# TMUA MOCK TEST 3
# Solution Book

## Paper 1 Styled

### Calculus

ThrivingScholars

1. Compute the integral $\int_0^1 \frac{x-4}{\sqrt{x}(\sqrt{x}+2)} dx$.

A. $-3$
B. $-1$
C. $2$
D. $3$
E. $1$

**A**

Let $I = \int_0^1 \frac{x-4}{\sqrt{x}(\sqrt{x}+2)}\, dx.$

Substitute $t = \sqrt{x} \Rightarrow x = t^2,\ dx = 2t\, dt$ and limits $t : 0 \to 1$:

$$I = \int_0^1 \frac{t^2-4}{t(t+2)} \cdot 2t\, dt = \int_0^1 2\, \frac{(t-2)(t+2)}{t+2}\, dt = \int_0^1 2(t-2)\, dt.$$

$$I = 2\left[\frac{t^2}{2} - 2t\right]_0^1 = 2\left(\frac{1}{2} - 2\right) = -3.$$

2. Given $f(x) = \left(9x^2 + 12 + \frac{4}{x^2}\right)^{\frac{1}{2}}$ and $\frac{d^n f}{dx^n}(2) = -\frac{3}{4}$, find $n$.

A. 1
B. 2
C. 3
D. 4
E. 5

**C**

Note that the inside of the bracket is a perfect square and is simply $\left(3x + \frac{2}{x}\right)^2$. So $f(x) = 3x + \frac{2}{x}, f'(x) = 3 - \frac{2}{x^2}, f''(x) = \frac{4}{x^3}, f'''(x) = -\frac{12}{x^4}$, and so $f'''(2) = -\frac{3}{4}$.

3. For $p > 0$, find the area enclosed by the curves $y = px^2$ and $x = py^2$.

A. $\frac{1}{3p}$

B. $3p$

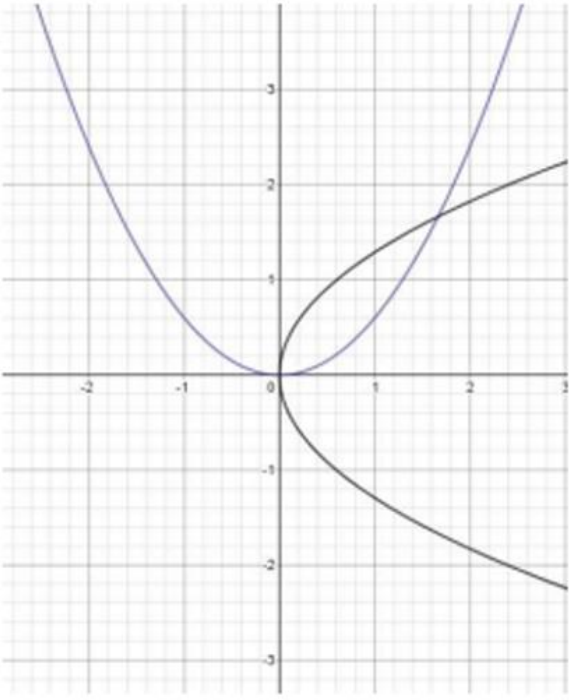
C. $\frac{1}{3p^2}$

D. $\frac{1}{2}p^2$

E. $3p$

**C**

A good sketch will help.



First find the intercept. At the intercept $x = py^2 = p(px^2)^2 = p^3x^4$, and so $x(p^3x^3 - 1) = 0$. Excluding the 0 solution, we see that $x = \frac{1}{p}$. Now we must compute the area, which is the difference of two integrals. We must invert $x = py^2$ to get $y = \sqrt{\frac{x}{p}}$, from which we compute the area as $\int_0^{1/p} \sqrt{\frac{x}{p}} - px^2 dx$, which one can compute to give $\frac{1}{3p^2}$.

4. What is the complete set of values for which $\frac{x^2+2x}{\sqrt{x^3}}$ is increasing?

A. $x > 3$
B. $x < 2$
C. $2 < x < 4$
D. $x > 2$
E. $0 < x < 2$

**D**

We can rewrite $\frac{x^2+2x}{\sqrt{x^3}}$ as $x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}$. Now differentiating we get $\frac{1}{2}x^{-\frac{1}{2}} - x^{-\left(\frac{3}{2}\right)}$. We want this to be positive, i.e. $\frac{1}{2\sqrt{x}} - \frac{1}{x\sqrt{x}} > 0 \to \frac{1}{2} - \frac{1}{x} > 0 \to x > 2$ .

5. Compute the shortest distance between the curves
$x^2 + 4x + y^2 + 6y + 10 = 0$ and $x^2 - 4x + y^2 - 8y + 12 = 0$

A. $\sqrt{65} - \sqrt{2} - \sqrt{3}$
B. $\sqrt{65} - 2\sqrt{2} - 2\sqrt{3}$
C. $\sqrt{65} - 2\sqrt{2} - \sqrt{3}$
D. $\sqrt{65}$
E. $\sqrt{65} - 4\sqrt{2} - 2\sqrt{3}$

**C**

These are clearly circles. Completing the square on both gives equations of the circles $(x-2)^2 + (y-4)^2 = 8$ and $(x+2)^2 + (y+3)^2 = 3$, so we have centre (2,4) radius $2\sqrt{2}$, and centre (-2,-3) radius $\sqrt{3}$. The shortest distance clearly lies on the line between the two centres, which has length $\sqrt{4^2 + 7^2} = \sqrt{65}$. Now subtract the two radii. So the answer is $\sqrt{65} - 2\sqrt{2} - \sqrt{3}$.

6. Given that $\frac{dV}{dt} = (1+t)^4$, and $V(1) = 5$, what is $V(2)$?

A. $\frac{174}{5}$

B. $\frac{236}{5}$

C. $\frac{112}{3}$

D. $\frac{89}{4}$

E. $\frac{17}{32}$

**B**

If you know the chain rule, the integration is slightly less tedious, but if not, by expanding and then integrating we see that $V(t) = \frac{t^5}{5} + t^4 + 2t^3 + 2t^2 + t + c$. $V(1) = 5$ gives $c = -\frac{6}{5}$. Then $V(2) = \frac{236}{5}$.

7. $y = \left(\frac{1}{3}k^3 + 1\right)x^3 - \left(2k^2 + 3\right)x^2 + \left(31 - 25k\right)x + 43$

has a turning point, that is a maximum, when $x = 1$, precisely for

**(a)** $-4 < k < 7$

**(b)** $k = 1$ or $7$

**(c)** $k > 7$

**(d)** $k = -4$ or $7$

**(e)** $k < 1$

**(f)** $k = -4$ or $1$

**(g)** $k < -4$

**(h)** $k = -4$, $1$ or $7$

**F**

Answer: **(f)** $k = -4$ **or** $1$.

Sketch:

- $y' = 3\left(\frac{1}{3}k^3 + 1\right)x^2 - 2(2k^2 + 3)x + (31 - 25k)$.
- Turning point at $x = 1 \Rightarrow y'(1) = 0$:

  $$k^3 - 4k^2 - 25k + 28 = 0 \;\Rightarrow\; (k-1)(k-7)(k+4) = 0$$

  so $k \in \{-4, 1, 7\}$.
- Maximum at $x = 1 \Rightarrow y''(1) < 0$. Since

  $$y''(1) = 6\left(\tfrac{1}{3}k^3 + 1\right) - 4k^2 - 6 = 2k^3 - 4k^2 = 2k^2(k-2),$$

  we need $k < 2$. This keeps $k = -4, 1$ and excludes $k = 7$.

Hence, $x = 1$ is a maximum precisely for $k = -4$ or $k = 1$.

8. Which of the following integrals has the largest value?

A $\displaystyle\int_0^{\frac{\pi}{3}} \cos\left(\frac{\pi}{2}-x\right) dx$

B $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos^2 x \, dx$

C $\displaystyle\int_{\frac{2\pi}{3}}^{\pi} -\sin\left(\frac{\pi}{2}-x\right) dx$

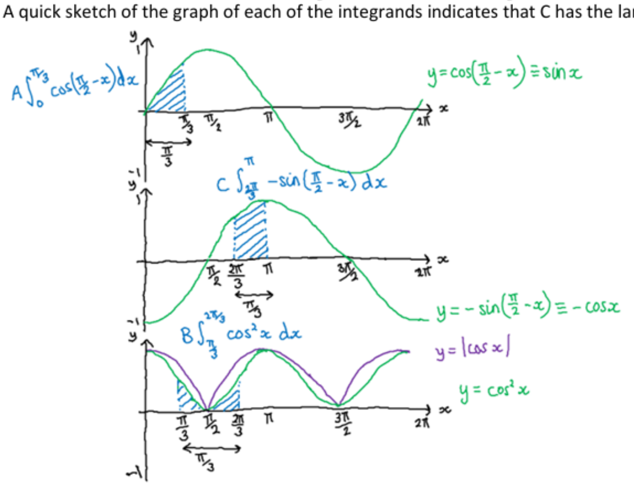
D $\displaystyle\int_0^{\pi} \sin\left(\frac{\pi}{2}-x\right) dx$

**C**

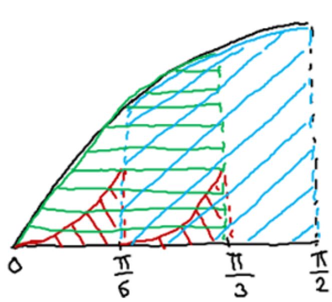
For A and C, the identity $\sin\left(\frac{\pi}{2}-\theta\right) \equiv \cos\theta$ helps.

$\int_0^{\frac{\pi}{3}} \cos\left(\frac{\pi}{2}-x\right) \, dx = \int_0^{\frac{\pi}{3}} \sin x \, dx$ and $\int_{\frac{2\pi}{3}}^{\pi} -\sin\left(\frac{\pi}{2}-x\right) \, dx = \int_{\frac{2\pi}{3}}^{\pi} -\cos x \, dx$.

A quick sketch of the graph of each of the integrands indicates that C has the largest value.




C has the largest area. From a sketch this is less obvious than from a drawing that is to scale. The following diagram may help, in which each of three areas have been placed inside the area represented by $\int_0^{\frac{\pi}{2}} \sin x \, dx$.



9. Find the value of $\int_{-1}^{1} \frac{(x^2 - 2)^2}{\sqrt[3]{x}} \, dx$

A 0

B 1

C 2

D 3

E $\frac{3}{7}$

F 12

**A**

You could compute this directly, although this is fiddly and it is easy to make a mistake. It is always worth making a quick check first of all to see if there is any other easier approach.

Let $f(x) = \frac{(x^2-2)^2}{\sqrt[3]{x}}$. Then $f(-x) = \frac{((-x)^2-2)^2}{\sqrt[3]{-x}} = \frac{(x^2-2)^2}{-\sqrt[3]{x}} = -f(x)$. Any function, $f(x)$, such that $f(-x) = -f(x)$, is called an odd function. Although this is not currently included on school syllabuses, it is helpful to know a bit about odd and even functions, and why they are useful.

In particular, the graph of any odd function, $y = f(x)$, since if $(x, y)$ is a point on the graph then so is $(-x, f(-x) = -f(x) = -y)$, has rotational symmetry of order two about the origin. This means that $\int_{-a}^{a} f(x)\ \mathrm{d}x = 0$ for any odd function and any real value of $a$, since the area to the right of the $y$-axis is the negative of the area to the left.

As soon as you realise the above, you instantly know that the answer must be zero. The answer is A.

The integral is actually an improper integral (consider what happens when $x = 0$ in $\frac{(x^2-2)^2}{\sqrt[3]{x}}$).

10. A curve has equation $y = (2a - x)(2a + x)$ where $a$ is a constant.

A different curve has the equation $y = (2a - x)(3a + x)$.

Find the value(s) of $a$ for which the maximum point on the first curve is closest to the maximum point on the second.

A $0$

B $\pm\frac{1}{2}$

C $\pm\frac{\sqrt{38}}{10}$

D $\pm 0.8$

E $\pm 1$

F $\pm\frac{2\sqrt{5}}{5}$

**A**

The first curve has equation $y = -x^2 + 4a^2$. The maximum point on this curve is $(0, 4a^2)$.
The second curve has equation $y = -x^2 - ax + 6a^2 = -\left(x + \frac{a}{2}\right)^2 + 6a^2 - \frac{a^2}{4} = -\left(x + \frac{a}{2}\right)^2 + \frac{23a^2}{4}$. The maximum point on this curve is $\left(-\frac{a}{2}, \frac{23a^2}{4}\right)$.
At this stage, you would normally be using Pythagoras' Theorem to find expressions for the distance between these two points and finding the value of $a$ that minimises this expression. However, pause and look at the pair of co-ordinates. Each ordinate is either always zero or a multiple of $a$, so, in particular, setting $a = 0$ would make all ordinates zero at the same time. Since when $a = 0$ the two points are the same (namely, the origin), it follows that the shortest possible distance between them is zero. The answer is A.

This is another good example of a question from which you benefit if you pause before diving into the usual method!

You could even have spotted this by considering the original equations with $a = 0$ as both being $y = -x^2$.

11. A function f is such that $\int_0^4 f(x)\ dx = 10$,

$\int_{-1}^4 f(x)\ dx = 13$ and $\int_{-4}^3 f(-x)\ dx = 2$.

The value of $\int_{-3}^0 f(x)\ dx$ is

A 2

B 3

C 5

D 8

E 11

F 12

**D**

The last of the three equations can also be written as $\int_{-3}^4 f(x)\ dx = 2$, since drawing the graph of $y = f(-x)$ from $x = -4$ to $x = 3$ is the same as drawing the graph of $y = f(x)$ from $x = 4$ to $x = -3$, so the areas are the same.

$$\int_{-1}^0 f(x)\ dx = 13 - 10 = 3$$

$$\int_{-3}^{-1} f(x)\ dx = \int_{-3}^4 f(x)\ dx - \int_{-1}^4 f(x)\ dx = 2 - 13 = -11$$

$$\int_{-3}^0 f(x)\ dx = \int_{-3}^{-1} f(x)\ dx + \int_{-1}^0 f(x)\ dx = -11 + 3 = -8$$

So

$$-\int_{-3}^0 f(x)\ dx = 8$$

Then $\int_{-3}^0 f(x)\ dx = \int_{-3}^4 f(x)\ dx - \int_0^4 f(x)\ dx = 2 - 10 = -8$. The answer is D.

12. The trapezium rule approximation for

$\int_0^6 |x(x-3)(x-6)|\ dx$ using four trapezia is equal to

A $\frac{3^5}{2^3}$

B $\left(\frac{3}{2}\right)^4$

C $\frac{3^5}{4}$

D $\frac{3^3}{2^5}$

E $\left(\frac{3}{2}\right)^5$

F 30

**A**

Let $y = |x(x-3)(x-6)|$.
When $x = 0$, $y = 0$.
When $x = \frac{3}{2}$, $y = \left|\frac{3}{2} \times -\frac{3}{2} \times -\frac{9}{2}\right| = \frac{81}{8}$.
When $x = 3$, $y = 0$.
When $x = \frac{9}{2}$, $y = \left|\frac{9}{2} \times \frac{3}{2} \times -\frac{3}{2}\right| = \frac{81}{8}$.
When $x = 6$, $y = 0$.
Applying the Trapezium Rule to $\int_0^6 |x(x-3)(x-6)|\ dx$, with four trapezia of equal width, gives:
$\frac{1}{2} \times \frac{3}{2}\left(0 + 2 \times \frac{81}{8} + 2 \times 0 + 2 \times \frac{81}{8} + 0\right) = \frac{3}{4} \times \frac{81}{2} = \frac{3^5}{2^3}$.
The answer is A.

13. Find the complete set of values of $k$ for which the cubic equation $2x^3 - 3x^2 - 36x + k = 0$ has three distinct real solutions.

**A** $x < 44$

**B** $x < 81$

**C** $-81 < x < 44$

**D** $-44 < x < 81$

**E** $x > 44$

**F** $x > 81$

**D**

Consider the function $y = 2x^3 - 3x^2 - 36x$. Then $\frac{dy}{dx} = 6x^2 - 6x - 36 \equiv 6(x^2 - x - 6) \equiv 6(x+2)(x-3) = 0$ precisely when $x = -2$ or $x = 3$.
If $x = -2$ then $y = -16 - 12 + 72 = 44$.
If $x = 3$ then $y = 54 - 27 - 108 = -81$. The two turning points are at $(-2, 44)$ and $(3, -81)$.
It follows that the equation
$2x^3 - 3x^2 - 36x + k = 0 \Leftrightarrow 2x^3 - 3x^2 - 36x = -k$ has three distinct real solutions when $-81 < -k < 44 \Leftrightarrow -44 < k < 81$. The answer is D.

14. Solve the differential equation, given that when $y = 5$ and $x = 3$. What is the value of the constant of integration?

$$\frac{dy}{dx} = \frac{(7x+3)y^{2/5}}{x}$$

A. $8(5)^{2/5}$

B. $\frac{1}{3}(5)^{8/5} - 21 - 3ln3$

C. $\frac{1}{8}(5)^{4/5} - 7 + ln3$

D. $\frac{1}{8}(5)^{4/5} - 21 - 3ln3$

E. $\frac{5}{3}(5)^{3/5} + 21 + ln3$

F. $-8(5)^{2/5}$

**B**

$$\frac{dy}{dx} = \frac{(7x+3)y^{\frac{2}{5}}}{x}$$

$$\int y^{-\frac{2}{5}}\,dy = \int \frac{(7x+3)}{x}\,dx$$

$$\frac{5}{3}y^{\frac{3}{5}} = 7x + 3lnx + c$$

When y = 5, x = 3:

$$\frac{5}{3}(5)^{\frac{3}{5}} = 21 + 3ln3 + c$$

Here, we can add the 5 from the 5/3 into the indices, as $5 = 5^{\frac{5}{5}}$,

therefore $\frac{5}{3}(5)^{\frac{3}{5}} = \frac{1}{3}(5)^{\frac{8}{5}}$

$$C = \frac{1}{3}(5)^{8/5} - 21 - 3ln3$$

15. What is the value of the integral below?

$$\int_0^2 |x-1|(3\sqrt{x}-x\sqrt{x})dx$$

A. $\frac{48-4\sqrt{2}}{35}$

B. $\frac{4\sqrt{2}}{35}$

C. $\frac{24}{35}$

D. $\frac{48+4\sqrt{2}}{35}$

E. $\frac{24+4\sqrt{2}}{35}$

F. $\frac{24-4\sqrt{2}}{35}$

**D**

With integrals with modulus signs in, we typically split the integral range into the parts where the modulus function is defined properly i.e.

$$\int_0^2 |x-1|(3\sqrt{x}-x\sqrt{x})dx$$

$$= \int_0^1 (1-x)(3\sqrt{x}-x\sqrt{x})dx + \int_1^2 (x-1)(3\sqrt{x}-x\sqrt{x})dx$$

You can notice here that the integrand is identical except for a minus sign in the 2 integrals on the right. This means we don't have to integrate two different functions, only 1, and change what we substitute in as limits.

If we take the first integral, $\int_0^1 (1-x)(3\sqrt{x}-x\sqrt{x})dx = \int_0^1 x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + 3x^{\frac{1}{2}}\,dx = \left[\frac{2x^{\frac{7}{2}}}{7} - \frac{8x^{\frac{5}{2}}}{5} + 2x^{\frac{3}{2}}\right]_0^1$. But we then know that the second integral is just $\left[\frac{2x^{\frac{7}{2}}}{7} - \frac{8x^{\frac{5}{2}}}{5} + 2x^{\frac{3}{2}}\right]_2^1$. So, we simply calculate $2\left(\frac{2}{7} - \frac{8}{5} + 2\right) - \left(\frac{2\times 2^{\frac{7}{2}}}{7} - \frac{8\times 2^{\frac{5}{2}}}{5} + 2\times 2^{\frac{3}{2}}\right) = 2\left(\frac{24}{35}\right) + \sqrt{2}\left(\frac{4}{35}\right) = \frac{48+4\sqrt{2}}{35}$.

16. For a function $f(x)$, it is given that $\left(\int_0^2 f(x)dx\right)\left(\int_0^1 f(x)dx + 4\right)$=24. You are also told that, for this function, $f(x+1) = f(1-x)$.

Which of the following could the value of $\int_1^2 f(x)dx$ be?

I. $-6$

II. $8$

III. $2$

A. None

B. I only

C. II only

D. III only

E. I & II only

F. I & III only

G. II & III only

H. All three

**F**

The function is symmetric about $1$, which means $\int_1^2 f(x)dx = \int_0^1 f(x)dx$. This implies that $\int_0^2 f(x)dx = 2\int_0^1 f(x)dx$. Making this substitution, we get a quadratic in $\int_0^1 f(x)dx$ with $\left(\int_0^1 f(x)dx\right)^2 + 4\int_0^1 f(x)dx - 12 = 0 = (\int_0^1 f(x)dx + 6)(\int_0^1 f(x)dx - 2)$ . This gives us $\int_0^1 f(x)dx = \int_1^2 f(x)dx = 2$ or $\int_1^2 f(x)dx = -6$.

17. $\frac{dx}{dt} = 3\left(\frac{1}{1-\sin t} + \frac{1}{1+\sin t}\right)$. If $x = 0$ when $t = \frac{\pi}{6}$, find $x$ when $t = \frac{\pi}{3}$. [the integral of $\frac{1}{\cos^2 t}$ *is* $\tan t$]

A. $4\sqrt{3}$
B. $6\sqrt{3}$
C. $\frac{6}{\sqrt{3}}$
D. $3\sqrt{3}$
E. $\sqrt{3}$
F. $\frac{4}{\sqrt{3}}$
G. $12\sqrt{3}$

**A**

The right-hand side can be converted, by combining the fractions, into $\frac{6}{1-\sin^2 t}$, which is $6\sec^2 t$. Then, by integrating, we get $x = 6\tan t + A$.

Using the initial condition, $A = -\frac{6}{\sqrt{3}}$. Then, substituting $x = 6\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)$ at $t = \frac{\pi}{3}$. This simplifies to $4\sqrt{3}$, by combining and rationalising the denominator.

18. Given that

$$\int_0^4 \left(x^3 + ax^2 + bx + 1\right) dx = 0,$$

and that $a \le 0,\ b \le 0$, what is the maximum value of

$$\int_0^3 \left(x^3 + ax^2 + bx + 1\right) dx\ ?$$

A. $-15$

B. $15$

C. $100$

D. $-\dfrac{87}{16}$

E. $\dfrac{123}{2}$

F. $-\dfrac{123}{2}$

**D**

From $\int_0^4 (x^3 + ax^2 + bx + 1)\, dx = 0$ we get
$b = -\frac{17}{2} - \frac{8}{3}a.$

$\int_0^3 (x^3 + ax^2 + bx + 1)\, dx = -15 - 3a.$

Constraints $a \le 0,\ b \le 0$ give $a \in \left[-\frac{51}{16}, 0\right]$.
Since the expression decreases with $a$, max at $a = -\frac{51}{16}$, $b = 0$

$$\boxed{-\frac{87}{16}}$$

19. Let $f(x) = \frac{\left(x^2 - \frac{1}{x}\right)^2}{x^{\frac{1}{3}}}$. It is given also that

$$f'(x) = \frac{x^{\frac{2}{3}}}{3}[ax^p + bx^q + cx^r]$$

What is the value of $abc + pqr$?

A. 388
B. $\frac{520}{3}$
C. 288
D. 316
E. 300
F. $-80$

**D**

$f(x) = \frac{x^4 - 2x + x^{-2}}{x^{\frac{1}{3}}} = x^{\frac{11}{3}} - 2x^{\frac{2}{3}} + x^{-\frac{7}{3}}$. Then, $f'(x) = \frac{11x^{\frac{8}{3}}}{3} - \frac{4x^{-\frac{1}{3}}}{3} - \frac{7x^{-\frac{10}{3}}}{3} = \frac{x^{\frac{2}{3}}}{3}[11x^2 - 4x^{-1} - 7x^{-4}]$. Thus, $abc + pqr = 11(-4)(-7) + 2(-1)(-4) = 316$.

20. Consider the function:

$g(x) = \int_0^x (t-p)\, f(t)\, dt$

where $0 \le p \le 7$.

Find:

$$\int_0^7 |x-p|\, f(x)\, dx$$

A. $g(7) - g(0)$
B. $g(7) - p\,g(0)$
C. $g(0) - p\,g(7)$
D. $2g(0) - g(7) + g(p)$
E. $g(7) + g(0) - 2g(p)$
F. $g(p) + g(7) - 2g(0)$

**E**

- Key fact: $g'(x) = (x-p)f(x)$, so
  $$\int_a^b (x-p)f(x)\,dx = g(b) - g(a).$$
- Think of $g$ as the *signed* accumulation of $(x-p)f(x)$ about the pivot $p$. On $[0,p]$ the factor $(x-p) < 0$; on $[p,7]$ it's $\ge 0$.
- The absolute value just *flips the sign* of the left-hand part:
  $$\int_0^7 |x-p| f(x)\,dx = -\int_0^p (x-p)f(x)\,dx + \int_p^7 (x-p)f(x)\,dx.$$
- Convert each piece with $g$:
  $$-\int_0^p (x-p)f(x)\,dx = -(g(p) - g(0)) = g(0) - g(p),$$
  $$\int_p^7 (x-p)f(x)\,dx = g(7) - g(p).$$
- Add them:
  $$\int_0^7 |x-p| f(x)\,dx = [g(0) - g(p)] + [g(7) - g(p)] = g(7) + g(0) - 2g(p).$$