TMUA MOCK TEST 4
Solution Book

Paper 2 Styled

- If, Only If, If and Only If
- Necessary and Sufficient
- Identifying errors in Proofs

ThrivingScholars

1. The polynomial $x^3 + (a-3)x^2 + (b-3a)x - 3b$ has exactly two roots. Which of the following is true?

A. $a^2 - 4b = 0$ is necessary
B. $a^2 - 4b = 0$ is sufficient
C. $a^2 - 4b = 0$ is necessary and sufficient
D. $b^2 - 4a = 0$ is necessary
E. None of the above

A

Note that 3 is an obvious root: $(x-3)(x^2+ax+b) = x^3 + (a-3)x^2 + (b-3a)x - 3b$. So now we require $x^2 + ax + b = 0$ to have one solution, but note it **can't** be 3. The discriminant $a^2 - 4b = 0$ is this a necessary condition but not sufficient. We would require additionally the condition $9 + 3a + b \neq 0$.

2. Which of the following statements are true?

I being a square is a sufficient condition for being a rectangle.
II being less than 12 is a sufficient condition for being less than 20.
III being a rectangle is a necessary condition for being a square.
IV having 4 equal length sides is necessary and sufficient for being a square.
V an integer being less than 19.5 is sufficient, but not necessary for being less than 20.

A. II and IV
B. I, II and V
C. I, II and III
D. I only
E. II only

C

I is true, as being a square implies being a rectangle. II is true, as being less than 12 implies being less than 20. III is true, all squares are rectangles. IV is false, a rhombus is a counterexample to sufficiency. V is false, the set of all integers less than 19.5 is the same as the set of all integers less than 20, so the condition is both necessary and sufficient.

3. Let I, II, III, IV be some statements. Suppose that $I \rightarrow II \rightarrow III$ and $IV \rightarrow Not\ III$ and $Not\ I \rightarrow II$, where $a \rightarrow b$ means if a is true, then b is true. $not\ a$ is just the opposite to a, so if a is true, $not\ a$ is false and vice versa.

Suppose II is a true statement. What can we say about the rest of the statements?

	I	III	IV
A	true	true	true
B	could be either	true	could be either
C	could be either	true	false
D	false	true	false
E	true	false	false

C

As $II \rightarrow III$, III is true. $as\ IV \rightarrow not\ III$ the contrapositive is $III \rightarrow not\ IV$ so IV is false. The contrapositive of the implication $Not\ I \rightarrow II$ is $Not\ II \rightarrow I$. But $Not\ II$ is false, so we can't say anything about I.

4. Let I and II be two statements. You are asked to show that I if and only if II. Which of the following does not prove the statement?

A. II if I, and I if II
B. $not\ I$ if II, and $not\ II$ if I
C. $not\ II$ if $not\ I$, and $not\ I$ if $not\ II$
D. $not\ I$ if $not\ II$, and I if II

B

We have the two conditions $I \rightarrow II$, ie $II\ if\ I$, and $II \rightarrow I$, ie $I\ if\ II$. The contrapositive is equivalent to each, ie the first is equivalent to $not\ II \rightarrow not\ I$, and the second to $not\ I \rightarrow not\ II$. So B is not an equivalent formulation.

5. Which of the following are necessary and sufficient conditions for equations $y = x - 4$ and $x^2 - 2y^2 = a$ to have solutions?

A. $a < 32$
B. $a \leq 32$
C. $a > 32$
D. $a \geq 32$
E. $a < 16$
F. $a \leq 16$
G. $a > 16$
H. $a \geq 16$

B

Substitute the linear equation into the quadratic, to get an equation in x only: $x^2 - 16x + (32 + a) = 0$. Use the positivity of the discriminant to get the condition $32 \geq a$.

6. Consider the statement: "If n is an integer and n^2 is divisible by 4, then n is divisible by 4"

How many counterexamples are there to this in the range $50 \leq n \leq 100$.

A. 24
B. 25
C. 11
D. 12
E. 13

E

Clearly we need to count the number of multiples of two, but not of four, in the range. So 50,54,58,…,98. There are 13.

7. Find a necessary and sufficient condition on a, such that

$$\sqrt{a-\sqrt{a-\sqrt{a-\cdots}}}=\frac{1}{a-\frac{1}{a-\frac{1}{a-\cdots}}}$$

A. True for all real a

B. $a=0$

C. $a=1$

D. $a=2$

E. $a=\sqrt{2}$

D

Let's call $x=\sqrt{a-\sqrt{a-\sqrt{a-\cdots}}}=\frac{1}{a-\frac{1}{a-\frac{1}{a-\cdots}}}$, so that $x=\sqrt{a-x}=\frac{1}{a-x}$.

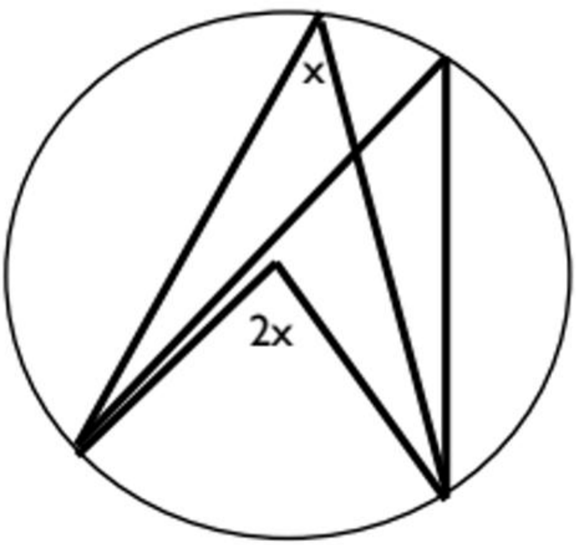
Now $x=\sqrt{a-x}\rightarrow x^2=a-x$, and $x=\frac{1}{a-x}\rightarrow x^2=ax-1$. Combining these, we see that $ax-1=a-x\rightarrow x(a+1)=(a+1)$. a clearly isn't -1, so we can divide through and see $x=1$. Substituting back into a previous equation we see that $a=2$ is necessary. To establish sufficiency, we need to check they are indeed equal when a=2, but this is fairly obvious.

8. Cisco is trying to prove the circle theorem that angles from the same arc in the same segment are equal.

Which of the following circle theorem rules should he use?

A. "The angle at the centre of a circle is twice the angle at the circumference of the circle from the same arc"
B. "the angle formed in a semicircle is always a right angle"
C. "opposite angles of a cyclic quadrilateral add up to 180 degrees"
D. "two tangents drawn from a point to a circle are equal"
E. "The angle between a tangent and a chord is equal to the angle at the circumference in the alternate segment"
F. "the perpendicular line from the centre of a circle to a chord bisects the chord"

A



9. A maths teacher suggests that the following proof contains 3 errors, on which lines are the errors?

Line 1: $2\log 5x = 3$
Line 2: $\leftrightarrow \log 5(x^2) = 3$
Line 3: $\rightarrow x\text{^}2 = 53 = 125$
Line 4: $\leftrightarrow x = \sqrt{125} = 5\sqrt{5}$

A. lines 1, 2, 3
B. lines 1, 2, 4
C. lines 2, 3, 4
D. lines 1, 3, 4
E. lines 2, 4
F. There are no errors
G. There is only one error on line 4
H. lines 1, 2

C

- line 2, the symbol should be $\rightarrow$ not $\leftrightarrow$, because if x is negative, the first line is not possible.
- line 3 the symbol should be $\leftrightarrow$ not $\rightarrow$
- line 4, the solution should be $\pm\sqrt{125}$, or the arrow should just be $\rightarrow$ not $\leftrightarrow$.

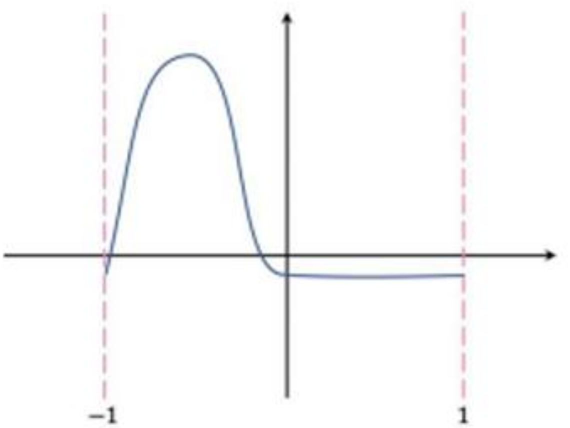
10. Let f be a function such that $f(x) \le 0$ for all $x \ge 0$. Which one of the following is necessary for $\int_{-1}^{1} f(x)dx > 0$?

A. $f(x) = -f(-x)$ for all x
B. $f(0) = 0$
C. $f(x) \ge 0$ for all $x < 0$
D. $\int_{-a}^{0} f(x)dx > 0$ for some a
E. $f(x) = f(-x)$ for all x
F. $f(-1) > 0$

D

Usually, the easiest way to show a condition is "not necessary" is to find, or convince yourself you could find, a function which satisfies the condition that does not satisfy the alleged "necessary condition". However here, we know that $\int_{0}^{1} f dx \le 0$ because $x \le 0$ for all the values of x we are considering. As $\int_{-1}^{1} f dx = \int_{0}^{1} f dx + \int_{-1}^{0} f dx$, we also know that $\int_{-1}^{0} f dx > 0$. This automatically satisfies D, for $a = -1$, so D is necessary, and we are only seeking one answer.

A is very nearly true, but any function which satisfies A actually has an integral of 0 from -1 to 1 (or from $-a$ to a for any a for that matter) because everything to the right of $x = 0$ perfectly cancels with that to the left.



For B, C and F consider the graph below. It has all the features of B,C and F, yet clearly the total area below the line is positive between -I and I.

For E, this in fact can never be true, as it forces all x values everywhere to be ≤ 0.

The only condition left is D.

11. Consider the statement:

A number p is prime only if it can be expressed as 3 more than a multiple of 4.
i.e. $p = 4k + 3$ for some k.

Which of the following are counterexamples to this?

I. $p = 19$
II. $p = 27$
III. $p = 11$

A. None
B. I only
C. II only
D. III only
E. I & II only
F. I & III only
G. II & III only
H. All three

A

"**A only if B**" is equivalent to the statement "**A implies B**" or "**if A then B**" for properties **A** and **B**. So, we are looking for a p which satisfies **A** and **not B**. In this case, that is a prime number which is **not** 3 more than a multiple of 4. But **none** of the given numbers satisfy that: 19 and 11 are both primes congruent to 3 mod 4, and 27 is not prime. Therefore, there are **no counterexamples** to the statement.

12. Consider the statement:

For all prime numbers p, if $p = 4k + 1$ for some positive integer k, it is true that there are positive integers a & b such that $p = a^2 + b^2$.

Which of the following, if they were true, would prove this statement false?

A. For any p prime, if $p = a^2 + b^2$ for some a, b, then $p = 4k + 1$.
B. For any p prime, if $p \neq a^2 + b^2$ for all a, b, then $p = 4k + 1$.
C. For any p prime, $p = 4k + 1$ if only if $p = a^2 + b^2$ for some a, b.
D. There exists a p prime such that $p = 4k + 1$ and $p \neq a^2 + b^2$ for all a, b.
E. There exists a p prime such that $p = a^2 + b^2$ for some a, b or $p = 4k + 1$.

D

For any "if A then B" statement, what makes it false is if there is an entity which is both A and not B. In this case, that is a number p, which is prime, that is both expressible as $p = 4k + 1$ and not expressible as $p = a^2 + b^2$. This last part is the same as saying $p \neq a^2 + b^2$ for all a, b. i.e. The answer is D.

13. Consider the following proof of the alternate segment theorem:

(I) Let a tangent touch a circle at A and let B, C be two points on the circumference of the circle. Let the angle between the line CA and the tangent be $\alpha°$.

(II) By drawing radii from C & A to the centre of the circle, which we will call O, we know that the angle between OA and the tangent is $90°$.

(III) This means the angle BAO is $(90 - \alpha)°$.

(IV) As the triangle OBA is isosceles, the angle OBA is also $(90 - \alpha)°$.

(V) Then the angle AOB is $180° - (180 - 2\alpha)° = 2\alpha°$.

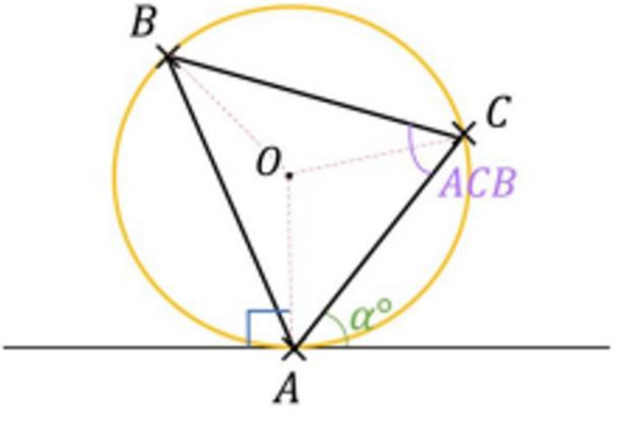
(VI) This means the angle ACB is $\left(\frac{2\alpha}{2}\right)° = \alpha°$.

Where is the first error?

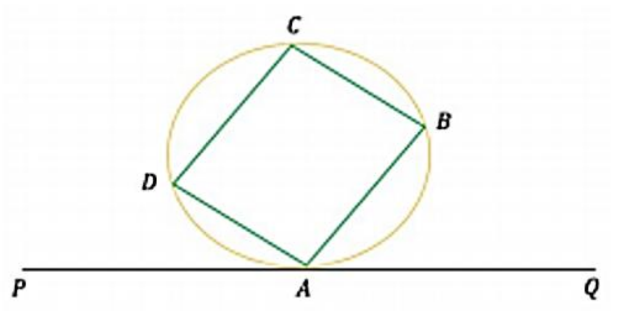
A. The proof is correct.
B. Line (II)
C. Line (III)
D. Line (IV)
E. Line (V)
F. Line (VI)

C

By drawing the circle and tangent, the error becomes relatively clear. The angle that would be equal to α using the alternate segment theorem would be ABC, not ACB as is calculated in the question, so the result is clearly false. To find the error, notice that angle BAO has nothing to do with α, and it is in fact angle CAO that is $(90 - \alpha)°$. So, the error is on line III.



14. Consider the following diagram:



Which of the following are necessary for $ABCD$ to be a rhombus?

I. Angle $PAD = 45°$
II. Length $CA = 2r$ (r is the radius of the circle)
III. Lengths $CD = CB$

A. None
B. I only
C. II only
D. III only
E. I & II only
F. I & III only
G. II & III only
H. All three

H

III must be true just because a rhombus has 4 equal side lengths.

Because of the fact that opposite angles in a cyclic quadrilateral add to $180°$, and the fact that opposite angles in a rhombus are equal, all the angles in this rhombus must be $90°$, which actually makes it a square. Then, because angle ABC is $90°$, the line CA must be a diameter. This shows that $CA = 2r$. And finally, the triangle ADC is therefore also isosceles, and angle ADC being $90°$ then implies that angle $DAC = 45°$. Because tangents and radii are perpendicular to each other, angle PAC is $90°$. Thus, angle $PAD = 45°$. So, all of them are in fact necessary.

15. A student tried to solve the following problem:

$$\frac{2^{18x}}{2^{3x^2}4^6} > 1$$

Here is his solution:

(I) $\frac{2^{18x}}{2^{3x^2}4^6} = 2^{18x-3x^2}4^{-6} > 1$

(II) $8^{6x-3x^2}4^{-6} > 1$

(III) $8^{6x-x^2-3} > 1$

(IV) $6x - x^2 - 3 > 0$

(V) Critical values are $x = \frac{6 \pm \sqrt{36-12}}{2} = 3 \pm \sqrt{6}$

(VI) So $3 - \sqrt{6} < x < 3 + \sqrt{6}$

Where is the first error?

A. The proof is correct

B. Line (I)

C. Line (II)

D. Line (III)

E. Line (IV)

F. Line (V)

G. Line (VI)

C

The first error occurs in Line (II). The student incorrectly rewrites the expression 2^{18x-3x^2-12} as $8^{6x-3x^2} \cdot 4^{-6}$, but this results in an incorrect exponent calculation. The correct transformation would require factoring powers of 2 carefully. Since this mismatch leads to an invalid inequality, the error is in Line 2. i.e. The answer is **C**.

16. Consider the following proof by contradiction that $\sqrt{12}$ is irrational.

(I) Suppose $\sqrt{12}$ is rational. The we can express it in the form $\sqrt{12} = \frac{a}{b}$ for some a and b with no common divisors.

(II) $12 = \frac{a^2}{b^2}$

(III) $12b^2 = a^2$

(IV) 12 divides a^2, which means that 12 must also divide a i.e. $a = 12c$ for some integer c.

(V) This means $144c^2 = 12b^2$ i.e. $b^2 = 12c^2$

(VI) Using the logic of line (IV), $b = 12d$ for some integer d.

(VII) But then a and d both have 12 as a divisor, which is a contradiction.

Where is the first error?

A. The proof is correct.
B. Line (I)
C. Line (II)
D. Line (IV)
E. Line (V)
F. Line (VI)
G. Line (VII)

D

12 dividing a^2 does not imply that 12 divides a, take for example 6 squared. Thus, the error is in the line (IV).

17. Consider the graphs of the two functions $y = mx + 10$ and $y = \log_2 x$. Which of the following correctly identifies the sufficiency and necessity of

I. $m < 0$
II. $m > 10$

in relation to the statement that there are intersection points of the two graphs?

A. Both necessary and sufficient.
B. Both necessary.
C. Both sufficient.
D. 1 is sufficient, the other is necessary.
E. Only 1 is necessary.
F. Only 1 is sufficient.
G. Neither is necessary or sufficient.

F

We want the two graphs to intersect, but equating the expressions directly may not help much. Instead, sketching both graphs is more insightful.

The line now intersects the y-axis at -10, which lies below the entire graph of $\log_2 x$. The logarithmic graph increases slowly for $x > 0$, while the straight line has a constant gradient m.

If $m > 10$, the line climbs very steeply, but since it starts far below the curve at -10, it may rise too quickly and miss the slowly increasing curve altogether. So, while large values of m might still intersect the curve, this is not guaranteed. Also, smaller positive values of m may still result in intersections. Therefore, condition II is neither sufficient nor necessary.

Now consider $m < 0$. In this case, the line slopes downward from the point $(0, -10)$. Since the curve eventually reaches any height (because $\log_2 x \to \infty$ as $x \to \infty$), even a very gently decreasing line will eventually intersect it. Thus, for all negative m, the two graphs must intersect.

Therefore, condition I is sufficient, while condition II is neither necessary nor sufficient.

18. Consider the following proof by induction that $3 \times 7^n + 6$ is divisible by 9 for all non-negative n:

(I) Check the base case, $n = 1$; $3 \times 7 + 6 = 27$, which is indeed divisible by 9.

(II) First, we suppose $3 \times 7^n + 6$ is divisible by 9 for $n \leq k$. i.e. $3 \times 7^k + 6 = 9M$ for some integer M.

(III) Then $3 \times 7^{k+1} + 6 = 7 \times (3 \times 7^k) + 6$

(IV) This means $7 \times (3 \times 7^k) + 6 = 7(9M - 6) + 6$

(V) Which means $3 \times 7^{k+1} + 6 = 9(7M - 4)$

(VI) So $3 \times 7^{k+1} + 6$ is also divisible by 9, which means $3 \times 7^n + 6$ is divisible by 9 for all non-negative n

Where is the first error?

A. The proof is correct.
B. Line (I)
C. Line (II)
D. Line (III)
E. Line (IV)
F. Line (V)
G. Line (VI)

B

This is a relatively simple question if you pay very close attention, as the first error is in line I and if you spot it, you don't even need to check any other line due to the wording of the question. The question states "for non-negative n", but the base case checked is $n = 1$. This is not the smallest non-negative number, 0 is. In fact, for $n = 0$, this conjecture is false, as 6 is not divisible by 9. The answer is therefore line (I).

19. Which of the following is a counterexample to the statement below?

If a function has 3 distinct real roots, it is cubic.

A. $\sin x$ in the range $0 < x < 2pi$
B. $(x-1)(x-2)(x-3)(x-4)$
C. $\cos x$ in the range $0 < x < 2pi$
D. $(x-1)^2(x-2)(x-3)$
E. $(x+1)(x^2-x+5)$
F. $(x-3)(x+1)(x-4)$

D

We seek a function which has 3 distinct roots and is not cubic. D. $(x-1)^2(x-2)(x-3)$ satisfies this as its only roots are 1,2 & 3, but it is quartic. We need not consider any others.

20. Consider this student's attempt at finding the solutions to $\sqrt{x+3} = 3x - 1$:

(I) $x + 3 = (3x - 1)^2$
(II) $x + 3 = 9x^2 - 6x + 1$
(III) $0 = 9x^2 - 7x - 2$
(IV) $0 = (x - 1)(9x + 2)$
(V) So $x = 1 \text{ and } -2/9$

Is this correct?

A. Both answers are correct.
B. Only one is right and it's due to an error in line (I).
C. Only one is right and it's due to an error in lines (II) and (III).
D. Only one is right and it's due to an error in line (IV).
E. Neither is right and it's due to an error in line (I).
F. Neither is right and it's due to an error in lines (II) and (III).
G. Neither is right and it's due to an error in line (IV).

B

It is easy to check which of the roots are right by substituting into the original equation. We see that $x = 1$ is fine, but if $x = -2/9$, the right-hand side is less than 0, but the $\sqrt{x}$ symbol denotes taking the positive root. Thus, only $x = 1$ is correct. The error occurred in line (I) due to assuming no roots were generated from squaring both sides. This is what created the 2^{nd} root.