (1) may also be rearranged as

$$66s = 100t - 85r. \qquad (3)$$

Since 100 and 85 are both divisible by 5, it follows from (3) that $66s$ is divisible by 5. Therefore s is divisible by 5. Hence $s \geq 5$.

We now note that when $r = 2$ and $s = 5$, we have

$$85r + 66s = 85 \times 2 + 66 \times 5 = 170 + 330 = 500.$$

Therefore equation (1) has the solution $r = 2$, $s = 5$ and $t = 5$. Because $r = 2$ and $s = 5$ are the least possible values for r and s, it follows that $r + s$ has the least possible value among the solutions of (1) in which r, s and t are positive integers.

Since, in this case, $r + s = 2 + 5 = 7$, we deduce that the least number of stamps that should be purchased is 7.

For investigation

- Today a first class postage stamp costs 95p, and a second class postage stamp costs 68p. In order to spend an exact number of pounds and to buy at least one of each type, what is the smallest total number of stamps that should be purchased?
- Find the solution of the equation

$$45r + 56s = 100t$$

in which r, s and t are positive integers and $r + s$ is as small as possible.

9. The number x is the solution to the equation $3^{(3^x)} = 333$.

Which of the following is true?

A $0 < x < 1$ B $1 < x < 2$ C $2 < x < 3$ D $3 < x < 4$
E $4 < x < 5$

Solution **B**

The first few powers of 3 are

$$3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81,\ 3^5 = 243 \text{ and } 3^6 = 729.$$

Because $3^{(3^x)} = 333$, it follows that

$$3^5 < 3^{(3^x)} < 3^6$$

and hence

$$5 < 3^x < 6.$$

Therefore

$$3^1 < 3^x < 3^2$$

and hence

$$1 < x < 2.$$

For investigation

The number x is the solution to the equation $2^{2^x} = 10^6$. Find the integer n such that $n < x < n + 1$.

10. A light-nanosecond is the distance that a photon can travel at the speed of light in one billionth of a second. The speed of light is $3 \times 10^8\,\mathrm{ms^{-1}}$.

How far is a light-nanosecond?

A 3 cm B 30 cm C 3 m D 30 m E 300 m

SOLUTION **B**

One billionth of a second is $\frac{1}{10^9}$ seconds.

Hence, in one billionth of a second light travels

$$\frac{1}{10^9} \times (3 \times 10^8)\,\mathrm{m} = \frac{3}{10}\,\mathrm{m} = 30\,\mathrm{cm}.$$

Therefore a light-nanosecond is 30 cm.

FOR INVESTIGATION

A light-minute is the distance that a photon travels in one minute at the speed of light. The mean distance of the Earth from the Sun is approximately 150 million kilometres.

How many light-minutes is that?

11. What is the 100th term of the sequence 1, 5, 7, 11, 13, 17, 19, 23, ... whose terms are consecutive odd numbers but with all the multiples of 3 removed?

A 201 B 203 C 247 D 299 E 301

Solution **D**

Method 1

The terms of this sequence occur in pairs where each pair consists of the integers 2 less and 2 more than successive odd multiples of 3.

That is, we can rewrite the terms of this sequence as

$$3-2,\ 3+2,\ \ 9-2,\ 9+2,\ \ 15-2,\ 15+2,\dots,$$

where $3, 9, 15, \dots$ is the sequence of odd multiples of 3.

The 100th term of the sequence therefore is 2 more than the 50th odd multiple of 3.

The 50th odd number is $2 \times 50 - 1 = 99$. Hence the 50th odd multiple of 3 is $3 \times 99 = 297$.

Therefore the 100th term of the sequence is $297 + 2 = 299$.

Method 2

Each positive integer can be expressed as either $6n$, $6n+1$, $6n+2$, $6n+3$, $6n+4$ or $6n+5$, where n is a non-negative integer. Of these only $6n+1$ and $6n+5$ are odd positive integers that are not multiples of 3. Therefore the first 100 numbers in this sequence are the numbers $6n+1$ and $6n+5$ for $0 \leq n \leq 49$. Hence the 100th number in the sequence is $6 \times 49 + 5 = 299$.

For investigation

The sequence

$$1,\ 3,\ 7,\ 9,\ 11,\ 13,\ 17,\ 19\dots$$

consists of the positive odd integers that are not multiples of 5, in numerical order.

What is the 100th term of this sequence?

The sequence

$$1,\ 3,\ 5,\ 9,\ 11,\ 13,\ 15,\ 17\dots$$

consists of the positive odd integers that are not multiples of 7, in numerical order.

What is the 100th term of this sequence?

The sequence

$$7,\ 21,\ 49,\ 63,\ 77,\ 91,\ 119\dots$$

consists of the positive odd integers that are multiples of 7 but not multiples of 5, in numerical order.

What is the 100th term of this sequence?

12. For what value of x is $\sqrt{(\sqrt{(\sqrt{x}+1)}+1)}+1=3$?

A 4096 B 64 C 8 D 3 E 0

Solution **B**

From the given equation, we have $\sqrt{(\sqrt{(\sqrt{x}+1)}+1)}=2$.

Therefore, squaring both sides, $\sqrt{(\sqrt{x}+1)}+1=4$.

Hence $\sqrt{(\sqrt{x}+1)}=3$.

Therefore, squaring again, $\sqrt{x}+1=9$.

Hence $\sqrt{x}=8$.

Finally, by squaring again, we deduce that $x=64$. [In Problem 6.2 you are asked to check that squaring three times has not introduced a spurious solution.]

For investigation

- For what value of x is $\sqrt{\sqrt{(\sqrt{(\sqrt{x}+1)}+1)}+1}+1=3$?
- Check that when $x=64$ the value of $\sqrt{(\sqrt{(\sqrt{x}+1)}+1)}+1$ is 3,

13. A three-piece suit consists of a jacket, a pair of trousers and a waistcoat. Two jackets and three pairs of trousers cost £380. A pair of trousers costs the same as two waistcoats.

What is the cost of a three-piece suit?

A £150 B £190 C £200 D £228 E more information needed

Solution **B**

We let the cost of a jacket, a pair of trousers and a waistcoat be £J, £T and £W, respectively.

From the information given in the question we can deduce that

$$2J + 3T = 380 \qquad (1)$$

and

$$T = 2W \qquad (2)$$

Note: In this problem we have three unknowns J, T and W, but only two equations. We don't have enough information to enable us to find the values of J, T and W. However, we can deduce the value of $J + T + W$ which is what we need to know. We give two methods for doing this.

Method 1

If we subtract equation (2) from equation (1) we obtain

$$2J + 2T = 380 - 2W.$$

Hence

$$2J + 2T + 2W = 380.$$

Hence, dividing both sides of this last equation by 2, we obtain

$$J + T + W = 190.$$

Therefore the cost of a three piece suit is £190.

Method 2

It follows from equation (2) that $W = \frac{1}{2}T$. Hence,

$$\begin{aligned} J + T + W &= J + T + \tfrac{1}{2}T \\ &= J + \tfrac{3}{2}T \\ &= \tfrac{1}{2}(2J + 3T) \\ &= \tfrac{1}{2}(380), \quad \text{by equation (1)}, \\ &= 190. \end{aligned}$$

Therefore the cost of a three-piece suit is £190.

14. Which of these is the mean of the other four?

A $\sqrt{2}$ B $\sqrt{18}$ C $\sqrt{200}$ D $\sqrt{32}$ E $\sqrt{8}$

SOLUTION **D**

We use the fact that one of the options is the mean of the other four options provided that it is the mean of all five of the options. Your are asked to check this fact in Problem 13.2.

The mean of the five numbers given as options is

$$\begin{aligned}\frac{\sqrt{2}+\sqrt{18}+\sqrt{200}+\sqrt{32}+\sqrt{8}}{5} &= \frac{\sqrt{2}+3\sqrt{2}+10\sqrt{2}+4\sqrt{2}+2\sqrt{2}}{5}\\ &= \left(\frac{1+3+10+4+2}{5}\right)\sqrt{2}\\ &= \left(\frac{20}{5}\right)\sqrt{2}\\ &= 4\sqrt{2}\\ &= \sqrt{32}.\end{aligned}$$

Therefore the correct option is D.

FOR INVESTIGATION

(a) Find the mean of the primes 5, 7, 11, 13 and 19.

(b) Hence show that this mean is one of these primes.

(c) Check that this mean is also the mean of the other four primes.

(a) Show that if the number p is the mean of the five numbers p, q, r, s and t, then p is also the mean of the four numbers q, r, s and t.

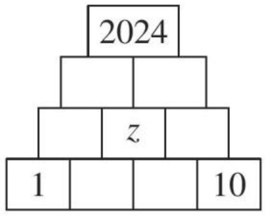
(b) Show that if the number p is the mean of the four numbers q, r, s and t, then p is also the mean of the five numbers p, q, r, s and t.

Show that the result of Problem 13.2 generalizes to the case of a set of n numbers, for each integer n with $n \geq 2$. That is, show that given a set of n numbers, a number p in the set is the mean of the other $n - 1$ numbers in the set if and only if it is the mean of all the n numbers in the set.

Which of the seven primes consecutive 7, 11, 13, 17, 19, 23 and 27 is the mean of the other six primes in the list?

Which of the seven consecutive primes 101, 103, 107, 109, 113, 127 and 131 is the mean of the other six primes in the list?

15. The diagram shows a partially completed number pyramid. When correctly completed, the number on any brick above the bottom row should be the sum of the two numbers on the two bricks on which it rests.



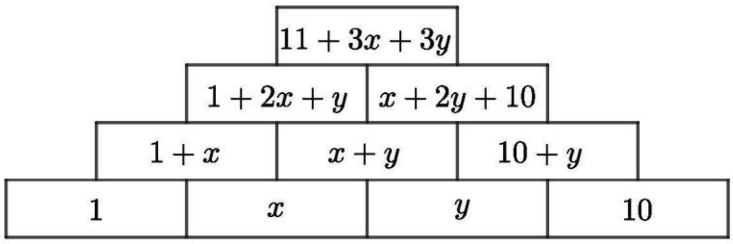
What number should appear on the brick marked 'z'?

A 176 B 617 C 671 D 716 E 761

Solution **C**

We let the numbers that are in the second and third boxes in the bottom row be x and y.

Then the remaining numbers are as shown in the diagram on the right.



Then $z = x + y$ and $11 + 3x + 3y = 2024$.

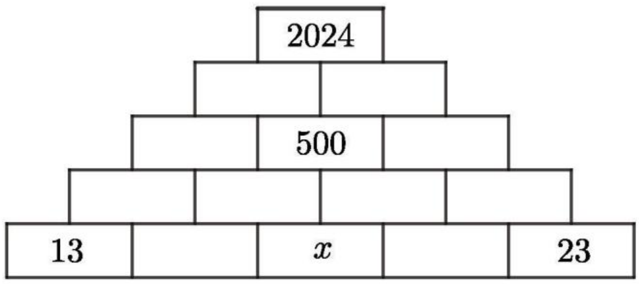
It follows from the second of these equations that $3x + 3y = 2013$. Hence $x + y = 2013 \div 3 = 671$. Hence $z = 671$.

For investigation

Suppose that the numbers in the bottom row of the number pyramid are w, x, y and z, from left to right. What number is in top box? What do you notice?

The diagram on the right shows a partially completed number pyramid.

When correctly completed, the number on any brick above the bottom row should be the sum of the two numbers on the two bricks on which it rests.



Which number should appear on the bricked marked with an x ?

Suppose you are given the partially completed number pyramid as shown in the diagram of Problem 13.2. In this diagram you are given the numbers on four of the bricks.

What is the smallest number of additional bricks on which you would need to know the number in order to work out which number should be on each of the bricks?

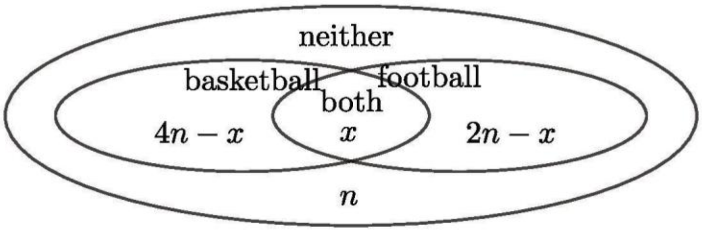
16. In Bethany's class of 30 students, twice as many people played basketball as played football. Twice as many played football as played neither.

Which of the following options could have been the number of people who played both?

A 19 B 14 C 9 D 5 E 0

SOLUTION **D**

Let x be the number of students who played both basketball and football and let n be the number of students who played neither.



Then $2n$ students played football and hence $2 \times 2n = 4n$ students played basketball.

Hence there are $4n - x$ students who played basketball, but not football, and $2n - x$ students who played football but not basketball.

We can deduce from this that the number of students who played basketball or football or both was $(4n - x) + x + (2n - x) = 6n - x$.

Because there are 30 students in the class, the sum of the number of students who played basketball or football or both, and the number who played neither is 30. That is, $(6n - x) + n = 30$. Therefore

$$x = 7n - 30. \qquad (1)$$

The number x cannot be negative. It follows, by (1), that $4 < n$.

The number of students who played football but not basketball is $2n - x$. This number cannot be negative. Hence $x \le 2n$. Therefore, by (1), $7n - 30 \le 2n$. Hence $5n \le 30$ and so $n \le 6$.

Therefore, the only possible values of n are 5 and 6. It follows, by (1), that the only possible values of x are 5 and 12. Hence, only option D gives a possible number of students who played both.

FOR INVESTIGATION

In Claire's class of 30 students, twice as many play neither cricket nor tennis, as play both.

The ratio of those playing cricket to those playing tennis is 7 : 5.

How many in Claire's class play cricket?

17. The hare and the tortoise had a race over 100 m, in which both maintained constant speeds. When the hare reached the finish line, it was 75 m in front of the tortoise. The hare immediately turned around and ran back towards the start line.

How far from the finish line did the hare and the tortoise meet?

A 54 B 60 C 64 D $66\frac{2}{3}$ E 72

Solution **B**

The tortoise is 75 m behind the hare when the hare has run the full 100 m. Therefore the tortoise runs 25 m in the same time as the hare runs 100 m. Hence the hare runs at four times the speed of the tortoise.

When the hare turns round, the hare and the tortoise are 75 m apart and running towards each other.

Since the hare is running at four times the speed of the tortoise, when they meet the hare has run $\frac{4}{5}$ths of the 75 m they were apart when the hare turned around. The tortoise has run $\frac{1}{5}$th of this distance.

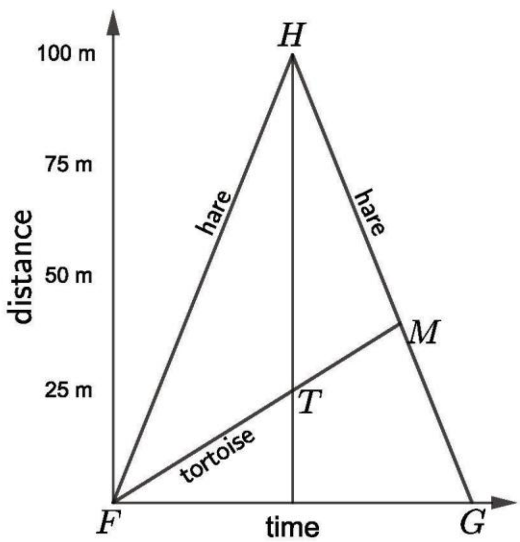
Therefore they meet at a distance $\frac{4}{5} \times 75$ m, that is, 60 m, from the finish line.

For investigation

The distance-time graph below illustrates the paths of the hare and the tortoise.

The point H corresponds to the position of the hare when the hare reaches the finish line and turns around. The point T corresponds to the position of the tortoise at this time. The point M corresponds to their common position when they meet. The points F and G are as shown.

Use the geometry of the diagram to work out how far from the finish line are the hare and tortoise when they meet.



[*Hint*: Extend the line FM to meet the line through G which is perpendicular to FG.]

18. The numbers p, q, r and s satisfy the following equations:

$$p + 2q + 3r + 4s = k \qquad 4p = 3q = 2r = s.$$

What is the smallest value of k for which p, q, r and s are all positive integers?

A 20 B 24 C 25 D 77 E 154

SOLUTION **D**

From the set of equations $4p = 3q = 2r = s$, we have $q = \frac{4}{3}p, r = 2p$ and $s = 4p$.

We require p and q to be positive integers. Therefore, from the equation $q = \frac{4}{3}p$ we deduce that p has to be an integer which is a multiple of 3. We also note that if p is a positive integer, then so also will be r and s.

It also follows that

$$\begin{aligned} k &= p + 2q + 3r + 4s = p + 2(\tfrac{4}{3}p) + 3(2p) + 4(4p) \\ &= p + \tfrac{8}{3}p + 6p + 16p = \tfrac{77}{3}p. \end{aligned}$$

Because p is a positive integer which is a multiple of 3, its smallest value is 3. Therefore the smallest value of k is $\frac{77}{3} \times 3$, which equals 77.

19. The simultaneous equations $x + \frac{1}{y} = 2$ and $y + \frac{1}{x} = \frac{9}{4}$ have two pairs of real solutions.

What is the difference between the possible values of x?

A $\frac{1}{3}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D 1 E $\frac{3}{2}$

Solution **B**

From the equation $x + \frac{1}{y} = 2$, we have $\frac{1}{y} = 2 - x$ and therefore $y = \frac{1}{2-x}$.

By substituting $\frac{1}{2-x}$ for y in the equation $y + \frac{1}{x} = \frac{9}{4}$, we deduce that

$$\frac{1}{2-x} + \frac{1}{x} = \frac{9}{4}.$$

This last equation may be rearranged to give

$$9x^2 - 18x + 8 = 0. \qquad (1)$$

[You are asked to check this in Problem 14.1.]

The left hand side of this last equation factorizes to give

$$(3x - 4)(3x - 2) = 0.$$

Therefore either $3x - 4 = 0$ or $3x - 2 = 0$. Hence the solutions of (1) are $x = \frac{4}{3}$ and $x = \frac{2}{3}$.

Therefore the difference between the two possible values of x is $\frac{4}{3} - \frac{2}{3} = \frac{2}{3}$.

For investigation

Check that the equation $\frac{1}{2-x} + \frac{1}{x} = \frac{9}{4}$ may be rearranged to give $9x^2 - 18x + 8 = 0$.

Let x and y be as in this question.

Put $z = xy$.

Find a quadratic equation satisfied by z. Use this equation to find the values of z and hence the values of x and y.

Find the solution of the following system of simultaneous equations.

$$x + \frac{1}{y} = 2$$
$$y + \frac{1}{z} = 1$$
$$z + \frac{1}{x} = 5.$$

20. Let x and y be positive integers such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$. What is the maximum possible value of y?

A 40 B 60 C 240 D 420 E 480

Solution **D**

The maximum possible value of y corresponds to the minimum possible value of $\frac{1}{y}$.

Since $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$, the minimum possible value of $\frac{1}{y}$ corresponds to the maximum possible value of $\frac{1}{x}$, and hence to the minimum possible value of x.

Now $\frac{1}{x} = \frac{1}{20} - \frac{1}{y} < \frac{1}{20}$ because $y > 0$. Therefore $x > 20$.

Since x is a positive integer, it follows that the minimum possible value of x is 21.

When $x = 21$, we have $\frac{1}{y} = \frac{1}{20} - \frac{1}{21} = \frac{21 - 20}{20 \times 21} = \frac{1}{420}$ and hence $y = 420$.

Therefore the maximum possible value of y is 420.

For investigation

20.1 (a) Rearrange the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$ to express y in terms of x.

(b) Use this rearranged equation to show that the maximum possible value of y is 420.

20.2 Let x and y be positive integers such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{25}$.

What is the maximum possible value of y?

20.3 Let x, y and z be positive integers such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$.

In terms of z, what is the maximum possible value of y?

20.4 Find all the the solutions of the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$, where x and y are positive integers with $x < y$.

20.5 Find all the solutions of the equation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, where x, y and z are positive integers with $x < y < z$.

20.6 Find all the solutions of the equation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}$, where x, y and z are positive integers with $x < y < z$.

21. The letters p, q, r, s and t represent different positive single-digit numbers such that $p - q = r$ and $r - s = t$.

How many different values could t have?

A 6 B 5 C 4 D 3 E 2

Solution **A**

Because $r = p - q$, we have $t = r - s = (p - q) - s = p - (q + s)$.

It follows that the maximum value of t occurs when p takes its maximum value, and $q + s$ takes its minimum value.

The maximum value of p is 9.

Because q and s have different values, the minimum value of $q + s$ is $1 + 2 = 3$.

Therefore the maximum possible value of t is $9 - 3 = 6$.

The following table shows that t can take all the positive integer values from 1 to 6.

p	q	r	s	t
9	1	8	2	6
9	1	8	3	5
9	2	7	3	4
9	2	7	4	3
9	3	6	4	2
9	3	6	5	1

Therefore the number of values that t could take is 6.

For investigation

- In how many different ways is it possible to choose different single-digit values for p, q, r and s so that $t = 1$?
- The letters p, q, r and s represent different positive single-digit numbers such that $p - q = r$ and $q - r = s$. How many different values could s have?
- The letters p, q, r and s represent different positive single-digit numbers such that $p - q = r$ and $p - r = s$. How many different values could s have?

22. The expression $\dfrac{7n+12}{2n+3}$ takes integer values for certain integer values of n.

What is the sum of all such integer values of the expression?

A 4 B 8 C 10 D 12 E 14

Solution **E**

We have

$$\frac{7n+12}{2n+3} = \frac{7}{2} + \frac{3}{2(2n+3)} = \frac{1}{2}\left(7 + \frac{3}{2n+3}\right).$$

It follows that $\dfrac{7n+12}{2n+3}$ is an integer provided that $7 + \dfrac{3}{2n+3}$ is an even integer and hence provided that $\dfrac{3}{2n+3}$ is an odd integer.

Now $\dfrac{3}{2n+3}$ is an integer provided that $2n+3$ is a factor of 3.

Therefore the only possible values of $2n+3$ are -3, -1, 1 and 3. In all these cases $\dfrac{3}{2n+3}$ is a factor of 3 and hence is odd.

When $2n+3=-3$, we have $n=-3$ and $\dfrac{7n+12}{2n+3} = \dfrac{1}{2}\left(7+\dfrac{3}{2n+3}\right) = \dfrac{1}{2}\left(7+\dfrac{3}{-3}\right) = \dfrac{1}{2}\left(7-1\right) = 3.$

When $2n+3=-1$, we have $n=-2$ and $\dfrac{7n+12}{2n+3} = \dfrac{1}{2}\left(7+\dfrac{3}{2n+3}\right) = \dfrac{1}{2}\left(7+\dfrac{3}{-1}\right) = \dfrac{1}{2}\left(7-3\right) = 2.$

When $2n+3=1$, we have $n=-1$ and $\dfrac{7n+12}{2n+3} = \dfrac{1}{2}\left(7+\dfrac{3}{2n+3}\right) = \dfrac{1}{2}\left(7+\dfrac{3}{1}\right) = \dfrac{1}{2}\left(7+3\right) = 5.$

When $2n+3=3$, we have $n=0$ and $\dfrac{7n+12}{2n+3} = \dfrac{1}{2}\left(7+\dfrac{3}{2n+3}\right) = \dfrac{1}{2}\left(7+\dfrac{3}{3}\right) = \dfrac{1}{2}\left(7+1\right) = 4.$

We see that n is an integer whenever $2n+3$ is equal to either -3, -1, 1 or 3. Therefore the sum of the integer values of $\dfrac{7n+12}{2n+3}$ that correspond to integer values of n is $3+2+5+4=14$.

For investigation

- Check that $\dfrac{7n+12}{2n+3} = \dfrac{7}{2} + \dfrac{3}{2(2n+3)}$.
- What is the sum of all the integer values taken by the expression

$$\frac{6n+5}{2n+9}$$

when n is an integer?

23. How many pairs of integers (x, y) satisfy the equation $\sqrt{x - \sqrt{x + 23}} = 2\sqrt{2} - y$?				
A 0	B 1	C 4	D 8	E infinitely many

SOLUTION **B**

Suppose that (x, y) is a pair of integers that satisfies the equation given in the question. That is, suppose that x and y are integers such that

$$\sqrt{x - \sqrt{x + 23}} = 2\sqrt{2} - y. \quad (1)$$

Squaring both sides of equation (1), we obtain

$$x - \sqrt{x + 23} = 8 - 4\sqrt{2}y + y^2. \quad (2)$$

We can rewrite equation (2) as

$$\sqrt{x + 23} = 4\sqrt{2}y - (8 + y^2 - x). \quad (3)$$

Since x and y are integers, $8 + y^2 - x$ is an integer. For convenience we put $z = 8 + y^2 - x$. Then equation (3) becomes

$$\sqrt{x + 23} = 4\sqrt{2}y - z. \quad (4)$$

By squaring both sides of equation (4), we obtain

$$x + 23 = 32y^2 - 8\sqrt{2}yz + z^2. \quad (5)$$

Equation (5) may be rearranged to give

$$8\sqrt{2}yz = 32y^2 + z^2 - x - 23. \quad (6)$$

Now, if $yz \neq 0$, equation (6) implies that

$$\sqrt{2} = \frac{32y^2 + z^2 - x - 23}{8yz}. \quad (7)$$

Because x, y and z are integers, equation (7) implies that $\sqrt{2}$ is rational. Since $\sqrt{2}$ is irrational (see Problem 22.1), we deduce that $yz = 0$. It follows that $y = 0$ or $z = 0$.

Suppose first that $y = 0$. It then follows from equation (2) that

$$x - \sqrt{x + 23} = 8. \quad (8)$$

We leave it as an exercise for the reader to show that equation (8) does not have an integer solution (see Problem 22.2).

Now suppose that $z = 0$. Then

$$8 + y^2 - x = 0. \quad (9)$$

It now follows from equations (3) and (9) that

$$\sqrt{x + 23} = 4\sqrt{2}y. \quad (10)$$

24. How many pairs (x, y) of positive integers satisfy the equation $4^x = y^2 + 15$?

A 0 B 1 C 2 D 4
E an infinite number

SOLUTION **C**

The equation $4^x = y^2 + 15$ may be rearranged as $4^x - y^2 = 15$. Now $4^x = (2^2)^x = (2^x)^2$. Hence $4^x - y^2$ may be factorized, using the standard factorization of the difference of two squares. This enables us to rewrite the equation as

$$(2^x - y)(2^x + y) = 15.$$

It follows that, for (x, y) to be a pair of positive integers that are solutions of the original equation, $2^x - y$ and $2^x + y$ must be positive integers whose product is 15, and with $2^x - y < 2^x + y$.

The only possibilities are therefore that either

$$2^x - y = 1 \quad \text{and} \quad 2^x + y = 15,$$

or

$$2^x - y = 3 \quad \text{and} \quad 2^x + y = 5.$$

In the first case $2^x = 8$ and $y = 7$, giving $x = 3$ and $y = 7$.

In the second case $2^x = 4$ and $y = 1$, giving $x = 2$ and $y = 1$.

Therefore there are just two pairs of positive integers that satisfy the equation $4^x = y^2 + 15$, namely, $(3, 7)$ and $(2, 1)$.

FOR INVESTIGATION

(a) Check that if $2^x - y = 1$ and $2^x + y = 15$, then $2^x = 8$ and $y = 7$.

(b) Check that if $2^x - y = 3$ and $2^x + y = 5$, then $2^x = 4$ and $y = 1$.

How many pairs of positive integers (x, y) are there which satisfy the equation $4^x = y^2 + 31$?

How many pairs of positive integers (x, y) are there which satisfy the equation $4^x = y^2 + 55$?

How many pairs of positive integers (x, y) are there which satisfy the equation $4^x = y^2 + 35$?

What can you say in general about those integers k for which there is at least one pair of positive integers (x, y) which satisfy the equation $4^x = y^2 + k$?

25. A function f satisfies $y^3 f(x) = x^3 f(y)$ and $f(3) \neq 0$. What is the value of $\dfrac{f(20) - f(2)}{f(3)}$?

A 6 B 20 C 216 D 296 E 2023

Solution **D**

By putting $x = 20$ and $y = 3$ in the equation $y^3 f(x) = x^3 f(y)$, we have $27f(20) = 8000f(3)$. Hence

$$f(20) = \frac{8000}{27} f(3).$$

By putting $x = 2$ and $y = 3$ in the same equation, we have $27f(2) = 8f(3)$. Hence

$$f(2) = \frac{8}{27} f(3).$$

Therefore,

$$\begin{aligned} f(20) - f(2) &= \frac{8000}{27} f(3) - \frac{8}{27} f(3) \\ &= \left(\frac{8000}{27} - \frac{8}{27}\right) f(3) \\ &= \frac{7992}{27} f(3) \\ &= 296 f(3). \end{aligned}$$

Therefore, because $f(3) \neq 0$, it follows that

$$\frac{f(20) - f(2)}{f(3)} = 296.$$

For investigation

- What is the value of

$$\frac{f(46) - f(23)}{f(23)}?$$

- Show that there are infinitely many positive integers x, y and z which satisfy the equation

$$\frac{f(x) - f(y)}{f(z)} = 7.$$

- Show that for each positive integer n the equation

$$\frac{f(x) - f(y)}{f(z)} = n$$

has either infinitely many solutions in which x, y and z are positive integers, or no such solutions.

- Show that a function g satisfies the equation $y^3 g(x) = x^3 g(y)$ for all real numbers x and y if, and only if, there is a constant k such that $g(x) = kx^3$, for all real numbers x.