AMC 10
MOCK TEST 3
Solution Book

Arithmetic and Algebra

ThrivingScholars

1. Cicely had her 21st birthday in 1939.

When did she have her 100th birthday?

A 2020 B 2019 C 2018 D 2010 E 2008

SOLUTION **C**

Cicely had her 21st birthday in 1939. Since 1939 – 21 = 1918, it follows that she was born in 1918.

Now, 1918 + 100 = 2018.

Therefore Cicely's 100th birthday was in 2018.

FOR INVESTIGATION

- The mathematician Augustus De Morgan was born and died in the 19th century. On one birthday he noticed that the square of his age was the same as the year number.

  In which year was Augustus De Morgan born?

- Determine for which values of n a person born in year n could have the same experience as Augustus De Morgan if they lived long enough, that is, they would have a birthday on which the square of their age was the same as the year number.

2. The sequence, formed from the sequence of primes by rounding each to the nearest ten, begins 0, 0, 10, 10, 10, 10, 20, 20, 20, 30, ...

When continued, how many terms in this sequence are equal to 40?

A 1   B 2   C 3   D 4   E 5

Solution **C**

The integers that round to 40 are those in the range from 35 to 44.

The primes in this range are 37, 41 and 43.

Therefore there are 3 primes that round to 40.

For investigation

- How many primes are rounded to 50?
- What is the largest number of primes that round to the same multiple of 10?

3. What is the value of $\frac{1}{1.01} + \frac{1}{1.1} + \frac{1}{1} + \frac{1}{11} + \frac{1}{101}$?

A 2.9 B 2.99 C 3 D 3.01 E 3.1

SOLUTION **C**

Note:
The key to an efficient solution is to regroup the terms so as to simplify the arithmetic.

We have

$$\begin{aligned}
\frac{1}{1.01} + \frac{1}{1.1} + \frac{1}{1} + \frac{1}{11} + \frac{1}{101} &= \left(\frac{1}{1.01} + \frac{1}{101}\right) + \left(\frac{1}{1.1} + \frac{1}{11}\right) + \frac{1}{1} \\
&= \left(\frac{100}{101} + \frac{1}{101}\right) + \left(\frac{10}{11} + \frac{1}{11}\right) + \frac{1}{1} \\
&= \frac{101}{101} + \frac{11}{11} + \frac{1}{1} \\
&= 1 + 1 + 1 \\
&= 3.
\end{aligned}$$

FOR INVESTIGATION

What is the value of $\frac{1}{1.24} + \frac{1}{6.2} + \frac{1}{31}$?

4. The numbers p, q, r and s satisfy the equations $p = 2$, $p \times q = 20$, $p \times q \times r = 202$ and $p \times q \times r \times s = 2020$.

What is the value of $p + q + r + s$?

A 32 B 32.1 C 33 D 33.1 E 34

Solution **B**

We have

$$\begin{aligned} p &= 2, \\ q &= \frac{p \times q}{p} = \frac{20}{2} = 10, \\ r &= \frac{p \times q \times r}{p \times q} = \frac{202}{20} = 10.1, \end{aligned}$$

and

$$s = \frac{p \times q \times r \times s}{p \times q \times r} = \frac{2020}{202} = 10.$$

Therefore $p + q + r + s = 2 + 10 + 10.1 + 10 = 32.1$.

5. One light-year is nearly 6×10^{12} miles. In 2016, the Hubble Space Telescope set a new cosmic record, observing a galaxy 13.4 thousand million light-years away.

Roughly how many miles is that?

A 8×10^{20} B 8×10^{21} C 8×10^{22} D 8×10^{23} E 8×10^{24}

SOLUTION **C**

One thousand million is $1000 \times 1\,000\,000 = 10^3 \times 10^6 = 10^{3+6} = 10^9$. Therefore 13.4 thousand million light-years is 13.4×10^9 light-years. Therefore, because a light-year is nearly 6×10^{12} miles, 13.4 thousand million light-years is approximately

$$(6 \times 10^{12}) \times (13.4 \times 10^9) \text{ light-years.}$$

Now

$$(6 \times 10^{12}) \times (13.4 \times 10^9) = (6 \times 13.4) \times (10^{12} \times 10^9).$$

Now 6×13.4 is approximately 80, therefore, $(6 \times 13.4) \times (10^{12} \times 10^9)$ is approximately

$$80 \times (10^{12} \times 10^9).$$

Finally, we have

$$80 \times (10^{12} \times 10^9) = 8 \times 10 \times 10^{12+9} = 8 \times 10 \times 10^{21} = 8 \times 10^{22}.$$

Therefore 13.4 thousand million light-years is approximately 8×10^{22} miles.

6. Which of the following numbers is the largest?

A $\frac{397}{101}$ B $\frac{487}{121}$ C $\frac{596}{153}$ D $\frac{678}{173}$ E $\frac{796}{203}$

Solution **B**

Commentary

Without the use of a calculator it is not feasible in the time available to answer this question by calculating the values of these fractions to an appropriate number of decimal places.

It would also not be reasonable to decide the relative sizes of, for example, $\frac{397}{101}$ and $\frac{487}{121}$ by using the fact that

$$\frac{397}{101} < \frac{487}{121} \Leftrightarrow 397 \times 121 < 487 \times 101.$$

Instead, the best approach here is to notice that all five fractions are close to 4, and then to decide which of them are less than 4, and which are greater than 4.

We note that

$$\frac{397}{101} < \frac{404}{101} = 4,$$
$$\frac{487}{121} > \frac{484}{121} = 4,$$
$$\frac{596}{153} < \frac{612}{153} = 4,$$
$$\frac{678}{173} < \frac{692}{173} = 4,$$

and

$$\frac{796}{203} < \frac{812}{203} = 4.$$

These calculations show that the fraction given as option B is the only one that is greater than 4, and hence is the largest of the given numbers.

For investigation

Which of the following numbers is the smallest?

$$\frac{527}{105}, \frac{617}{123}, \frac{707}{141}, \frac{803}{161}, \frac{917}{183}.$$

7. Official UK accident statistics showed that there were 225 accidents involving teapots in one year. However, in the following year there were 47 such accidents.

What was the approximate percentage reduction in recorded accidents involving teapots from the first year to the second?

A 50 B 60 C 70 D 80 E 90

Solution **D**

The reduction in the number of teapot accidents in the second year was $225 - 47 = 178$.

178 as a percentage of 225 is

$$\frac{178}{225} \times 100 \approx \frac{180}{225} \times 100 = \frac{20}{25} \times 100 = 20 \times 4 = 80.$$

8. In the following expressions, x is non-zero. When one of these expressions is removed, the mean of the remaining four is $11x$.

Which expression is removed?

A $4x$ B $8x$ C $12x$ D $16x$ E $20x$

SOLUTION **D**

Because the mean of the remaining four numbers is $11x$, their sum is $4 \times 11x$, that is, $44x$.

The sum of all five of the numbers is $4x + 8x + 12x + 16x + 20x$, that is, $60x$.

Since $60x - 44x = 16x$, the number that is removed is $16x$.

FOR INVESTIGATION

In the following expressions, the number x is non-zero: x, $5x$, $9x$, $13x$, $17x$. When one of these expressions is removed, the mean of the remaining four is $8x$.

Which expression is removed?

In the following expressions, the number x is a positive integer: x^2, x, $2x$, $3x$, $4x$. When one of these expressions is removed, the mean of the remaining four is 4.

(a) Which expression is removed?

(b) What is the value of x?

9. In 2018, a racing driver was allowed to use the Drag Reduction System provided that the car was within 1 second of the car ahead. Suppose that two cars were 1 second apart, each travelling at 180 km/h (in the same direction!).

How many metres apart were they?

A 100 B 50 C 10 D 5 E 1

SOLUTION **B**

The distance apart of the cars was the distance that a car travelling at 180 km/h travels in 1 second.

There are 1000 metres in one kilometre. Hence there are 180×1000 metres in 180 km.

There are 60×60 seconds in each hour.

It follows that at 180 km/h a car travels 180×1000 metres in 60×60 seconds.

Therefore the number of metres that it travels in 1 second is

$$\frac{180 \times 1000}{60 \times 60} = \frac{3 \times 1000}{60} = \frac{1000}{20} = 50.$$

Therefore when the cars are 1 second apart, they are 50 metres apart.

FOR INVESTIGATION

- The Highway Code gives the following table of typical *stopping distances* in metres for motor vehicles travelling at different velocities.

velocity	stopping distance
32 km/h	12 m
48 km/h	23 m
64 km/h	36 m
80 km/h	53 m
96 km/h	73 m
112 km/h	96 m

For each velocity, how far apart in seconds should two cars travelling in the same direction at that velocity be so that their distance apart is the same as the corresponding stopping distance given in the above table?

- *Note*: The Highway Code adds that "The distances shown are a general guide. The distance will depend on your attention (thinking distance), the road surface, the weather conditions and the condition of your vehicle at the time. "

The Highway Code gives a thinking distance of 12 m for a car travelling at 64 km/h. How much thinking time does that correspond to?

10. The teenagers Sam and Jo notice the following facts about their ages:

The difference between the squares of their ages is four times the sum of their ages.

The sum of their ages is eight times the difference between their ages.

What is the age of the older of the two?

A 15 B 16 C 17 D 18 E 19

SOLUTION **D**

Suppose that the ages of the teenagers are a and b, with $a > b$.

Because the difference between the squares of their ages is four times the sum of their ages

$$a^2 - b^2 = 4(a + b).$$

By factorizing its left hand side, we may rewrite this last equation as

$$(a - b)(a + b) = 4(a + b)$$

Because $a + b \neq 0$, we may divide both sides of this last equation by $a + b$ to give

$$a - b = 4. \qquad (1)$$

Because the sum of their ages is eight times their difference

$$a + b = 8(a - b)$$

Hence, by (1)

$$a + b = 32. \qquad (2)$$

By adding equations (1) and (2), we obtain

$$2a = 36$$

and hence

$$a = 18.$$

Therefore the age of the older of the two teenagers is 18.

FOR INVESTIGATION

What is the age of the younger teenager in Question 11?

Suppose that the difference between the squares of the ages of two teenagers is six times the sum of their ages, and the sum of their ages is five times the difference of their ages.

What are their ages in this case?

Suppose that the difference between the squares of the ages of the two teenagers is k times the sum of their ages, and the sum of their ages in n times the difference of their ages.

Find a formula in terms of k and n for the ages of the teenagers. Check that your formula gives the correct answer to Question 11 and Problems 11.1 and 11.2.

11. On a training ride, Laura averages speeds of 12 km/h for 5 minutes, then 15 km/h for 10 minutes and finally 18 km/h for 15 minutes.

What was her average speed over the whole ride?

A 13 km/h B 14 km/h C 15 km/h D 16 km/h E 17 km/h

Solution **D**

To determine Laura's average speed over the whole ride we calculate the total distance she travels, and the total time that she takes.

Because 5 minutes is $\frac{1}{12}$ of an hour, Laura travels $\frac{1}{12} \times 12\,\text{km} = 1\,\text{km}$ when she travels for 5 minutes at 12 km/h.

Because 10 minutes is $\frac{1}{6}$ of an hour, Laura travels $\frac{1}{6} \times 15\,\text{km} = 2.5\,\text{km}$ when she travels for 10 minutes at 15 km/h.

Because 15 minutes is $\frac{1}{4}$ of an hour, Laura travels $\frac{1}{4} \times 18\,\text{km} = 4.5\,\text{km}$ when she travels for 15 minutes at 18 km/h.

Therefore Laura travels at total of $1\,\text{km} + 2.5\,\text{km} + 4.5\,\text{km} = 8\,\text{km}$ in $5 + 10 + 15$ minutes, that is, in 30 minutes, which is half an hour.

Because Laura travels 8 km in half an hour, her average speed is 16 km/h.

For investigation

- Suppose Laura had ridden for a further 20 minutes at 21 km/h. What would then have been her average speed for the whole ride?
- Suppose Laura had extended her training ride by 30 minutes. How fast would she have had to ride in this 30 minutes to make her average speed for the whole hour equal to 20 km/h?

12. Anish has a number of small congruent square tiles to use in a mosaic. When he forms the tiles into a square of side n, he has 64 tiles left over. When he tries to form the tiles into a square of side $n + 1$, he has 25 too few.

How many tiles does Anish have?

A 89 B 1935 C 1980 D 2000 E 2019

SOLUTION **D**

Because Anish has 64 tiles left over when he forms a square of side n, he has $n^2 + 64$ tiles.

Because Anish has 25 tiles too few to make a square of side $n + 1$, he has $(n + 1)^2 - 25$ tiles.

Therefore $n^2 + 64 = (n + 1)^2 - 25$. Now

$$\begin{aligned} n^2 + 64 = (n+1)^2 - 25 &\Leftrightarrow n^2 + 64 = (n^2 + 2n + 1) - 25 \\ &\Leftrightarrow 2n = 64 + 25 - 1 \\ &\Leftrightarrow 2n = 88 \\ &\Leftrightarrow n = 44. \end{aligned}$$

Because $n = 44$, the number of tiles that Anish has is given by $44^2 + 64 = 1936 + 64 = 2000$.

FOR INVESTIGATION

- For $n = 44$, check that $(n + 1)^2 - 25$ also equals 2000.
- Anish has exactly enough square 1×1 tiles to form a square of side m. He would need 2019 more tiles to form a square of side $m + 1$.

  How many tiles does Anish have?
- Anish has exactly enough $1 \times 1 \times 1$ cubes to form a cube of side m. He would need 397 more cubes to form a cube of side $m + 1$.

  How many cubes does Anish have?

13. The numbers x, y and z satisfy the equations $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$.

What is the mean of x, y and z?

A 10 B 11 C 12 D 13 E 14

Solution **A**

Commentary

The mean of x, y and z is $\frac{1}{3}(x + y + z)$. Therefore to answer this question we need to find the value of $x + y + z$. We are given just two equations for the three unknowns x, y and z. It follows that if these equations have a solution, they will have an infinite number of solutions.

A systematic method for answering the question would be to use the two equations to find expressions for two of the unknowns in terms of the third unknown. For example, we could find x and y in terms of z, and thus work out $x + y + z$ in terms of z.

However, the wording of the question suggests that $x + y + z$ is independent of z. Thus a good starting point is to try to find a way to use the two equations we are given to find a value for $x + y + z$ without the need to find x and y in terms of z.

We are given that

$$9x + 3y - 5z = -4 \qquad (1)$$

and

$$5x + 2y - 2z = 13 \qquad (2)$$

If we multiply equation (2) by 2, and subtract equation (1) we obtain

$$2(5x + 2y - 2z) - (9x + 3y - 5z) = 2(13) - (-4),$$

that is,

$$10x + 4y - 4z - 9x - 3y + 5z = 26 + 4,$$

that is,

$$x + y + z = 30.$$

We deduce that $\frac{1}{3}(x + y + z) = 10$.

Hence the mean of x, y and z is 10.

For investigation

(a) Use the equations $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$ to find expressions for y and z in terms of x.

(b) Use your answers to part (a) to show that for all values of x, we have $x + y + z = 30$.

The numbers x, y and z satisfy the equations $3x - 5y + 7z = 4$ and $4x - 8y + 10z = 6$. What is the mean of x, y and z?

14. Amy, Beth and Claire each has some sweets. Amy gives one third of her sweets to Beth. Beth gives one third of all the sweets she now has to Claire. Then Claire gives one third of all the sweets she now has to Amy. All the girls end up having the same number of sweets.

Claire begins with 40 sweets.

How many sweets does Beth have originally?

A 20 B 30 C 40 D 50 E 60

SOLUTION **D**

We work backwards from the final situation when Amy, Beth and Claire end up with the same number of sweets. We let s be the number of sweets they all end up with.

Claire ends up with s sweets. Therefore, before she gave one third of her sweets to Amy, Claire had t sweets, where $\frac{2}{3}t = s$. This gives $t = \frac{3}{2}s$. We deduce that, before giving one-third of her sweets to Amy, Claire has $\frac{3}{2}s$ sweets and gives one third of these, namely $\frac{1}{2}s$ sweets, to Amy. Hence, before she received these sweets, Amy had $\frac{1}{2}s$ sweets.

Similarly, Beth ends up with s sweets after she has given one third of her sweets to Claire. Hence, before this she had $\frac{3}{2}s$ sweets. As Claire has $\frac{3}{2}s$ sweets after receiving $\frac{1}{2}s$ from Beth, she had s sweets before this.

It follows that after Amy has given one third of her sweets to Beth, Amy has $\frac{1}{2}s$ sweets, Beth has $\frac{3}{2}s$ sweets, and Claire has s sweets.

Therefore, before Amy gives one third of her sweets to Beth, Amy has $\frac{3}{4}s$ sweets. She gives $\frac{1}{4}s$ sweets to Beth. Hence, before receiving these, Beth had $\frac{3}{2}s - \frac{1}{4}s = \frac{5}{4}s$ sweets.

We can sum this up by the following table.

Stage	Amy	Beth	Claire
Final distribution of sweets, after Claire gives one third of her sweets ($\frac{1}{2}s$ sweets) to Amy.	s	s	s
Distribution of sweets after Beth gives one third of her sweets ($\frac{1}{2}s$ sweets) to Claire.	$\frac{1}{2}s$	s	$\frac{3}{2}s$
Distribution of sweets after Amy gives one third her sweets ($\frac{1}{4}s$ sweets) to Beth.	$\frac{1}{2}s$	$\frac{3}{2}s$	s
Initial distribution of sweets.	$\frac{3}{4}s$	$\frac{5}{4}s$	s

We are told that Claire begins with 40 sweets. Therefore $s = 40$. So Beth begins with $\frac{5}{4}(40) = 50$ sweets.

FOR INVESTIGATION

An alternative method is to suppose that, say, Amy begins with a sweets and Beth with b sweets, and then to work out how many sweets they all end up with. Use this method to

15. The arithmetic mean, A, of any two positive numbers x and y is defined to be $A = \frac{1}{2}(x+y)$ and their geometric mean, G, is defined to be $G = \sqrt{xy}$.

For two particular values x and y, with $x > y$, the ratio $A : G = 5 : 4$.

For these values of x and y, what is the ratio $x : y$?

A $5 : 4$ B $2 : 1$ C $5 : 2$ D $7 : 2$ E $4 : 1$

Solution **E**

We are told that

$$\frac{\frac{1}{2}(x+y)}{\sqrt{xy}} = \frac{5}{4}.$$

Therefore

$$2(x+y) = 5\sqrt{xy}.$$

By squaring both sides of this equation we deduce that

$$4(x+y)^2 = 25xy.$$

By expanding the left-hand side of this last equation, we have

$$4(x^2 + 2xy + y^2) = 25xy,$$

or, equivalently,

$$4x^2 + 8xy + 4y^2 = 25xy.$$

Hence

$$4x^2 - 17xy + 4y^2 = 0.$$

The left-hand of this last equation factorizes to give

$$(4x - y)(x - 4y) = 0.$$

Hence

$$4x = y \text{ or } x = 4y.$$

It follows that, because $x > y$,

$$x = 4y.$$

Hence

$$x : y = 4 : 1.$$

For investigation

- Suppose that x and y are positive integers, with $x > y$ and $A : G = 5 : 3$.

  What is the ratio $x : y$ in this case?

- Do there exist positive integers x and y such that $A < G$, that is, such that $\frac{1}{2}(x+y) < \sqrt{xy}$?

16. The numbers x and y are such that $3^x + 3^{y+1} = 5\sqrt{3}$ and $3^{x+1} + 3^y = 3\sqrt{3}$.

What is the value of $3^x + 3^y$?

A $\sqrt{3}$ B $2\sqrt{3}$ C $3\sqrt{3}$ D $4\sqrt{3}$ E $5\sqrt{3}$

Solution **B**

Note that $3^{x+1} = 3(3^x)$ and $3^{y+1} = 3(3^y)$.

Therefore the equations given in the question may be rewritten as

$$3^x + 3(3^y) = 5\sqrt{3} \qquad (1)$$

and

$$3(3^x) + 3^y = 3\sqrt{3}. \qquad (2)$$

Adding equations (1) and (2) gives

$$4(3^x + 3^y) = 8\sqrt{3}. \qquad (3)$$

Therefore

$$3^x + 3^y = 2\sqrt{3}. \qquad (4)$$

For investigation

(a) Use equations (1) and (2) to find the values of 3^x and 3^y.

(b) Check that the values for 3^x and 3^y that you have found satisfy equation (4).

(For those who know about logarithms.)

(a) Use your answer to Problem 18.1 (a) to show that $x = \frac{1}{2} - \frac{\ln 2}{\ln 3}$.

(b) Find a similar expression for the value of y.

Find the values of x and y that satisfy both of the equations

$$4^{x+1} + 5^y = 281$$

and

$$4^x + 5^{y+1} = 189.$$

17. The positive integers m, n and p satisfy the equation $3m + \dfrac{3}{n + \frac{1}{p}} = 17.$

What is the value of p?

A 2 B 3 C 4 D 6 E 9

Solution **A**

Because $\dfrac{3}{n + \frac{1}{p}} = 17 - 3m$, where m is an integer, it follows that $\dfrac{3}{n + \frac{1}{p}}$ is an integer.

Because n and p are positive integers, $1 < n + \frac{1}{p}$ and therefore $0 < \dfrac{3}{n + \frac{1}{p}} < 3$. Therefore $\dfrac{3}{n + \frac{1}{p}}$ is either 1 or 2.

If $\dfrac{3}{n + \frac{1}{p}} = 1$, then $3m = 17 - \dfrac{3}{n + \frac{1}{p}} = 17 - 1 = 16$. This implies that $m = \frac{16}{3}$ contradicting the fact that m is an integer. Hence $\dfrac{3}{n + \frac{1}{p}} = 2$. Therefore $n + \frac{1}{p} = \frac{3}{2} = 1 + \frac{1}{2}$. Hence $n = 1$ and $p = 2$. Also $3m = 17 - \dfrac{3}{1 + \frac{1}{2}} = 17 - 2 = 15$ and so $m = 5$. We therefore see that $m = 5$, $n = 1$ and $p = 2$.

18. The following twelve integers are written in ascending order:

$$1, x, x, x, y, y, y, y, y, 8, 9, 11.$$

The mean of these twelve integers is 7. What is the median?

A 6 B 7 C 7.5 D 8 E 9

SOLUTION **D**

Because the mean of the given integers is 7, we have

$$1 + x + x + x + y + y + y + y + y + 8 + 9 + 11 = 12 \times 7.$$

We may rewrite this last equation as $1 + 3x + 5y + 28 = 84$ and therefore $3x + 5y = 55$.

It follows that $3x = 55 - 5y = 5(11 - y)$. Therefore, because x and y are integers, $3x$ is a multiple of 5 and hence x is a multiple of 5.

Because the integers are written in ascending order, $1 \leq x \leq 8$. Hence $x = 5$ and therefore $15 + 5y = 55$. It follows that $5y = 40$ and therefore $y = 8$.

Therefore the twelve integers in ascending order are 1, 5, 5, 5, 8, 8, 8, 8, 8, 8, 9, 11. We now see that the median is 8.

19. Bethany has 11 pound coins and some 20p coins and some 50p coins in her purse. The mean value of the coins is 52 pence.

Which could not be the number of coins in the purse?

A 35 B 40 C 50 D 65 E 95

SOLUTION **B**

We suppose that numbers of 20p and 50p coins that Bethany has are m and n, respectively. (Note that m and n are positive integers.)

Then Bethany has c coins, where $c = 11+m+n$. The total value of these coins is $11+0.20m+0.50n$ pounds, which equals $1100 + 20m + 50n$ pence.

Because the mean value of Bethany's coins is 52 pence, $\frac{1100 + 20m + 50n}{11 + m + n} = 52$. It follows that $1100 + 20m + 50n = 52(11 + m + n)$ and hence $1100 + 20m + 50n = 572 + 52m + 52n$. This equation may be rearranged as $2n = 528 - 32m$, which simplifies to $n = 264 - 16m$.

It follows that $c = 11 + m + n = 11 + m + (264 - 16m) = 275 - 15m$. Therefore $275 - c = 15m$. Thus $275 - c$ is an integer multiple of 15.

We now consider the different values for c given by the options in the question.

We have $275 - 35 = 240 = 15 \times 16$, $275 - 50 = 225 = 15 \times 15$, $275 - 65 = 210 = 15 \times 14$ and $275 - 95 = 180 = 15 \times 12$. Hence Bethany could have 35 or 50 or 65 or 95 coins in her purse.

However $275 - 40 = 235$ which is not an integer multiple of 15. Hence Bethany could not have 40 coins in her purse.

20. A function f satisfies the equation $(n-2019)f(n)-f(2019-n)=2019$ for every integer n.

What is the value of $f(2019)$?

A 0 B 1 C 2018×2019 D 2019^2 E 2019×2020

Solution **C**

Putting $n=0$ in the equation

$$(n-2019)f(n)-f(2019-n)=2019 \qquad (1)$$

gives

$$-2019f(0)-f(2019)=2019 \qquad (2)$$

from which it follows that

$$f(2019)=-2019f(0)-2019. \qquad (3)$$

Putting $n=2019$ in equation (1) gives

$$-f(0)=2019 \qquad (4)$$

and hence

$$f(0)=-2019. \qquad (5)$$

Substituting from (5) in (3) gives

$$\begin{aligned} f(2019) &= -2019\times -2019-2019\\ &= 2019\times 2019-2019\\ &= 2019(2019-1)\\ &= 2019\times 2018. \end{aligned}$$

Therefore, the value of $f(2019)$ is 2018×2019.

For investigation

- What is the value of $f(1)$?
- Find the general formula for $f(n)$ in terms of n.

21. The numbers m and k satisfy the equations $2^m + 2^k = p$ and $2^m - 2^k = q$.

What is the value of 2^{m+k} in terms of p and q?

A $\frac{p^2 - q^2}{4}$ B $\frac{pq}{2}$ C $p + q$ D $\frac{(p-q)^2}{4}$ E $\frac{p+q}{p-q}$

Solution **A**

We have

$$2^m + 2^k = p \qquad (1)$$

and

$$2^m - 2^k = q. \qquad (2)$$

Adding equations (1) and (2), we obtain

$$2(2^m) = p + q$$

and hence

$$2^m = \frac{p+q}{2}. \qquad (3)$$

Subtracting equations (2) from equation (1), we obtain

$$2(2^k) = p - q$$

and hence

$$2^k = \frac{p-q}{2}. \qquad (4)$$

Therefore, by (3) and (4)

$$\begin{aligned} 2^{m+k} &= 2^m \times 2^k \\ &= \left(\frac{p+q}{2}\right) \times \left(\frac{p-q}{2}\right) \\ &= \frac{(p+q)(p-q)}{4} \\ &= \frac{p^2 - q^2}{4}. \end{aligned}$$

For investigation

Use the equations $p = 2^m + 2^k$ and $q = 2^m - 2^k$ to obtain expressions for p^2 and q^2.

Hence deduce that $\frac{p^2 - q^2}{4} = 2^{m+k}$.

The numbers a and b satisfy the equations

$$2^{a+b} = r$$

and

$$2^{a-b} = s.$$

Find $2^a + 2^b$ in terms of r and s.

22. Laura and Dina have a running race. Laura runs at constant speed and Dina runs n times as fast where $n > 1$. Laura starts s m in front of Dina.

What distance, in metres, does Dina run before she overtakes Laura?

A $\frac{ns}{n-1}$ B ns C $\frac{s}{n-1}$ D $\frac{ns}{n+1}$ E $\frac{s}{n}$

Solution **A**

Suppose that Dina has run a distance of d metres when she overtakes Laura. Because Laura has a start of s metres, at this time Laura has run a distance of $d - s$ metres.

Because they have been running for the same amount of time when Dina overtakes Laura, at this time the ratio of the distances they have run is the same as the ratio of their speeds. That is

$$d : d - s = n : 1.$$

It follows that

$$\frac{d}{d-s} = \frac{n}{1}.$$

Hence

$$d = n(d - s).$$

This last equation may be rearranged as

$$ns = d(n - 1).$$

Therefore

$$d = \frac{ns}{n-1}.$$

For investigation

Suppose that when Dina overtakes Laura she has run twice as far as Laura.

What is the ratio of Dina's speed to Laura's speed?

23. The real numbers x and y satisfy the equations $4^y = \dfrac{1}{8(\sqrt{2})^{x+2}}$ and $9^x \times 3^y = 3\sqrt{3}$.

What is the value of 5^{x+y}?

A $5\sqrt{5}$ B 5 C $\sqrt{5}$ D $\frac{1}{5}$ E $\frac{1}{\sqrt{5}}$

Solution **E**

We have $4^y = (2^2)^y = 2^{2y}$, and $\dfrac{1}{8(\sqrt{2})^{x+2}} = \dfrac{1}{2^3(2^{\frac{1}{2}})^{x+2}} = 2^{-(3+\frac{1}{2}(x+2))}$.

Therefore from the equation $4^y = \dfrac{1}{8(\sqrt{2})^{x+2}}$ we deduce that $2y = -(3 + \frac{1}{2}(x+2))$.

Therefore

$$y = -\tfrac{1}{4}x - 2. \qquad (1)$$

Also, $9^x \times 3^y = (3^2)^x \times 3^y = 3^{2x+y}$ and $3\sqrt{3} = (3^1)(3^{\frac{1}{2}}) = 3^{1+\frac{1}{2}} = 3^{\frac{3}{2}}$.

Therefore, from the equation $9^x \times 3^y = 3\sqrt{3}$ we deduce that $2x + y = \frac{3}{2}$.

Therefore

$$y = -2x + \tfrac{3}{2}. \qquad (2)$$

From equations (1) and (2)

$$-\tfrac{1}{4}x - 2 = -2x + \tfrac{3}{2}.$$

This last equation may be rearranged to give

$$\tfrac{7}{4}x = \tfrac{7}{2}.$$

Hence $x = 2$.

Therefore, from equation (1), $y = -\frac{5}{2}$.

It follows that $x + y = 2 + (-\frac{5}{2}) = 2 - \frac{5}{2} = -\frac{1}{2}$.

Therefore $5^{x+y} = 5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$.

For investigation

The real numbers x and y satisfy the equations

$$(\sqrt{3})^x \times 3^y = \sqrt[5]{27}$$

and

$$8^x \times (\sqrt{2})^y = \sqrt[3]{4}.$$

Find the value of 8^{x+y}.

24. A ‘complete’ football kit consists of a shirt, a pair of shorts and a pair of socks. Three pairs of shorts and one pair of socks together cost the same as two shirts. Seven pairs of shorts and four pairs of socks together cost the same as five shirts. Eden has exactly the right amount of money to buy nine shirts. How many ‘complete’ football kits could be bought for the same amount of money?

A 3 B 4 C 5 D 6 E 7

Solution **C**

Let the cost of one shirt, one pair of shorts and one pair of socks be x, y and z pounds, respectively. Thus the cost of a full football kit is $x + y + z$ pounds.

From the information given in the question, we have

$$3y + z = 2x \qquad (1)$$
$$7y + 4z = 5x. \qquad (2)$$

Since $5 \times 2x = 2 \times 5x$, it follows from (1) and (2) that

$$5(3y + z) = 2(7y + 4x).$$

That is,

$$15y + 5z = 14y + 8z,$$

and hence

$$y = 3z. \qquad (3)$$

It follows from (1) and (3) that $9z + z = 2x$ and hence that

$$z = \frac{1}{5}x.$$

Hence, by (3),

$$y = \frac{3}{5}x.$$

It follows that

$$\begin{aligned} x + y + z &= x + \frac{3}{5}x + \frac{1}{5}x \\ &= \frac{9}{5}x. \end{aligned}$$

Therefore the cost of a complete football kit is $\frac{9}{5}x$ pounds.

Eden has exactly enough money to buy 9 shirts, and so has $9x$ pounds.

Therefore Eden has enough money to buy $9x \div \frac{9}{5}x = 5$ complete football kits.

For investigation

24.1 If, in this question, one pair of football socks costs £2.40, what is the cost of a complete football kit?

25. When $\frac{1}{x} - \frac{1}{y} = 2025$, what is the value of $\frac{x + 2026xy - y}{2y - 2025xy - 2x}$?

A 0 B 1 C $\frac{1}{2}$ D 2026 E $\frac{1}{2025}$

Solution **E**

Since $\frac{1}{x} - \frac{1}{y} = 2025$, it follows that $\frac{y - x}{xy} = 2025$, and hence $y - x = 2025xy$.

Therefore

$$\begin{aligned}\frac{x + 2026xy - y}{2y - 2025xy - 2x} &= \frac{2026xy - (y - x)}{2(y - x) - 2025xy} \\ &= \frac{2026xy - 2025xy}{2(2025xy) - 2025xy} \\ &= \frac{xy}{2025xy} \\ &= \frac{1}{2025}.\end{aligned}$$

For investigation

Show that for every integer n, if $\frac{1}{x} - \frac{1}{y} = n$, then

$$\frac{x + (n+1)xy - y}{2y - nxy - 2x} = \frac{1}{n}.$$

You are given that

$$\frac{5y + xy - 5x}{3y - xy - 3x} = 2.$$

What is the value of $\frac{1}{x} - \frac{1}{y}$?