2019

AP® Calculus BC

Free-Response Questions

CALCULUS BC

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

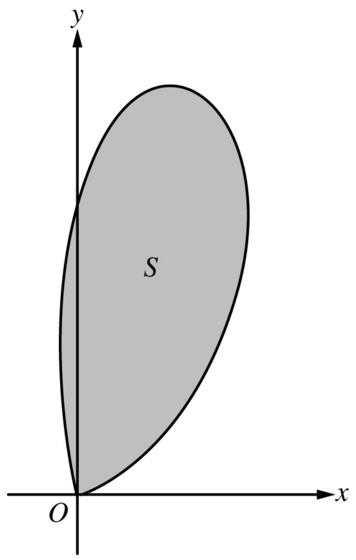
1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

   (a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)? Give your answer to the nearest whole number.

   (b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)?

   (c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

   (d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$)? Explain your reasoning.



2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

   (a) Find the area of S.

   (b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

   (c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

   (d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim_{k\to\infty} A(k)$.

END OF PART A OF SECTION II

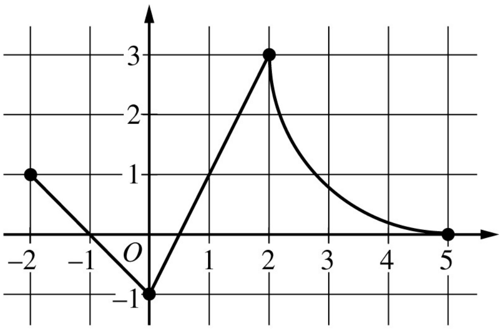
CALCULUS BC

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

   (a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

   (b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

   (c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

   (d) Find $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

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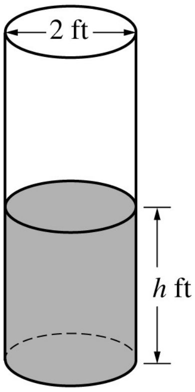


4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

   (a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

   (b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

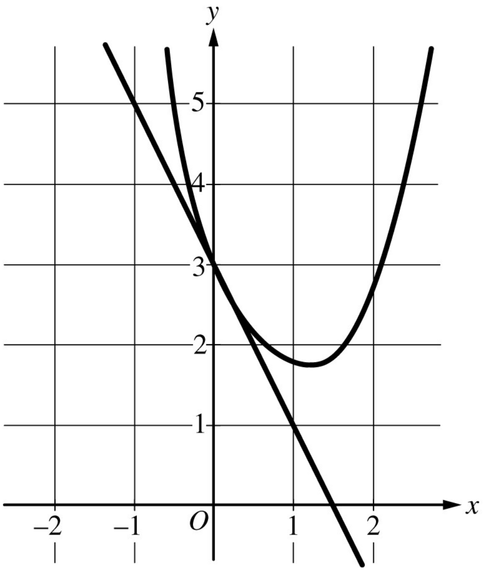
   (c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

   (a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

   (b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

   (c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

   (a) Write the third-degree Taylor polynomial for f about $x = 0$.

   (b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

   (c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

   (d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

STOP

END OF EXAM

2019

AP CollegeBoard

AP® Calculus BC

Scoring Guidelines

AP® CALCULUS AB/CALCULUS BC
2019 SCORING GUIDELINES

Question 1

(a) $\int_0^5 E(t)\,dt = 153.457690$

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{1}{5-0}\int_0^5 L(t)\,dt = 6.059038$

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) The rate of change in the number of fish in the lake at time t is given by $E(t) - L(t)$.

$E(t) - L(t) = 0 \Rightarrow t = 6.20356$

$E(t) - L(t) > 0$ for $0 \le t < 6.20356$, and $E(t) - L(t) < 0$ for $6.20356 < t \le 8$. Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

— OR —

Let $A(t)$ be the change in the number of fish in the lake from midnight to t hours after midnight.

$A(t) = \int_0^t (E(s) - L(s))\,ds$

$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$

t	$A(t)$
0	0
C	135.01492
8	80.91998

Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

$3 : \begin{cases} 1 : \text{sets } E(t) - L(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) $E'(5) - L'(5) = -10.7228 < 0$

Because $E'(5) - L'(5) < 0$, the rate of change in the number of fish is decreasing at time $t = 5$.

$2 : \begin{cases} 1 : \text{considers } E'(5) \text{ and } L'(5) \\ 1 : \text{answer with explanation} \end{cases}$

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2019 SCORING GUIDELINES

Question 2

(a) $\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta = 3.534292$

The area of S is 3.534.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{1}{\sqrt{\pi} - 0}\int_0^{\sqrt{\pi}} r(\theta) \, d\theta = 1.579933$

The average distance from the origin to a point on the curve $r = r(\theta)$ for $0 \le \theta \le \sqrt{\pi}$ is 1.580 (or 1.579).

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) $\tan\theta = \frac{y}{x} = m \;\Rightarrow\; \theta = \tan^{-1} m$

$$\frac{1}{2}\int_0^{\tan^{-1} m} (r(\theta))^2 \, d\theta = \frac{1}{2}\left(\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta\right)$$

$3 : \begin{cases} 1 : \text{equates polar areas} \\ 1 : \text{inverse trigonometric function applied to } m \text{ as limit of integration} \\ 1 : \text{equation} \end{cases}$

(d) As $k \to \infty$, the circle $r = k\cos\theta$ grows to enclose all points to the right of the y-axis.

$$\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\pi/2} (r(\theta))^2 \, d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2} \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2 \, d\theta = 3.324$$

$2 : \begin{cases} 1 : \text{limits of integration} \\ 1 : \text{answer with integral} \end{cases}$

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Question 3

(a) $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$

$$\Rightarrow 7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right)$$

$$\Rightarrow \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$$

3 : { 1 : $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$
1 : $\int_{-2}^{5} f(x)\,dx$
1 : answer }

(b) $\int_{3}^{5} (2f'(x) + 4)\,dx = 2\int_{3}^{5} f'(x)\,dx + \int_{3}^{5} 4\,dx$

$$= 2(f(5) - f(3)) + 4(5 - 3)$$
$$= 2\left(0 - \left(3 - \sqrt{5}\right)\right) + 8$$
$$= 2\left(-3 + \sqrt{5}\right) + 8 = 2 + 2\sqrt{5}$$

— OR —

$$\int_{3}^{5} (2f'(x) + 4)\,dx = [2f(x) + 4x]_{x=3}^{x=5}$$
$$= (2f(5) + 20) - (2f(3) + 12)$$
$$= (2\cdot 0 + 20) - \left(2\left(3 - \sqrt{5}\right) + 12\right)$$
$$= 2 + 2\sqrt{5}$$

2 : { 1 : Fundamental Theorem of Calculus
1 : answer }

(c) $g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \le x \le 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

3 : { 1 : $g'(x) = f(x)$
1 : identifies $x = -1$ as a candidate
1 : answer with justification }

(d) $\lim_{x\to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1}$

$$= \frac{10 - 3\cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}}$$

1 : answer

Question 4

(a) $V = \pi r^2 h = \pi (1)^2 h = \pi h$

$$\left.\frac{dV}{dt}\right|_{h=4} = \pi \left.\frac{dh}{dt}\right|_{h=4} = \pi\left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5} \text{ cubic feet per second}$$

$2: \begin{cases} 1: \dfrac{dV}{dt} = \pi \dfrac{dh}{dt} \\ 1: \text{answer with units} \end{cases}$

(b) $$\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$$

Because $\dfrac{d^2h}{dt^2} = \dfrac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

$3: \begin{cases} 1: \dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}} \\ 1: \dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} \\ 1: \text{answer with explanation} \end{cases}$

(c) $$\frac{dh}{\sqrt{h}} = -\frac{1}{10}\,dt$$

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$$

$$2\sqrt{h} = -\frac{1}{10}t + C$$

$$2\sqrt{5} = -\frac{1}{10}\cdot 0 + C \Rightarrow C = 2\sqrt{5}$$

$$2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$$

$$h(t) = \left(-\frac{1}{20}t + \sqrt{5}\right)^2$$

$4: \begin{cases} 1: \text{separation of variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration and uses initial condition} \\ 1: h(t) \end{cases}$

Note: $0/4$ if no separation of variables

Note: max $2/4$ [1-1-0-0] if no constant of integration

Question 5

(a) $f'(x) = \dfrac{-(2x-2)}{\left(x^2 - 2x + k\right)^2}$

$f'(0) = \dfrac{2}{k^2} = 6 \;\Rightarrow\; k^2 = \dfrac{1}{3} \;\Rightarrow\; k = \dfrac{1}{\sqrt{3}}$

$3 : \begin{cases} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{cases}$

(b) $\dfrac{1}{x^2 - 2x - 8} = \dfrac{1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$\Rightarrow\; 1 = A(x+2) + B(x-4)$

$\Rightarrow\; A = \dfrac{1}{6},\; B = -\dfrac{1}{6}$

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2} \right) dx$$

$$= \left[\frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2| \right]_{x=0}^{x=1}$$

$$= \left(\frac{1}{6}\ln 3 - \frac{1}{6}\ln 3 \right) - \left(\frac{1}{6}\ln 4 - \frac{1}{6}\ln 2 \right) = -\frac{1}{6}\ln 2$$

$3 : \begin{cases} 1 : \text{partial fraction decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$

(c) $$\int_0^2 \frac{1}{x^2 - 2x + 1}\,dx = \int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{b\to 1^+} \int_b^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{b\to 1^-} \left(-\frac{1}{x-1} \Big|_{x=0}^{x=b} \right) + \lim_{b\to 1^+} \left(-\frac{1}{x-1} \Big|_{x=b}^{x=2} \right)$$

$$= \lim_{b\to 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b\to 1^+} \left(-1 + \frac{1}{b-1} \right)$$

Because $\displaystyle\lim_{b\to 1^-} \left(-\frac{1}{b-1} \right)$ does not exist, the integral diverges.

$3 : \begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$

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Question 6

(a) $f(0) = 3$ and $f'(0) = -2$

The third-degree Taylor polynomial for f about $x = 0$ is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-\frac{23}{2}}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.$$

$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$

(b) The first three nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{1}{2!}x^2$.

The second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is

$$3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1 + x) + \frac{3}{2}x^2(1)$$
$$= 3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2$$
$$= 3 + x + x^2.$$

$2 : \begin{cases} 1 : \text{three terms for } e^x \\ 1 : \text{three terms for } e^x f(x) \end{cases}$

(c) $h(1) = \int_0^1 f(t)\,dt$

$$\approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt$$
$$= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_{t=0}^{t=1}$$
$$= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48}$$

$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(d) The alternating series error bound is the absolute value of the first omitted term of the series for $h(1)$.

$$\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \le \left|\frac{9}{20}\right| = 0.45$$

$3 : \begin{cases} 1 : \text{uses fourth-degree term of Maclaurin series for } f \\ 1 : \text{uses first omitted term of series for } h(1) \\ 1 : \text{error bound} \end{cases}$