AMC 10
MOCK TEST 8
Solution Book

Logic

ThrivingScholars

1. Two standard dice are placed on a table, with one on top of the other, so that only nine of the faces of the dice may be seen. The touching faces have the same number on them. The sum of the numbers on the visible faces is 33.

What is the number on the touching faces?

A 1 B 2 C 3 D 4 E 6

SOLUTION **B**

Let x be the number which is on the two touching faces. On a standard dice the numbers on opposite faces have sum 7. Hence the number on the bottom face which touches the table and hence is not visible is $7 - x$. Therefore the total of the numbers on the three faces that are not visible is $x + x + (7 - x) = 7 + x$.

The total of the numbers on all six faces of a standard dice is $1 + 2 + 3 + 4 + 5 + 6 = 21$.

It follows that the sum of the visible numbers is $2 \times 21 - (7 + x) = 35 - x$. Therefore $35 - x = 33$. Hence $x = 2$.

FOR INVESTIGATION

Suppose that three standard dice are placed on a table stacked one above the other. In this situation there are thirteen visible numbers.

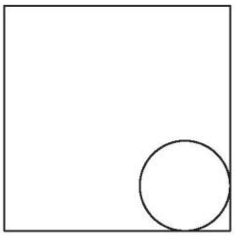
What is the highest possible total of the thirteen visible numbers?

2. The diagram shows a circle with radius 1 that rolls without slipping around the inside of a square with sides of length 5.

The circle rolls once around the square, returning to its starting point.

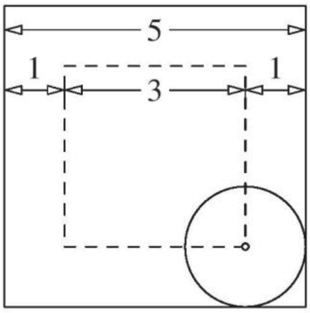
What distance does the centre of the circle travel?

A $16 - 2\pi$ B 12 C $6 + \pi$ D $20 - 2\pi$ E 20



SOLUTION **B**

As the circle rolls, its centre is always at a distance 1 from the square. Therefore, as shown in the diagram, the centre traces out a square whose side length is 2 less than the side length of the square. It follows that the centre travels a distance equal to the length of the perimeter of a square with side length 3.



We deduce that the distance that the centre travels is 12.

3. How many lines of three adjacent cells can be chosen from this grid, horizontally, vertically or diagonally, such that the sum of the numbers in the three cells is a multiple of three?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

A 30 B 24 C 18 D 12 E 6

Solution **B**

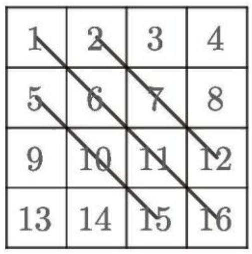
Suppose first that in a line of three numbers, adjacent numbers have the same difference. Let n be the first of these numbers, and d be the common difference. Then these numbers are n, $n + d$ and $n + 2d$. The sum of these numbers is $3n + 3d$ and therefore is a multiple of 3.

Adjacent numbers in each row have a common difference 1. Therefore, the sum of the numbers in three adjacent cells in the same row is always a multiple of 3.

There are two lines of three adjacent cells in each row, for example 1,2,3 and 2,3,4 in the top row.

Therefore, in the 4 rows there are $4 \times 2 = 8$ lines of three adjacent cells such that the sum of the numbers in these cells is a multiple of 3.

Adjacent numbers in each column differ by 4. Hence, it follows similarly, that there are 8 lines of three adjacent cells in the same column such that the sum of the numbers in these cells is a multiple of 3.



We now consider the diagonals from top left to bottom right. Adjacent numbers in these diagonals each column differ by 5. Therefore the sum of numbers in three adjacent cells in the same diagonal is always a multiple of 3.

One of these diagonals contains 4 numbers. There are 2 lines of three adjacent cells in this diagonal whose sum is a multiple of 3. There are 2 of these diagonals containing three numbers. Therefore, altogether, there are 4 lines of three adjacent numbers on theses diagonals whose sum is a multiple of 3.

Similarly, on the diagonals from top right to bottom left adjacent numbers have a common difference 3, and therefore there are 4 lines of three adjacent numbers on these diagonals whose sum is a multiple of 3.

Hence, in total, there are $8 + 8 + 4 + 4 = 24$ lines of three adjacent cells whose sum is a multiple of 3.

For investigation

- How many lines of three adjacent cells can be chosen from the grid, horizontally, vertically or diagonally, such that the sum of the numbers in the three cells is a multiple of six?
- How many lines of four adjacent cells can be chosen from the grid, horizontally, vertically or diagonally, such that the sum of the numbers in the four cells is a multiple of four?
- In how many ways can three different cells in any position, be chosen from the grid such that sum of the numbers in the three cells is a multiple of three?

4. Each of the factors of 100 is to be placed in a 3 by 3 grid, one per cell, in such a way that the products of the three numbers in each row, column and diagonal are all equal. The positions of the numbers 1, 2, 50 and x are shown in the diagram.

x	1	50
2		

What is the value of x?

A 4   B 5   C 10   D 20   E 25

Solution **D**

Let P be the common product of the three numbers in each row. Then $P \times P \times P$ is the product of all the numbers in all three rows. Therefore P^3 is the product of all the factors of 100.

Because $100 = 2^2 \times 5^2$, it has the nine factors 1, $2^1 = 2$, $2^2 = 4$, $5^1 = 5$, $2^1 5^1 = 10$, $2^2 5^1 = 20$, $5^2 = 25$, $2^1 5^2 = 50$ and $2^2 5^2 = 100$. The product of these factors is

$$\begin{aligned} 1 \times 2^1 \times 2^2 \times 5^1 \times 2^1 5^1 \times 2^2 5^1 \times 2^1 5^2 \times 5^2 \times 2^2 5^2 &= 2^{1+2+1+2+1+2} \times 5^{1+1+1+2+2+2} \\ &= 2^9 \times 5^9 \ (= 1\,000\,000\,000). \end{aligned}$$

It follows that

$$P^3 = 2^9 \times 5^9$$

and therefore $P = 2^3 \times 5^3 = 1000$.

From the first row we have

$$x \times 1 \times 50 = 1000$$

and hence $x = 20$.

Note

In the context of the SMC it is not necessary to check that it is possible to complete the grid with all the factors of 100, so as to meet the condition that the product of the three numbers in each row, column and diagonal are all equal. However, you are asked to do this in Exercise 6.1.

Note, also that our solution does not use the position of the factor 2. Exercise 6.1 asks you to show that with 2 in the bottom left-hand cell, there is just one way to complete the grid.

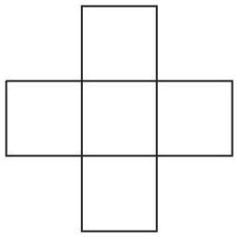
For investigation

- Show that it is possible to complete the grid with the factors of 100 so as to meet the required condition in just one way.
- How many ways are there to complete the grid if the factor 2 does not have to be in the bottom left-hand cell?
- Suppose that $n = p^2 q^2$, where p and q are different primes. Explain why n has nine factors.
- Suppose that $n = p^a q^b$, where p and q are different primes and a and b are non-negative integers. How many factors does n have?
- Find a general formula for the number of factors of a positive integer in terms of the exponents that occur in its prime factorization.

5. All the digits 2, 3, 4, 5 and 6 are placed in the grid, one in each cell, to form two three-digit numbers that are squares.

Which digit is placed in the centre of the grid?

A 2 B 3 C 4 D 5 E 6

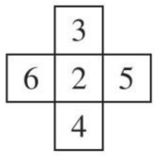


Solution **A**

The three-digit squares use just the digits 2, 3, 4, 5 and 6. The smallest square which uses these digits is $15^2 = 225$ and the largest is $25^2 = 625$. However, the squares that go in the grid cannot repeat digits. Therefore the only squares that we need consider are the three-digit squares that use three of digits 2, 3, 4, 5 and 6, once each. It may be checked (see Exercise 5.1, below) that there are just three squares which meet these conditions. They are $16^2 = 256$, $18^2 = 324$ and $25^2 = 625$.

The digit 2 occurs twice as the tens digit of these three squares. Therefore, the digit that goes in the central square of the grid is 2.

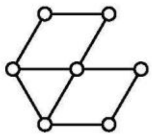
In the context of the SMC we could stop here. However, for a complete solution you would need to add that, as the squares 324 and 625 use each of the digits 3, 4, 5 and 6 just once, it is possible to place the digits in the grid so as to make these squares. The diagram shows one way to do this.



For investigation

- Check that 16^2, 18^2 and 25^2 are the only three-digit squares using just digits from the list 2, 3, 4, 5, 6, without any repeats.
- Arrange the digits 2, 4, 7, 8, 9 in the grid given in the question so as to form two three-digit squares.
- Arrange the digits 1, 2, 3, 4, 5 in the grid given in the question so as to form one three-digit square and one three-digit cube.
- Which three digits may be used to make three different three-digit squares?
- List all the three-digit squares. Which digits occurs least often and most often in this list?

6. The circles in the diagram are to be coloured so that any two circles connected by a line segment have different colours.

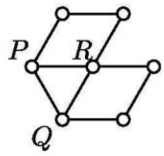


What is the smallest number of colours required?

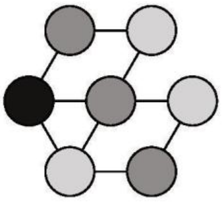
A 2 B 3 C 4 D 5 E 6

SOLUTION **B**

Each pair of the circles labelled P, Q and R in the figure on the right is connected by a line segment. Therefore these three circles must be coloured using different colours. So at least three colours are needed.



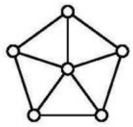
The figure on the right (with the circles enlarged for the sake of clarity) shows one way to colour the circles using three colours so that circles connected by a line segment have different colours.



Therefore 3 is the smallest number of colours required.

FOR INVESTIGATION

- In how many different ways is it possible to colour the circles in the diagram in the question, using the three colours red, green and blue, so that circles connected by a line segment have different colours?
- What is the smallest number of colours needed to colour the circles in the figure on the right so that circles connected by a line segment have different colours?



- It follows from the *Four Colour Theorem* that any arrangement of circles in the plane, connected by line segments that do not cross one another, may be coloured using at most four colours so that circles connected by a line segment have different colours.

  Find an arrangement of circles connected by line segments for which four colours are needed.

  What is the smallest number of circles in such an arrangement?

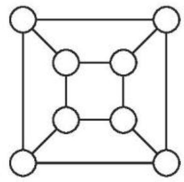
NOTE

The first proof of the Four Colour Theorem about maps drawn in the plane was published by Kenneth Appel and Wolfgang Haaken in 1977. Their proof reduced the general case to 1482 *unavoidable* configurations which needed to be checked separately. These configurations were generated and checked by a computer program. Since 1977 simpler proofs using a computer have been found. But no-one has yet found a proof which is simple enough for a human being to check it, just using pencil and paper, in a reasonable amount of time.

A good book on the Four Colour theorem is *Four Colours Suffice: How the Map Problem was Solved*, Robin Wilson, 2002.

7. Olive Green wishes to colour all the circles in the diagram so that, for each circle, there is exactly one circle of the same colour joined to it.

What is the smallest number of colours that Olive needs to complete this task?

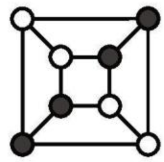


A 1 B 2 C 3 D 4 E 5

SOLUTION **B**

If Olive uses just one colour, then each circle would be joined to three circles with the same colour as it. So one colour is not enough.

However, as the diagram shows, it is possible using just two colours to colour the circles so that each white circle is joined to just one white circle, and each black circle is joined to just one black circle.

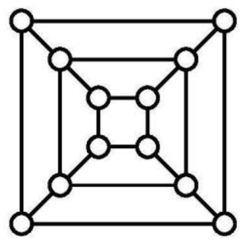


Therefore the smallest number of circles that Olive needs is two.

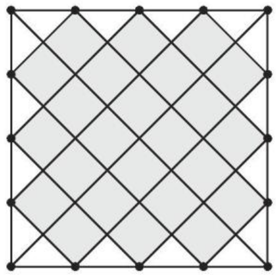
FOR INVESTIGATION

Cherry Red wishes to colour all the circles in the diagram so that for each circle, there is exactly one circle with the same colour joined to it.

What is the smallest number of colours that Cherry needs to complete this task?



8. Points are drawn on the sides of a square, dividing each side into n equal parts (so, in the example shown, $n = 4$). The points are joined in the manner indicated, to form several small squares (24 in the example, shown shaded) and some triangles.

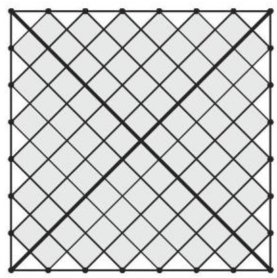


How many small squares are formed when $n = 7$?

A 56 B 84 C 140 D 840 E 5040

SOLUTION **B**

The main diagonals of the square shown in the question divide the square into four quarters each of which contains $1+2+3$ small squares. So the number of small squares when $n = 4$ is $4\times(1+2+3) = 4\times6 = 24$, as the question says.



It can be seen from the diagram on the right that, similarly, when $n = 7$ the number of small squares is given by $4 \times (1 + 2 + 3 + 4 + 5 + 6) = 4 \times 21 = 84$.

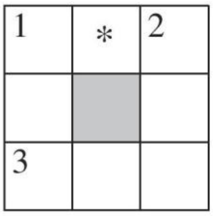
FOR INVESTIGATION

- What fraction of the area of the big square is covered by the small squares in the case when $n = 4$?
- What fraction of the area of the big square is covered by the small squares in the case when $n = 7$?
- Find a formula, in terms of n, for the number of small squares.
- Find a formula, in terms of n, for the fraction of the area of the square that is covered by the small squares. What happens to this fraction as n gets larger and larger?

9.

Across
1. A square
3. A fourth power

Down
1. Twice a fifth power
2. A cube



When completed correctly, the cross number is filled with four three-digit numbers.

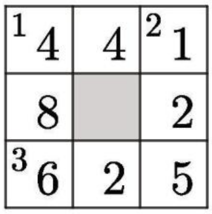
What digit is *?

A 0 B 1 C 2 D 4 E 6

Solution **D**

When you are faced with a crossnumber, the best strategy is to look for clues where it is easy to find a unique solution.

Among the three-digit integers there are more squares and cubes than fourth and fifth powers. So the best strategy is to begin with 1 Down and 3 Across.



The first few numbers that are twice fifth powers are $2 \times 1^5 = 2 \times 1 = 2$, $2\times2^5 = 2\times32 = 64$, $2\times3^5 = 2\times243 = 486$ and $2\times4^5 = 2\times1024 = 2048$. We can deduce from this that 486 which is the only three-digit number in this list is the answer for 1 Down.

It follows that 3 Across is a three-digit fourth power with 6 as its hundreds digit. We have $3^4 = 81$, $4^4 = 256$, $5^4 = 625$ and $6^4 = 1296$. We deduce that 3 Across is 625.

We now see that 2 Down is three-digit cube with units digit 5. Hence 2 Down is is $5^3 = 125$.

Finally, 1 Across is a three-digit square with hundreds digit 4 and units digit 1. Therefore 1 Across is $21^2 = 441$.

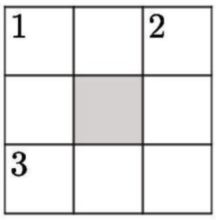
We can now deduce that * is 4.

For investigation

Complete this crossnumber in such a way that no two clues have the same answer.

Across
1. $3 \times$ a cube
3. $3 \times$ a square

Down
1. $3 \times$ a square
2. $3 \times$ a square



10. In the grid below each of the blank squares and the square marked X are to be filled by the mean of the two numbers in its adjacent squares.

Which number should go in the square marked X?

10			X		25

A 15 B 16 C 17 D 18 E 19

Solution **E**

We note first that if b is the mean of the numbers a and c, then $b = \frac{1}{2}(a + c)$. Hence $2b = a + c$ and therefore $b - a = c - b$. In other words, the numbers a, b, c form an arithmetic sequence. That is, their differences are equal.

We deduce that, because each number in the grid, other than 10 and 25, is the mean of the two adjacent numbers, the six numbers in the grid form an arithmetic sequence. Suppose that their common difference is d. Then the numbers in the grid will form the sequence $10, 10 + d, 10 + 2d, 10 + 3d, 10 + 4d, 10 + 5d$.

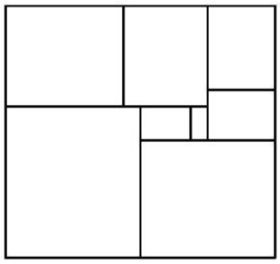
It follows that $10 + 5d = 25$. Hence $d = 3$.

The number which is in the square marked X is $10 + 3d$. Because $d = 3$, this number is 19.

11. The diagram shows an $n \times (n+1)$ rectangle tiled with $k \times (k+1)$ rectangles, where n and k are integers and k takes each value from 1 to 8 inclusive.

What is the value of n?

A 16 B 15 C 14 D 13 E 12



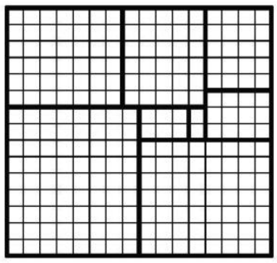
Solution **B**

The total area of the rectangles of size $k \times (k + 1)$, for $k = 1, 2, 3, 4, 5, 6, 7, 8$, is

$$\begin{aligned}1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + 5 \times 6 + 6 \times 7 + 7 \times 8 + 8 \times 9 &= 2 + 6 + 12 + 20 + 30 + 42 + 56 + 72 \\ &= 240 \\ &= 15 \times 16.\end{aligned}$$

Therefore $n = 15$.

In the context of the SMC the above calculation is sufficient to show that, if the smaller rectangles tile a rectangle of size $n \times (n + 1)$, for some integer n, then $n = 15$.



However, for a complete solution it is necessary to show that the eight smaller rectangles can be used to a tile a 15×16 rectangle.

It looks from the figure in the question that this is possible. The figure on the right confirms that the sizes of the rectangles are correct.

Note also that from this figure we can see directly that the large rectangle has size 15×16.

For investigation

(a) Find a formula in terms of s for the total area of the rectangles of size $k \times (k + 1)$ for all the integer values of k from 1 to s inclusive.

In other words, find a formula for the sum

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + s \times (s + 1).$$

[Note that using the Σ notation, we may write this sum as

$$\sum_{k=1}^{s} k \times (k + 1), \text{ or, suppressing the multiplication sign, as } \sum_{k=1}^{s} k(k + 1)].$$

(b) Check that your formula gives the answer 240 when $s = 8$.

(a) Can you find values of s, other than $s = 8$, such that for some integer n

$$\sum_{k=1}^{s} k(k + 1) = n(n + 1)?$$

(b) For the values of s that you have found in answer to part (a) is it possible to use the rectangles of size $k \times (k + 1)$, where k takes all integer values from 1 to s inclusive, to tile a rectangle of size $n \times (n + 1)$?

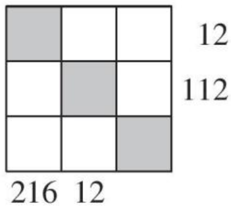
12. The digits from 1 to 9 are to be written in the nine cells of the 3×3 grid shown, one digit in each cell.

The product of the three digits in the first row is 12.

The product of the three digits in the second row is 112.

The product of the three digits in the first column is 216.

The product of the three digits in the second column is 12.



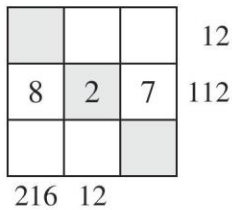
What is the product of the digits in the shaded cells?

A 24 B 30 C 36 D 48 E 140

Solution **B**

We note first that 112 is divisible by 7, but neither 216 nor 12 is divisible by 7. Therefore, the digit 7 is in the second row but not in the first or second columns. Hence the digit 7 is written in the cell in the second row and third column.

Because $112 = 7 \times 16$, the product of the other two digits in the second row is 16. Since the digits have to be different, these digits are 2 and 8 in some order. Because 8 is not a factor of 12, the digit 8 cannot be in the second column. Therefore 8 is written in the cell in the second row and first column, and 2 is written in the cell in the second row and second column.

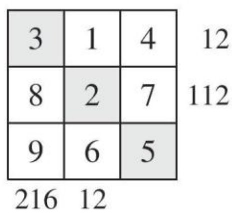


Therefore we know that the second row is as shown in the diagram.

Because $216 = 8 \times 27$, the product of the other two digits in the first column is 27. Hence these digits are 3 and 9. Because 9 is not a factor of 12, the digit 9 cannot be in the first row. Hence 9 is in the first column and third row, and 3 is in the first column and first row.

The product of the three digits in the second column is 12. One of these digits is 2. Hence, the product of the other two digits in the second column is 6. As 2 and 3 are already in the grid, these digits are 1 and 6.

The digit 6 cannot be in the first row, as otherwise the product of the digits in the first row would not be 12. It follows that the digit 1 is in the first row and second column, and 6 is in the third row and second column. The digit in the first row and third column is therefore 4. The remaining digit 5 is in the third row and third column.

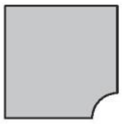


Therefore the digits are written as shown in the diagram. The digits in the shaded squares are 3, 2 and 5. The product of these digits is 30.

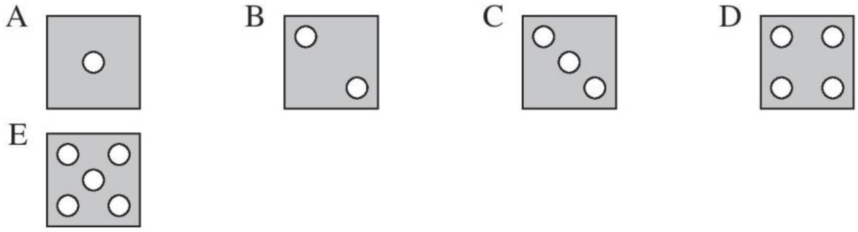
For investigation

- In the above argument, we showed that the digit in the third row and third column is 5 because it was the only digit we had not yet placed. However, it is possible to see that the digit 5 is in this cell before working out where any of the other digits go. Explain how.
- Invent more puzzles of a similar kind to that in this question.

13. A square of paper is folded in half four times to obtain a smaller square. Then a corner is removed as shown.



Which of the following could be the paper after it is unfolded?



Solution **D**

Each time the square piece of paper is folded in half the number of layers of paper doubles. Therefore, after it has been folded four times, there are 16 layers.

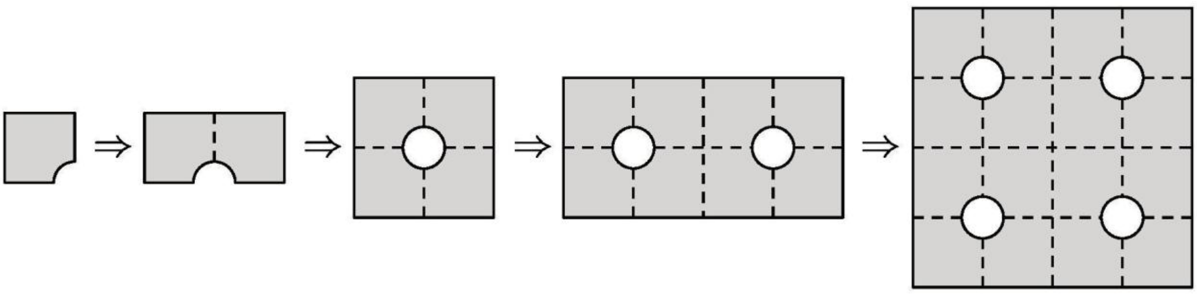
It follows that when the corner is then removed, altogether 16 quarter circles have been removed.

Hence, if these quarter circles come together to make complete circles after the paper has been unfolded, they will make four complete circles.

It follows that of the options given in the question, option D is the only one that is possible.

Note

In the context of the SMC it is sufficient to show that option D is the only possibility. For a complete answer it is also necessary to show that the pattern of option D can be achieved. This is shown by the diagram below. This shows that, provided none of the quarter-circles that are removed comes from the edge of paper, the paper will unfold to make the pattern of option D.



For investigation

What are the other possibilities for the paper after it has been unfolded?

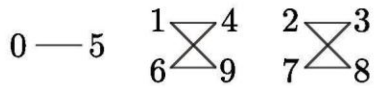
14. Alison has a set of ten fridge magnets showing the integers from 0 to 9 inclusive.

In how many different ways can she split the set into five pairs so that the sum of each pair is a multiple of 5?

A 1 B 2 C 3 D 4 E 5

Solution **D**

The number 0 can only be paired with 5.



The number 1 may be paired with 4 or with 9. If 1 is paired with 4, 6 has to be paired with 9. If 1 is paired with 9, 6 has to be paired with 4.

The number 2 may be paired with 3 or with 8. If 2 is paired with 3, 7 has to be paired with 8. If 2 is paired with 8, 7 has to be paired with 3.

These possibilities are shown in the diagram above. Thus the complete pairing is determined by first the choice which of 4 or 9 to pair with 1, giving two choices, and then the choice of which of 3 or 8 to pair with 2.

These choices are independent. It follows that there are $2 \times 2 = 4$ ways to split the set of numbers into five pairs so that the sum of each pair is a multiple of 5.

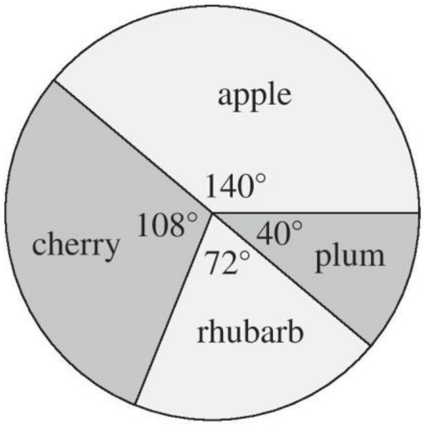
For investigation

- List the 4 different pairings that satisfy the condition that the sum of each pair is a multiple of 5.
- Bibi has a set of twenty fridge magnets showing the integers from 0 to 19, inclusive. In how many different ways can she split the set into ten pairs so that the sum of each pair is a multiple of 5?
- Chandra has a set of twenty-four fridge magnets showing the integers from 0 to 23, inclusive. In how many different ways can she split the set into twelve pairs so that the sum of each pair is a multiple of 5?

15. In a survey, people were asked to name their favourite fruit pie. The pie chart shows the outcome. The angles shown are exact with no rounding.

What is the smallest number of people who could have been surveyed?

A 45 B 60 C 80 D 90 E 180



SOLUTION **D**

Suppose that p people were surveyed.

The total of the angles is 360°. Therefore the proportion of the people who said that their favourite is apple pie is $\frac{140}{360} = \frac{7}{18}$. Hence the number who chose apple pie was $\frac{7}{18}p$. This is an integer. Therefore p is a multiple of 18.

Similarly, as the proportion who said their favourite is cherry pie is $\frac{108}{360} = \frac{3}{10}$, we deduce that p is a multiple of 10.

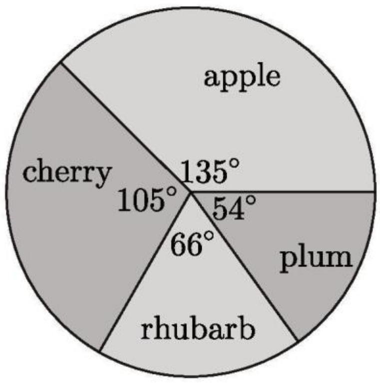
Likewise, as $\frac{72}{360} = \frac{1}{5}$ and $\frac{40}{360} = \frac{1}{9}$, we know that p is also a multiple of 5 and of 9.

Therefore the smallest possible value of p is the least common multiple of 18, 10, 5 and 9, which is 90. Hence the smallest number of people who could have been surveyed is 90.

FOR INVESTIGATION

The results of another survey about people's favourite fruit pies are shown in the pie chart on the right. Again, the angles are exact with no rounding.

What is the smallest number of people who could have been surveyed?



16. Alitta claims that if p is an odd prime then $p^2 - 2$ is also an odd prime.

Which of the following values of p is a counterexample to this claim?

A 3 B 5 C 7 D 9 E 11

Solution **E**

A counterexample to the claim is an odd prime p such that $p^2 - 2$ is *not* an odd prime.

3 is an odd prime, and $3^2 - 2 = 7$ is also an odd prime. So 3 is not a counterexample.

5 is an odd prime, and $5^2 - 2 = 23$ is also an odd prime. So 5 is not a counterexample.

7 is an odd prime, and $7^2 - 2 = 47$ is also an odd prime. So 7 is not a counterexample.

9 is not an odd prime. So 9 is not a counterexample

11 is an odd prime, but $11^2 - 2 = 119 = 7 \times 17$ is not an odd prime. Therefore 11 is a counterexample.

For investigation

11.1 For each of the following statements find a counterexample.

(a) If p is a prime, then $6p + 1$ is also a prime.

(b) If p is an integer with $p > 1$ and $6p + 1$ is a prime, then p is also a prime.

(c) If p is a prime, then $3^p + 20$ is also a prime.

(d) If p is a prime, then there is another prime between p and $p + 10$.

12. For how many positive integers N is the remainder 6 when 111 is divided by N?

A 5 B 4 C 3 D 2 E 1

Solution **A**

The remainder when 111 is divided by N is 6 provided that $111 = QN + 6$, where Q is a non-negative integer and $6 < N$. In other words, N is a factor of $111 - 6$ with $6 < N$.

Now $111 - 6 = 105$. The prime factorization of 105 is $3 \times 5 \times 7$. Therefore the factors of $111 - 6$ are 1, 3, 5, 7, 15, 21, 35 and 105.

Of these 8 factors all but 1, 3 and 5 are greater than 6.

Therefore there are 5 positive integers N which give a remainder 6 when 111 is divided by N.

For investigation

For how many positive integers N is the remainder 7 when 112 is divided by N?

17. Six friends Pat, Qasim, Roman, Sam, Tara and Uma, stand in a line for a photograph. There are three people standing between Pat and Qasim, two between Qasim and Roman and one between Roman and Sam. Sam is not at either end of the line.

How many people are standing between Tara and Uma?

A 4 B 3 C 2 D 1 E 0

SOLUTION **C**

We indicate each of the friends by the first letter of their name, and a person whose name we are not yet sure about by an asterisk (*).

We can assume, without loss of generality, that, from the point of view of the photographer, Qasim is to the right of Pat. Because there are three people standing between Pat and Qasim, the line is either

$$* \; P \; * \; * \; * \; Q \quad \text{or} \quad P \; * \; * \; * \; Q \; * \,.$$

There are two people between Qasim and Roman. Roman cannot be to the right of Quasim, because there is at most one friend to the right of Quasim. Therefore Roman is to the left of Quasim and the line is either

$$* \; P \; R \; * \; * \; Q \quad \text{or} \quad P \; R \; * \; * \; Q \; * \,.$$

There is one person between Roman and Sam. Therefore either Sam is immediately to the left of Pat, or immediately to the left of Qasim.

If Sam is immediately to the left of Pat, the line would be

$$S \; P \; R \; * \; * \; Q.$$

However this is impossible because Sam is not at either end of the line.

Therefore Sam is immediately to the left of Qasim. Hence the line is either

$$* \; P \; R \; * \; S \; Q \quad \text{or} \quad P \; R \; * \; S \; Q \; *$$

with Tara and Uma occupying the two places marked by asterisks.

We see that, however Tara and Uma occupy these two places, the number of people standing between Tara and Uma will be 2.

FOR INVESTIGATION

- Specify one additional piece of information that would make it possible to work out the exact order of the six friends from left to right, as seen by the photographer.

18. Fifty squares are drawn side by side in a line. The first and last squares are shaded. Othe squares in the line must be shaded such that both these rules apply: (a) no two adjacent squares are shaded and (b) there are no more than three consecutive unshaded squares.

What is the difference between the smallest and largest number of squares that can be shaded?

A 8 B 10 C 11 D 13 E 15

Solution **C**

We number the squares 1 to 50 in order from left to right so that we may refer to them.

Because of rule (a) at most one of each pair of consecutive squares can be shaded. So at most half, that is, 25, squares can be shaded. We can achieve this by shading all the odd numbered squares from 1 to 47 inclusive. Square 49 cannot be shaded because it is next to square 50 which is shaded. However, when the squares with odd numbers up to 47 and square 50 are shaded we have shaded 25 of the squares, as shown below.

Because of rule (b) in every block of four consecutive squares at least one must be shaded. So at least 12 of the squares from 2 to 49 must be shaded. This means that, together with squares 1 and 50, at least 14 squares must be shaded. One way to achieve this is to shade squares 1, 5, 9 and so on shading every fourth square up to square 45, and also squares 47 and 50, as shown below.

Therefore the largest number of squares that can be shaded is 25 and the smallest number is 14.

The difference between these two numbers is $25 - 14$ which is equal to 11.

For investigation

(a) In how many different ways is it possible to shade 25 squares, including squares 1 and 50, so that no two adjacent squares are shaded?

(b) In how many different ways is it possible to shade 25 squares so that no two adjacent squares are shaded, when the requirement that squares 1 and 50 must be shaded is dropped?

In how many different ways is it possible to shade 14 squares, including squares 1 and 50, so that there are no more than three consecutive unshaded squares?

If the requirement that squares 1 and 50 must be shaded is dropped, what is the smallest number of squares that need to be shaded so that there are no more than three consecutive unshaded squares?

19. Isobel: "Josh is innocent"   Genotan: "Tegan is guilty"

Josh: "Genotan is guilty"   Tegan: "Isobel is innocent"

Only the guilty person is lying; all the others are telling the truth.

Who is guilty?

A Isobel   B Josh   C Genotan   D Tegan
E More information required

SOLUTION **C**

There is only one guilty person, so either Genotan or Josh is lying. If Josh is lying, Genotan is innocent and is therefore telling the truth. Hence Tegan is guilty, contradicting the fact that there is just one guilty person. So Josh is not lying. Therefore Genotan is the guilty person.

FOR INVESTIGATION

- Check that the guilt of Genotan is consistent with all the information given in the question.

20. P, Q, R, S and T are the digits 1, 2, 3, 4 and 5 in some order. 'PRT' and 'QRS' are both three-digit primes.

Which digit is R?

A 1 B 2 C 3 D 4 E 5

Solution **B**

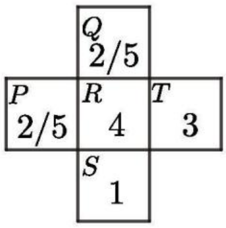
Of the available digits, only 1 and 3 can be the units digit of a three-digit prime. Without loss of generality we can assume that S is 1 and T is 3.

This leaves 2, 4 and 5 for the values of P, Q and R.

Suppose that R were 4. This would leave 2 and 5 for the values of P and Q.

Hence 'QRS' and 'PRT' would either be 241 and 543, or 541 and 243.

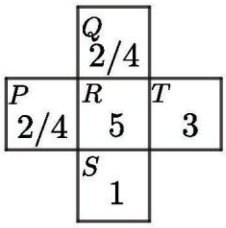
Neither of these is possible, as both 543 and 243 are divisible by 3, and hence they are not primes. So R cannot be 4.



Suppose next that R were 5. This would leave 2 and 4 for the values of P and Q. Hence 'QRS' and 'PRT' would either be 251 and 453, or 451 and 253.

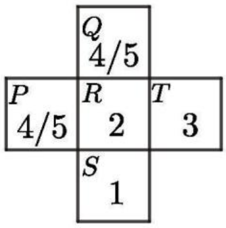
Neither of these is possible, as 453 is divisible by 3, and 253 is divisible by 11 and hence they are not primes. So R cannot be 5.

In the context of the SMC we can conclude that R is 2. [In Problem 14.1 you are asked to check that this is possible.]



For investigation

Show that when R is 2, the digits P, Q, S and T may be chosen to be 1, 3, 4 and 5 in some order, so that both 'QRS' and 'PRT' are primes.



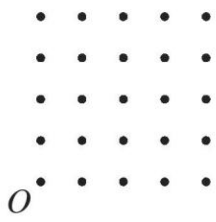
In how many way can P, Q, R, S and T be chosen to be the digits 2, 3, 4, 5 and 6 in some order so that 'PRT' and 'QRS' are both three-digit primes?

In how many way can P, Q, R, S and T be chosen to be the digits 3, 4, 5, 6 and 7 in some order so that 'PRT' and 'QRS' are both three-digit primes?

21. An array of 25 equally spaced dots is drawn in a square grid as shown. Point O is in the bottom left corner. Linda wants to draw a straight line through the diagram which passes through O and exactly one other point.

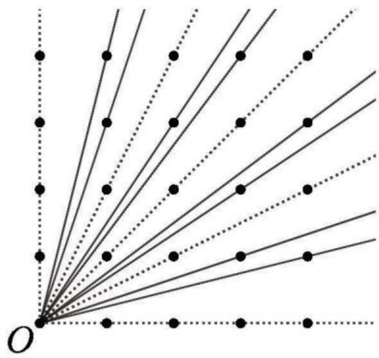
How many such lines can Linda draw?

A 4 B 6 C 8 D 12 E 24



Solution **C**

In the diagram on the right the solid lines go through O and exactly one other point, and the dotted lines go through O and at least two other points.



There is a line through every point so all possible lines have been considered.

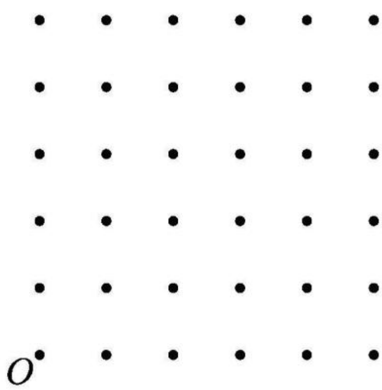
The solid lines are the lines that Linda can draw. We therefore see that the number of lines that Linda can draw is 8.

[*Note*: Because the diagram is symmetric about the bottom-left to top-right diagonal, it was only really necessary to draw half the lines in the diagram.]

For investigation

An array of 36 equally spaced dots is drawn in a square grid as shown. Mollie wants to draw a straight line which passes through the dot marked O and exactly one other dot.

How many of these lines can Mollie draw?



Naomi has a piece of paper on which are drawn 400 equally spaced dots in a square 20×20 grid.

Naomi wants to draw a straight line which passes through the bottom left-hand dot and exactly one other dot.

How many of these lines can Naomi draw?

Olivia has a piece of paper on which are drawn 10 000 equally spaced dots in a square 100×100 grid.

Olivia wants to draw a straight line which passes through the bottom left-hand dot and exactly one other dot.

How many of these lines can Olivia draw?

22. Five friends are dealt two cards each from a set of twelve cards. The cards are numbered 1 to 12 inclusive. In turn, the friends declare the sum of the values of their two cards. Paolo scores 4, Quinn scores 11. Romy scores 16. Stephen scores 19 and Thomas scores 20.

Which of the following statements is true?

A Paolo has card 2 B Quinn has card 3 C Romy has card 5
D Stephen has card 7 E Thomas has card 11

Solution **C**

The numbers on the cards are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12.

Since Paolo scored 4, he must have been dealt cards 1 and 3. It follows that both the statements A and B are false.

Because Paolo has cards 1 and 3, cards 2, 4, 5, 6, 7, 8, 9, 10, 11 and 12 are left for Quinn, Romy, Stephen and Thomas.

We now consider the possibility that statement C is true. In this case Romy's cards must be 5 and 11. This leaves cards 2, 4, 6, 7, 8, 9, 10 and 12 as possible cards held by Quinn, Stephen and Thomas.

Since Thomas scores 20 his cards must be 8 and 12. Then, as Stephen scores 19, his cards must be 9 and 10. This leaves 2, 4, 6 and 7. So Quinn's cards are 4 and 7.

We have therefore shown that it is possible for statement C to be true.

In the context of the SMC we can stop here. We can assume that only one of the statements could be true. So having shown that C could be true, we can conclude that C is the correct option.

For a complete answer it would be necessary to check that statements D and E cannot be true. You are asked to do this Problem 19.1.

For investigation

Determine which cards each of the five friends were dealt.

Hence, show that statements D and E are not true.

23.

Across
1. A multiple of 9
3. A square

Down
1. A multiple of 11
2. A multiple of 13 and of 19

1		2
3		

The crossnumber is to be filled with eight of the digits 1 to 9, which are each used once.

Which digit is not used?

A 9 B 8 C 5 D 3 E 2

Solution **B**

We begin with 2 Down because it looks as though there are fewer possible answers for this clue than for the other clues.

13 and 19 have no factors in common (other than 1). Hence a multiple of 13 and 19 is also a multiple of 13×19. That is, it is a multiple of 247.

The three-digit multiples of 247 are 247, 494, 741 and 988. 2 Down is not 247, because then 3 Across would have 7 as its units digit which is not possible for a square. 2 Down is neither 494 nor 988 because these numbers contain repeated digits. We may therefore deduce that 2 Down is 741.

It follows that 3 Across is a three-digit square with units digit 1. So the possibilities for 3 Across are $11^2 = 121$, $19^2 = 361$, $21^2 = 441$, $29^2 = 841$ and $31^2 = 961$.

We can rule out 121 and 441 because they have a repeated digit, and 841 because the digit 4 has already been used in 2 Down. This leaves 361 and 961 as possible values for 3 Across.

Suppose 3 Across is 361. Then the digits 2, 5, 8 and 9 have not yet been used.

For 1 Across to be a multiple of 9 its digits must add up to a multiple of 9. The units digit of 1 Across is 7, and its other digits are two of 2, 4, 8 and 9. The only possibilities are that the digits of 1 Across add up to 18, and that 1 Across is either 297 or 927.

Suppose 1 Across is 297. Then 1 Down is $2x3$ where x is either 5 or 8.

Now $253 = 11 \times 23$ and hence is a multiple of 11. Therefore we can complete the crossnumber as shown on the right.

In this completed crossnumber the digit that is not used is 8.

1 2	9	2 7
5		4
3 3	6	1

In the context of the SMC where we are entitled to assume that just one of the given options is correct, we can stop here now that we have found one solution. For a complete solution we would need to show that there is no other solution. You are asked to do this in Problem 21.1

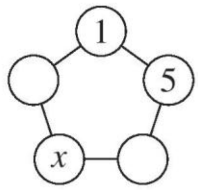
For investigation

Show that the solution given above is the only solution of the crossnumber.

24. The circles are filled with five integers so that any integer from 1 to 21 can be made, either by choosing one of the integers or by summing up to 5 adjacent integers.

When 1 and 5 are in the positions shown, what is the value of x?

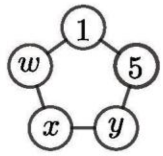
A 2 B 3 C 7 D 10 E 11



Solution **D**

We let w and y be the integers in the other two circles, as shown in the first diagram below.

There are just 21 ways in which we can use these integers to make an integer: each of the 5 integers can be used separately; there are 5 sums of two adjacent integers ($1 + w$, $w + x$, $x + y$, $y + 5$, $5 + 1$); 5 sums of three adjacent integers ($1 + w + x$, $w + x + y$, and so on); 5 sums of four adjacent integers ($1 + w + x + y$, and so on); and the sum, $1 + w + x + y + 5$, of all 5 of the integers.

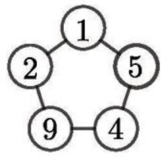


Because all the integers from 1 to 21 can be made, each of the 21 ways described above must have a different outcome. In particular, the integers in the circles must be different. It follow that 2 cannot be made from the sum $1 + 1$. Hence 2 is one of the integers in the circles.

We can also deduce that as the largest sum that can be made is 21, we have $1 + w + x + y + 5 = 21$ and hence $w + x + y = 15$.

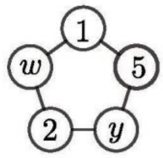
The number 2 must be either be w, x or y. We consider these cases in turn.

Suppose that w is 2. Then 4 cannot be made as $1 + 3$ or $2 + 2$, and must therefore be either x or y. It cannot be x since otherwise, two sums $5 + 1$ and $2 + 4$ would give 6. So 4 would have to be y. Then, as $w + x + y = 15$, we would have that x is 9, as shown on the right. However, in this case 7 cannot be made. We deduce that w is not 2.



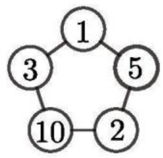
Next suppose that x is 2. Then 3 could not be made as the sum $1 + 2$. Hence either w or y is 3. However in each of these cases 5 would occur both as a single integer and as the sum $2 + 3$, so all the sums would not be different.

It follows that x is not 2. Therefore y is 2.



In this case 3 cannot be made as the sum $1 + 2$. Therefore 3 is either w or x. It cannot be x as 5 could then again be made in more than one way. So w is 3.

Since $w + x + y = 15$, it then follows that x is 10. This case is shown in diagram on the right. You are asked in Problem 17.1 to check that with this arrangement all the integers from 1 to 21 can be made.



For investigation

- Check that, in the situation of the final diagram above, all the integers from 1 to 21 can be made.
- Is there an other way to put five positive integers in the circles so that all the integers from 1 to 21 can be made?

25. Three friends make the following statements.

Ben says, “Exactly one of Dan and Cam is telling the truth.”

Dan says, “Exactly one of Ben and Cam is telling the truth.”

Cam says, “Neither Ben nor Dan is telling the truth.”

Which of the three friends is lying?

A Just Ben
B Just Dan
C Just Cam
D Each of Ben and Cam
E Each of Ben, Cam and Dan

SOLUTION **C**

If Cam’s statement is true, then both Ben and Dan are lying. But then exactly one of Ben and Cam is telling the truth. So Dan is telling the truth. This is a contradiction.

We deduce that Cam is lying.

Hence at least one of Ben and Dan is telling the truth.

If Ben’s statement is true, then exactly one of Ben and Cam is telling the truth. Hence Dan is telling the truth.

Similarly, if Dan’s statement is true, then Ben is telling the truth.

We deduce that Ben and Dan are telling the truth and that Cam is lying.

FOR INVESTIGATION

Four friends make the following statements.

Ben says, “Exactly one of Cam, Dan and Sam is telling the truth.”

Dan says, “Exactly one of Ben, Cam and Sam is telling the truth.”

Cam says, “Exactly one of Ben, Dan and Sam is telling the truth.”

Sam says, “None of Ben, Dan and Cam is telling the truth.”

What can you deduce about who is telling the truth and who is lying?