AMC 8 Formulas and Strategies

AMC 8 Contest

- Organized by MAA (Mathematical Association of America)
- 25 questions
- 40 minutes (1.6 min per question)
- Multiple choice, no negative marking
- Anyone in grade 8 or lower
- No lower grade limit, so even elementary schoolers can take the test
- Competition Date: Mid November (might change from 2021-22 school year)

- Memorize some important numbers:
  - Prime Factorization of 2020: $2^2 \times 5 \times 101$
  - There are 12 positive integers that are factors of 2020
  - Factors of 2020 are 1, 2, 4, 5, 10, 20, 101, 202, 404, 505, 1010, 2020
  - Prime Factorization of 2019: 3 x 673
  - Prime Factorization of 2021: 43 x 47
- Remember to do this for the year of the competition

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Combinatorics

Factorial

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1$$

Number of ways to arrange n objects in a line: $n!$

Number of ways to arrange n objects in a circle: $(n-1)!$ (where rotations are considered same)

Word Rearrangements

General Formula for the number of ways to arrange the letters of a Word:

$$\frac{n!}{d_1! \times d_2! \times d_3! \times \dots}$$

where n is the number of letters to arrange and where $d_1, d_2, d_3, \cdots$ are the number of times each of the letters that occur more than 1 time appear in the word.

Permutations

Formula for number of ways of assigning k distinct positions to n things:

$$P(n,k) = \frac{n!}{(n-k)!}$$

Combinations

Formula for choosing k objects from n objects:

$$\binom{n}{k} = \frac{n!}{k! \times (n-k)!}$$

Notice that:

$$\binom{n}{k} = \binom{n}{n-k}$$

from the formula. This is because the number of ways of choosing k objects is the same as the number of ways of k objects to choosing $n-k$ objects not to be selected.

Usually, the words **permute, order does matter, etc.** imply a permutation
while the words **choose, select, order doesn't matter, etc.** imply a combination.

Casework

Solving counting or probability problems by considering the different cases and adding them together.

Complementary Counting

Complementary counting is the problem solving technique of counting the opposite of what we want and subtracting that from the total number of cases.

Look for the keyword “at least”

Overcounting

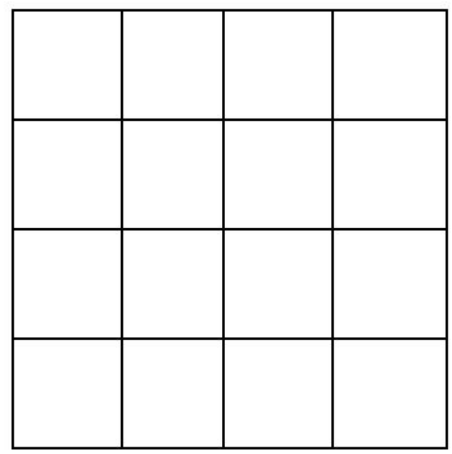
Overcounting is the process of counting more than what you need and then systematically subtracting the parts which do not belong.

Probability

Probability is the likelihood of something happening. To calculate probability, you need to know how many possible options or outcomes there are and how many right combinations there are.

$$\text{Probability} = \frac{\text{Total Successful Outcomes}}{\text{Total Possible Outcomes}}$$

Using Combinatorics for Grid Problems



The general formula for the number of squares of all sizes in a square grid with dimensions n x n is

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

The general formula for the number of rectangles of all sizes in a rectangular grid of size m x n is

$$\binom{m+1}{2} \times \binom{n+1}{2}$$

The general formula for the number of ways to get from (0, 0) to the point (x, y) in a grid where you can only go right or up along the grid lines:

$$\binom{x+y}{x}$$

Recursion

Start from 1st step, and try to define subsequent steps in terms of previous steps

Stars and Bars (Sticks and Stones)

Formula where n in the number of identical objects to distribute to k things:

$$\binom{n+k-1}{n}$$

Use Stars and Bars wherever you are distributing identical objects to people (or groups)

Algebra

Mean, Median, Mode

Mean = Average of all numbers

Mode = Most common Number
Note: There could be multiple modes. If the problem says “unique mode”, it means that there is only mode

After arranging the numbers in increasing or decreasing order:

If number of terms is odd, Median = middle number

If number of terms is even, Median = average of middle two numbers

Harmonic Mean

Harmonic Mean of Numbers $a_1, a_2, a_3, \ldots, a_n = \dfrac{1}{\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n}} = \dfrac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$

Telescoping

Expand the first few and last few terms, and cancel out any terms you see

Graph Problems

Best way to solve is to carefully analyze the graphs and eliminate answer choices

Speed, Time, and Distance

$$\text{Distance} = \text{Speed} \times \text{Time}$$

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\text{Average Speed} = \frac{\text{Total Distance}}{\text{Total Time}}$$

System of Equations

Set up equation and let unknowns be variables

Difference of Squares

$a^2 - b^2 = (a - b)(a + b)$

Arithmetic Sequences

An arithmetic sequence is a sequence of numbers with the same difference between consecutive terms.

Here is an example of an arithmetic sequence:

$$1, 4, 7, 10, 13, \ldots, 40$$

because there is always a difference of 3 between terms.

In general, the terms of an arithmetic sequence can be represented as:

$$a_1, a_2, a_3, a_4, \ldots, a_n$$

n is the number of terms in the sequence
d is the common difference between consecutive terms

Formula for calculating the nth term in an arithmetic sequence

$$a_n = a_1 + (n-1) \times d$$

More general form:

$$a_n = a_m + (n-m) \times d$$

Number of Terms in an arithmetic sequence

$$\text{Number of Terms} = \frac{\text{Last Term} - \text{First Term}}{\text{Common Difference}} + 1$$

$$n = \frac{a_n - a_1}{d} + 1$$

Average of Terms in an arithmetic sequence

$$\text{Average of Terms} = \frac{\text{First Term} + \text{Last Term}}{2}$$

$$\text{Average of Terms} = \frac{a_1 + a_n}{2}$$

$$\text{Average of Terms} = \frac{\text{Sum of all Terms}}{\text{Number of Terms}}$$

$$\text{Average of Terms} = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$$

If number of terms is odd, $\text{Average of Terms} = \text{middle term}$

If number of terms is even, $\text{Average of Terms} = \text{average of middle 2 term}$

Sum of all terms in an arithmetic sequence

$$\text{Sum of All Terms} = \text{Average of Terms} \times \text{Number of Terms}$$

$$S_n = \frac{a_1 + a_n}{2} \cdot n$$

Substituting $a_n = a_1 + (n-1)d$, we get

$$S_n = \frac{2a_1 + (n-1)d}{2} \cdot n$$

Special Series

$\text{Sum of first n numbers} = 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

$\text{Sum of first n odd numbers} = 1 + 3 + \cdots + (2n-3) + (2n-1) = n^2$

Geometric Sequences

A geometric sequence is a sequence of numbers with the same ratio between consecutive terms. Here is an example of a geometric sequence:

$$1, 2, 4, 8, 16, 32, \ldots, 1024$$

because there is always a ratio of 2 between terms.

In general, the terms of a geometric sequence can be represented as:

$$a_1, a_2, a_3, a_4, \ldots, a_n$$

n is the number of terms in the sequence
r is the common ratio between consecutive terms

nth term of a geometric sequence

$$a_n = a_1 \cdot r^{n-1}$$

More general form:

$$a_n = a_m \cdot r^{n-m}$$

Sum of a geometric sequence

If r $>$ 1,

$$S_n = a_1 \frac{(r^{n-1} - 1)}{(r - 1)}$$

If r $<$ 1,

$$S_n = a_1 \frac{(1 - r^{n-1})}{(1 - r)}$$

Sum of a geometric sequence with infinite number of terms (r $<$ 1)

$$S_\infty = \frac{a_1}{1 - r}$$

Number Theory

Primes

Primes are numbers that have exactly two factors: 1 and the number itself.

Ex. 2, 3, 5, 7, 11, 13, 17, 19, etc. are all primes

Note: 1 is not a prime and 2 is the only even prime.

Prime Factorization

Prime factorization is a way to express each number as a product of primes.
Example:
The prime factorization of 21 is 3×7
The prime factorization of 60 is $2^2 \times 3 \times 5$

Number of Factors of a Number

A number with prime factorization

$$p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$$

has $(e_1 + 1)(e_2 + 1)\ldots(e_k + 1)$ factors. Basically, in order to find the number of factors of a number:

1. Find the prime factorization of the number
2. Add 1 to all of the exponents
3. Multiply them together

Divisibility Rules

2 - Last digit is even
3 - Sum of digits is divisible by 3
4 - Last 2 digits divisible by 4
5 - Last digit is 0 or 5
6 - Divisible by 2 and 3
7 - Take out factors of 7 until you reach a small number that is either divisible or not divisible by 7
8 - Last 3 digits are divisible by 8
9 - Sum of digits is divisible by 9
10 - Last digit is 0
11 - Calculate the sum of odd digits (O) and even digits (E). If $|O - E|$ is divisible by 11, then the number is also divisible by 11
12 - Divisible by 3 and 4
15 - Divisible by 3 and 5

More on Prime Factorization and Divisibility

Modular Arithmetic

When given different remainder conditions, find how much more or less of a certain multiple each statement means and combine them together.

Digit Cycles

Calculate the first few values and look for a pattern

More on Modular Arithmetic and Digit Cycles

GCD/LCM

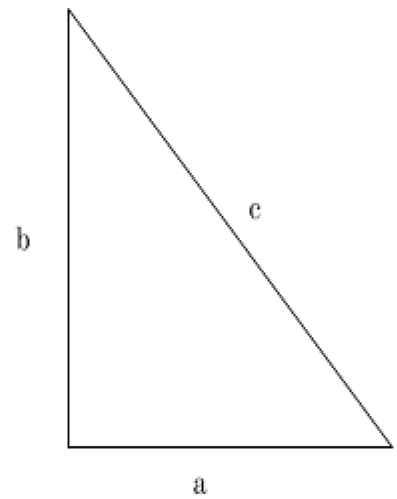
Greatest common factor of m and n $= GCD(m.n)$ can be found by taking the lowest prime exponents from m and n

Least common multiple of m and n $= LCM(m.n)$ can be found by taking the highest prime exponents from m and n

$GCD(m, n) \cdot LCM(m, n) = m \cdot n$

Geometry

Pythagorean Theorem:



A right triangle with legs a, b and hypotenuse c satisfies the following relation:

$$a^2 + b^2 = c^2$$

List of common pythagorean triples:

$3, 4, 5$

$5, 12, 13$

$7, 24, 25$

$8, 15, 17$

If all numbers in a pythagorean triple are multiplied by a number, it is still a pythagorean triple. These are all pythagorean triples:

$3, 4, 5$

$6, 8, 10$

$9, 12, 15$

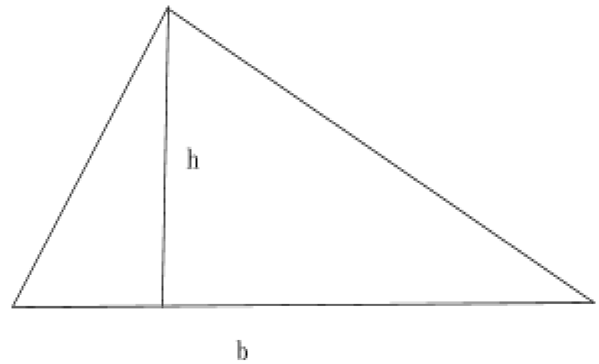
$12, 16, 20$

$15, 20, 25$

<u>Area of 2-D Shapes</u>

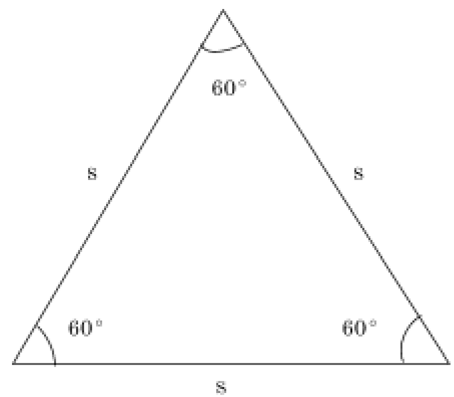
Area of a Triangle: Any triangle with base b and height h has an area of

$$\frac{bh}{2}$$



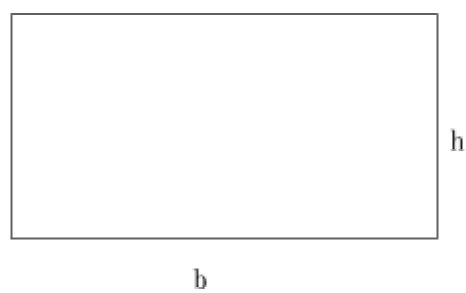
Area of an Equilateral Triangle:

$$\text{Area} = \frac{\sqrt{3}}{4}s^2$$



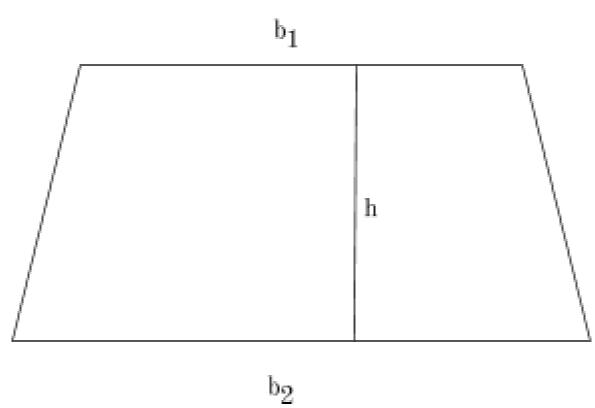
Area of a Rectangle: Any rectangle with base b and height h has an area of

$$bh$$



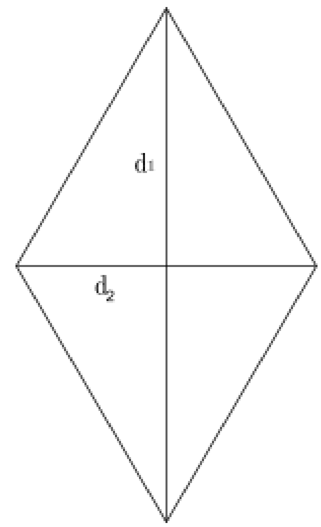
Area of a Trapezoid: A trapezoid with 2 bases b_1 and b_2 and a height h has an area of

$$\frac{b_1 + b_2}{2} \times h$$

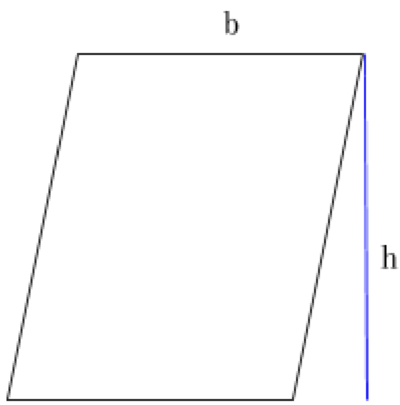


Rhombus: A rhombus with diagonals d_1 and d_2

$$\frac{d_1 d_2}{2}$$



Parallelogram: A parallelogram with base b and height h has an area of bh

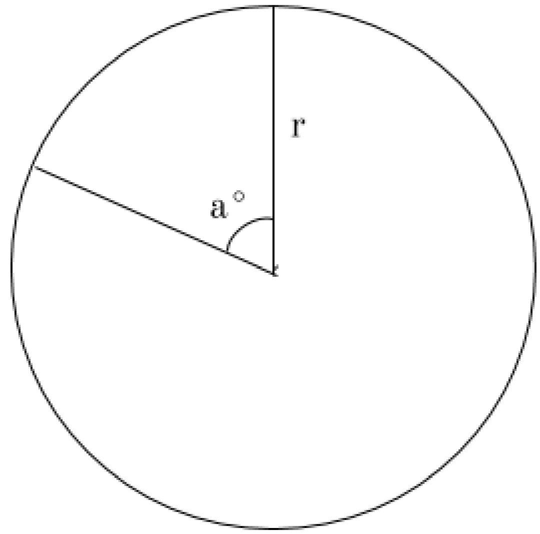


Circle:

A circle with radius r

$$\text{Area} = \pi r^2$$

$$\text{Circumference} = 2\pi r$$



Arcs of a circle: An arc of a circle with radius r and angle a°

$$\text{Area of a sector} = \pi r^2 \times \frac{a^\circ}{360}$$

$$\text{Length of the arc } = 2\pi r \times \frac{a^\circ}{360}$$

Finding Area of Complex Shapes

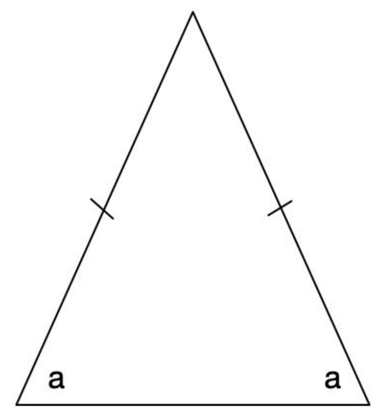
- Extend Lines
- Break up areas
- Look for “nicer” areas

Finding Length of Complex Shapes

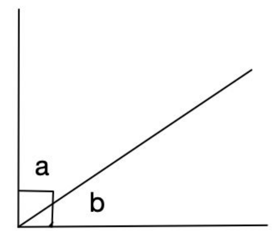
- Equal Lengths, Isosceles Triangle
- 90 degrees, use pythagorean theorem
- Split into multiple components

Angle Chasing

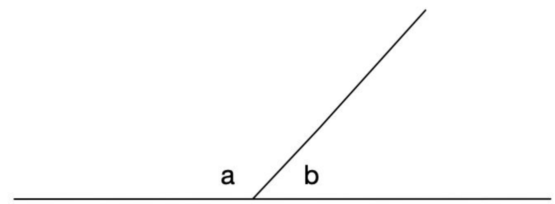
- Sum of Angles in Triangle is $180°$
- A triangle with 2 angles equal will have their corresponding sides equal and a triangle with 2 sides equal will have their corresponding angles equal (isosceles triangle)



Complementary Angle:

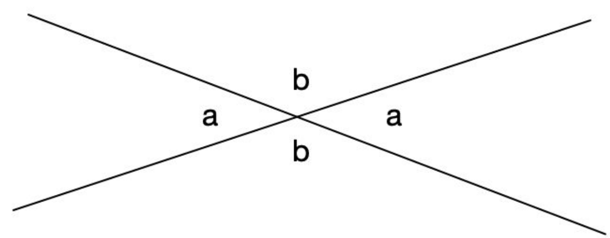


Supplementary Angle:



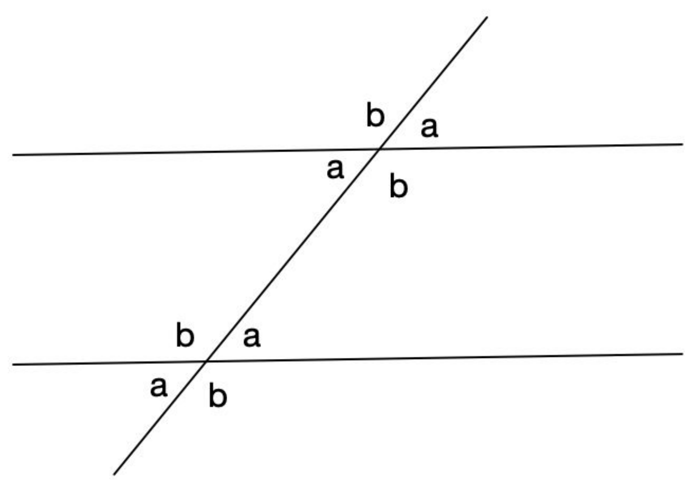
Intersecting lines:

Opposite angles equal

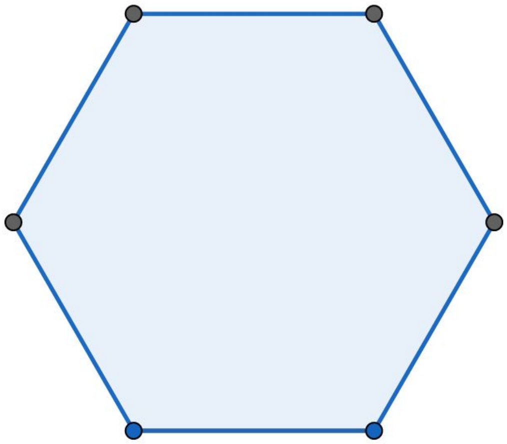


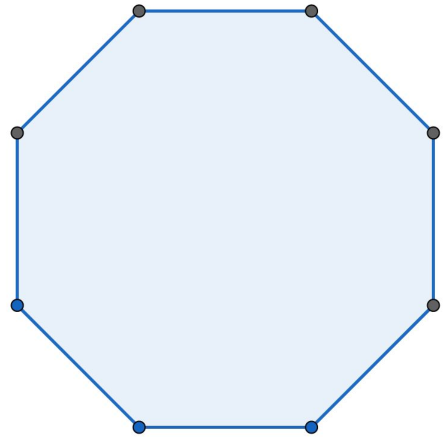
Parallel Lines:

Corresponding angles equal



Angles of a polygon:





Sum of interior angle of a polygon: $(n-2)\times 180$

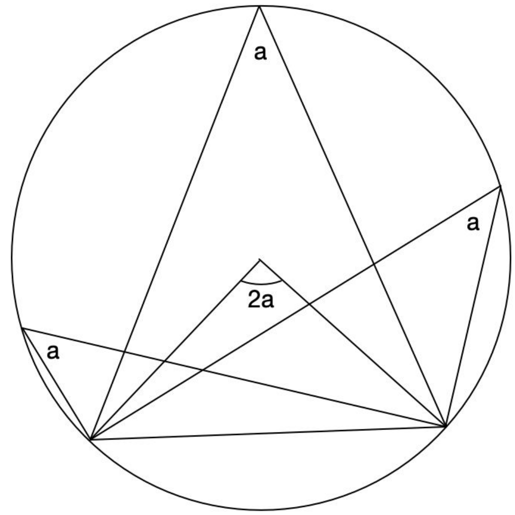
Interior angle of a regular polygon: $\frac{n-2}{n}\times 180$

Exterior angle of a regular polygon: $\frac{360}{n}$

Number of sides in regular polygon	**Interior angle of regular polygon**
3	60
4	90
5	108
6	120
8	135
9	140
10	144

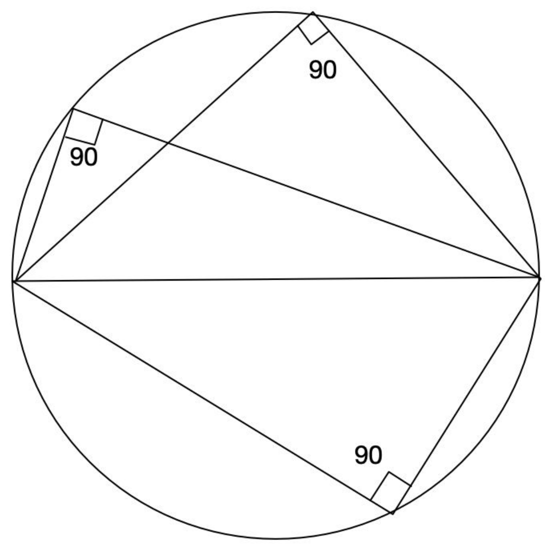
Circle Properties:

Angle formed by an arc in center double of the angle formed on the edge



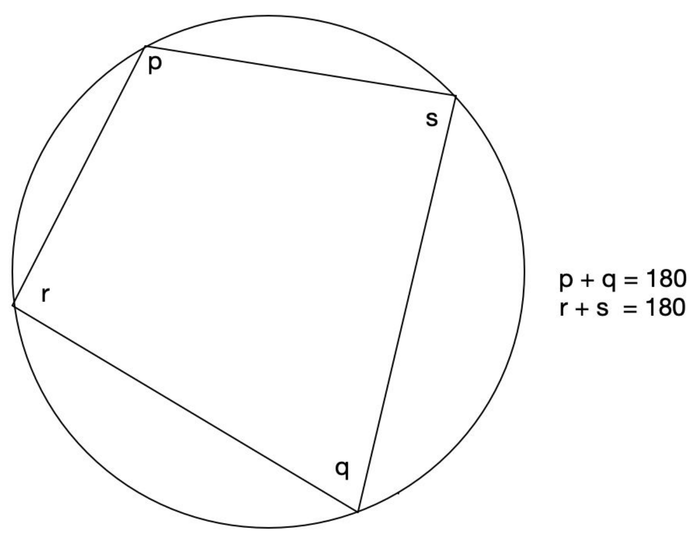
Inscribed triangle with diameter as one side

Always a right triangle

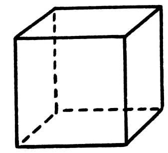
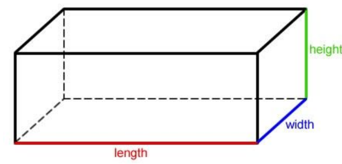
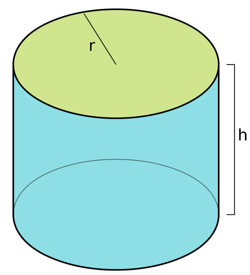
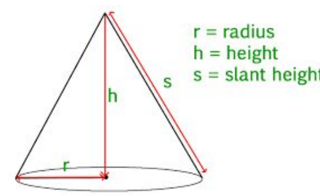
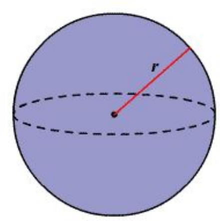
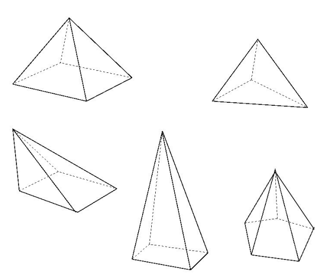


Cyclic quadrilateral:

sum of opposite angles 180

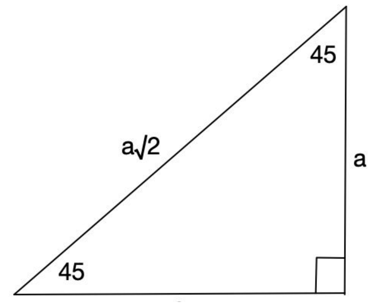


3D - Geometry

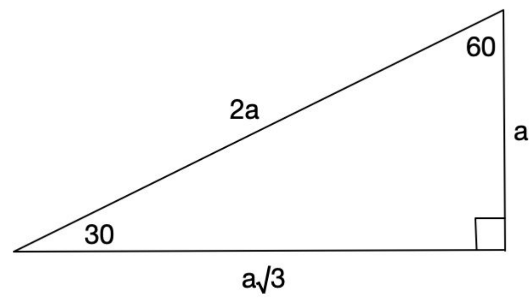
Cube	Surface Area $= 6s^2$ Volume $= s^3$	
Rectangular Prism	Surface Area $= 2(lw + wh + lh)$ Volume $= lwh$	
Cylinder	Surface Area $= 2(\pi r^2) + 2\pi rh$ $= 2\pi r(r + h)$ Volume $= r^2\pi h$	
Cone	Surface Area $= \pi r^2 + \pi rs$ $= \pi r(r + s)$ Volume $= \frac{1}{3}r^2\pi h$	
Sphere	Surface Area $= 4\pi r^2$ Volume $= \frac{4}{3}\pi r^3$	
Pyramid	$Volume = \dfrac{1}{3} \cdot \text{Area of base} \cdot \text{height}$	

Special Right Triangles

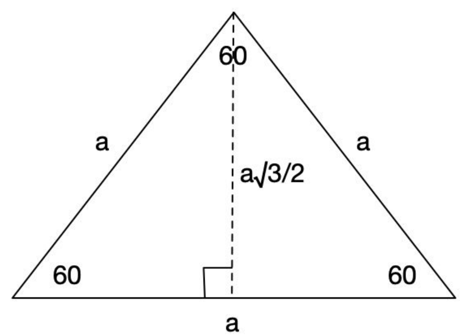
45-45-90 Triangle



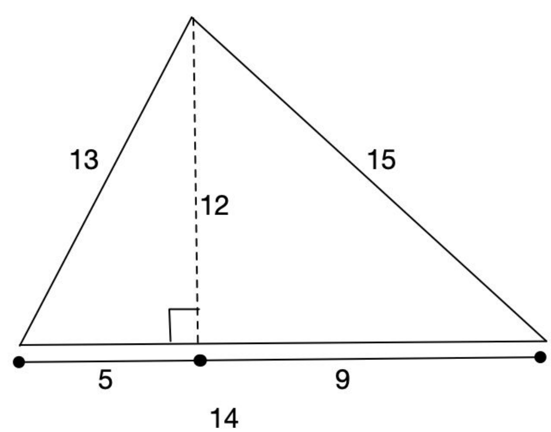
30-60-90 Triangle



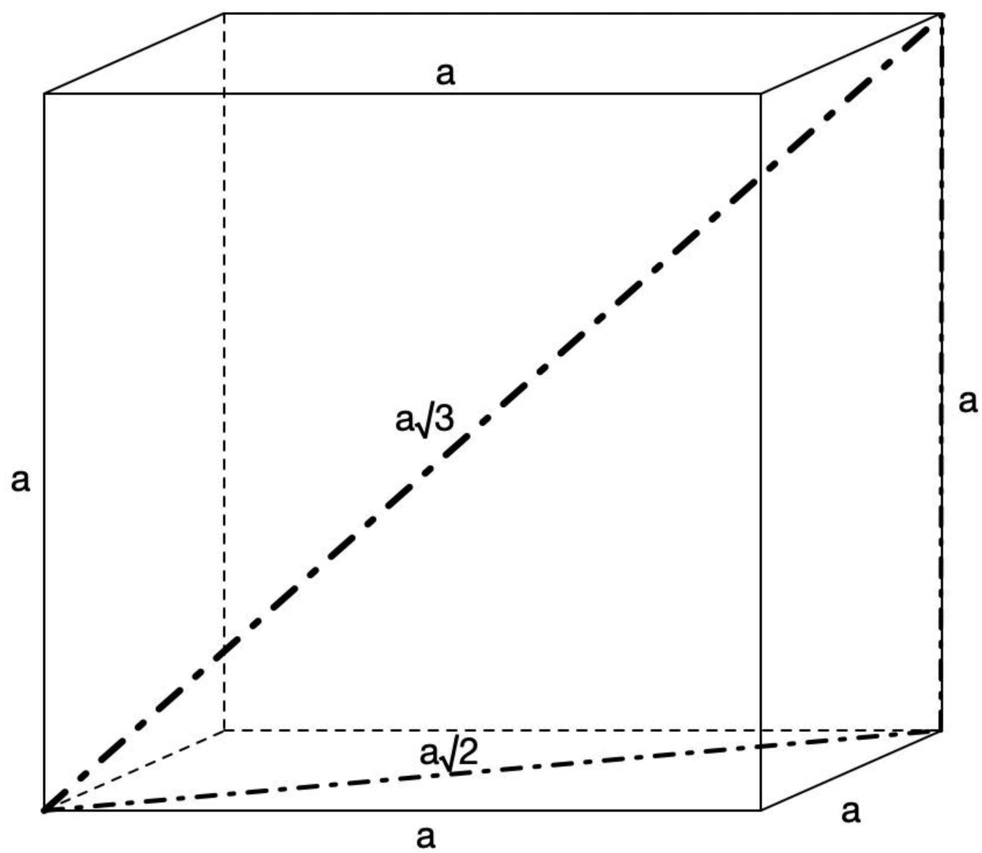
Equilateral (60-60-60) Triangle



13-14-15 Triangle



Cube Properties



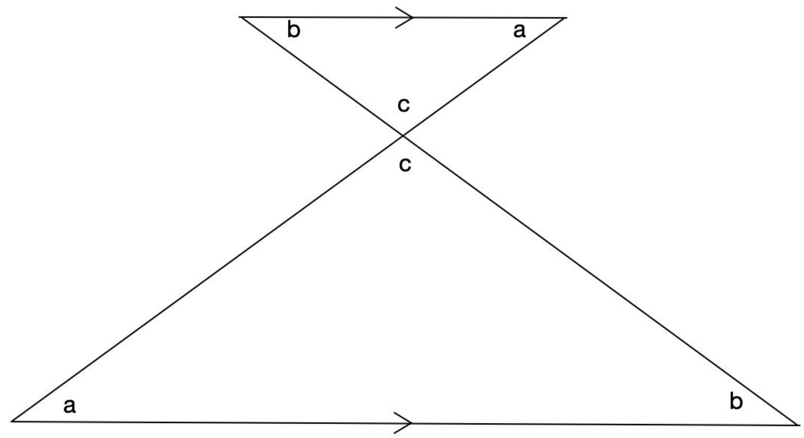
Similar Triangles

Triangles are similar if they are the same shape multiplied by a scale factor. In general, triangles are similar if

1) All angles of the triangle are the same (Or even just 2)
2) The bases of triangles are parallel and the side lengths of the triangles are in a line (see figure below)

For similar triangles:

- All the angles of the triangles are same
- All corresponding sides have same ratio
- Area ratio is the square of side length ratio



Test Taking Strategies

- Try solving the problem first
- Substitute answer choices
- Eliminate some choices (increase probability of guesses)
- Try to find patterns with smaller numbers

Guessing Strategies

- If you can't solve, here are some tips on guessing
- Sometimes you can eliminate
    - answer choices too large or too small
    - answer choices odd or even
    - answer choices divisible by 5, etc.
- In geometry problems, estimate the dimensions
    - Figures are not to scale, but pretty close
    - Graph paper, rulers, and protractors are allowed
- Make sure to mark an answer for every problem (20% chance of getting it right)

Tips to avoid Silly Mistakes

- After solving a problem, reread the question part of the problem to see if you are answering what the question is asking for
- If your answer doesn't match one of the option choices, check your work
- If you can see multiple ways to solve a problem, use alternate ways to validate answers
- After getting an answer, try plugging it into the question to make sure it's right
- Check if your answer is reasonable (i.e. if you get a car is traveling at 800 mph it probably isn't right)

Hope these tips and strategies will help you on the AMC 8...and other math competitions

Good Luck!