TMUA MOCK TEST 6
Solution Book

Paper 2 Styled

- All Topics

ThrivingScholars

Question 1

Which of the following is equivalent to "If P is true, then Q is false."?

(A) "P is true or Q is false."

(B) "If Q is false then P is true."

(C) "If P is false then Q is true."

(D) "If Q is true then P is false."

(E) "If Q is true then P is true."

Answer D

Remember that a statement is logically equivalent to its contrapositive, which is formed by first negating the hypothesis and conclusion and then switching them. In this case, the contrapositive of "If P is true, then Q is false." is "If Q is true, then P is false." $\boxed{D}$

Question 2

The remainder can be defined for all real numbers x and y with $y \neq 0$ by

$$\text{rem}(x, y) = x - y\left\lfloor \frac{x}{y} \right\rfloor$$

where $\left\lfloor \frac{x}{y} \right\rfloor$ denotes the greatest integer less than or equal to $\frac{x}{y}$. What is the value of $\text{rem}\left(\frac{3}{8}, -\frac{2}{5}\right)$?

(A) $-\frac{3}{8}$ **(B)** $-\frac{1}{40}$ **(C)** 0 **(D)** $\frac{3}{8}$ **(E)** $\frac{31}{40}$

Answer B

The value, by definition, is

$$\begin{aligned}
\text{rem}\left(\frac{3}{8}, -\frac{2}{5}\right) &= \frac{3}{8} - \left(-\frac{2}{5}\right)\left\lfloor \frac{\frac{3}{8}}{-\frac{2}{5}} \right\rfloor \\
&= \frac{3}{8} - \left(-\frac{2}{5}\right)\left\lfloor \frac{3}{8} \times \frac{-5}{2} \right\rfloor \\
&= \frac{3}{8} - \left(-\frac{2}{5}\right)\left\lfloor \frac{-15}{16} \right\rfloor \\
&= \frac{3}{8} - \left(-\frac{2}{5}\right)(-1) \\
&= \frac{3}{8} - \frac{2}{5} \\
&= \boxed{\textbf{(B)}\ -\frac{1}{40}}.
\end{aligned}$$

Question 3

Five cards are lying on a table as shown.

	P	Q
3	4	6

Each card has a letter on one side and a whole number on the other side. Jane said, "If a vowel is on one side of any card, then an even number is on the other side." Mary showed Jane was wrong by turning over one card. Which card did Mary turn over? (Each card number is the one with the number on it. For example card 4 is the one with 4 on it, not the fourth card from the left/right)

(A) 3 (B) 4 (C) 6 (D) P (E) Q

Answer A

Logically, Jane's statement is equivalent to its contrapositive,

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If an even number is not on one side of any card, then a vowel is not on the other side.
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For Mary to show Jane wrong, she must find a card with an odd number on one side, and a vowel on the other side. The only card that could possibly have this property is the card with 3, which is answer choice $\boxed{\textbf{(A)}}$

Question 4

Alan, Beth, Carlos, and Diana were discussing their possible grades in mathematics class this grading period. Alan said, "If I get an A, then Beth will get an A." Beth said, "If I get an A, then Carlos will get an A." Carlos said, "If I get an A, then Diana will get an A." All of these statements were true, but only two of the students received an A. Which two received A's?

(A) Alan, Beth (B) Beth, Carlos (C) Carlos, Diana

(D) Alan, Diana (E) Beth, Diana

Answer C

Let's say that Alan gets an A. Well, from his statement, then Beth would also get an A. But from her statement, Carlos would get an A. And from his statement, Diana would also get an A. So all 4 would get A's, but the problem said only 2 got A's.

Let's say that Beth gets an A. From her statement, we know that Carlos get an A, and from his statement we know that Diana gets an A. But that makes 3, which is not 2.

If Carlos gets an A, then Diana gets an A. That makes 2, so $\boxed{\text{C}}$ is the right answer. Note that although Beth said "If I get an A, then Carlos will get an A.", that does NOT mean that "If Carlos gets an A, then I will get an A."

Question 5

"If a whole number n is not prime, then the whole number $n-2$ is not prime." A value of n which shows this statement to be false is

$\textbf{(A)}\ 9 \qquad \textbf{(B)}\ 12 \qquad \textbf{(C)}\ 13 \qquad \textbf{(D)}\ 16 \qquad \textbf{(E)}\ 23$

Answer A

To show this statement to be false, we need a non-prime value of n such that $n-2$ is prime. Since 13 and 23 are prime, they won't prove anything relating to the truth of the statement.

Now we just check the statement for $n = 9, 12, 16$. If $n = 12$ or $n = 16$, then $n-2$ is 10 or 14, which aren't prime. However, $n = 9$ makes $n - 2 = 7$, which is prime, so $n = 9$ proves the statement false.

Therefore, the answer is $\boxed{\textbf{A}}$, 9.

Question 6

Abby, Bret, Carl, and Dana are seated in a row of four seats numbered #1 to #4. Joe looks at them and says:

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"Bret is next to Carl."
"Abby is between Bret and Carl."
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However each one of Joe's statements is false. Bret is actually sitting in seat #3. Who is sitting in seat #2?

(A) Abby (B) Bret (C) Carl (D) Dana (E) There is not enough information

Answer D

—— —— Bret ——

We know that Carl does not sit next to Bret, so he must sit in seat #1. Since Abby is not between Bret and Carl, she must sit in seat #4. Finally, Dana has to take the last seat available, which is #2.

D

Question 7

Andy the Ant lives on a coordinate plane and is currently at $(-20, 20)$ facing east (that is, in the positive x-direction). Andy moves 1 unit and then turns 90° left. From there, Andy moves 2 units (north) and then turns 90° left. He then moves 3 units (west) and again turns 90° left. Andy continues his progress, increasing his distance each time by 1 unit and always turning left. What is the location of the point at which Andy makes the 2020th left turn?

(A) $(-1030, -994)$ **(B)** $(-1030, -990)$ **(C)** $(-1026, -994)$ **(D)** $(-1026, -990)$ **(E)** $(-1022, -994)$

Answer B

Andy makes a total of 2020 moves: 1010 horizontal (505 left and 505 right) and 1010 vertical (505 up and 505 down).

The x-coordinate of Andy's final position is

$$-20+\overbrace{\underbrace{1-3}_{-2}+\underbrace{5-7}_{-2}+\underbrace{9-11}_{-2}+\cdots+\underbrace{2017-2019}_{-2}}^{1010\text{ terms, }505\text{ pairs}}=-20-2\cdot 505=-1030.$$

The y-coordinate of Andy's final position is

$$20+\overbrace{\underbrace{2-4}_{-2}+\underbrace{6-8}_{-2}+\underbrace{10-12}_{-2}+\cdots+\underbrace{2018-2020}_{-2}}^{1010\text{ terms, }505\text{ pairs}}=20-2\cdot 505=-990.$$

Together, we have $(x,y)=\boxed{\textbf{(B) } (-1030,-990)}$.

Question 8

Define a sequence recursively by $F_0 = 0,\ F_1 = 1$, and $F_n =$ the remainder when $F_{n-1} + F_{n-2}$ is divided by 3, for all $n \geq 2$. Thus the sequence starts $0, 1, 1, 2, 0, 2, \ldots$ What is $F_{2017} + F_{2018} + F_{2019} + F_{2020} + F_{2021} + F_{2022} + F_{2023} + F_{2024}$?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Answer D

A pattern starts to emerge as the function is continued. The repeating pattern is $0, 1, 1, 2, 0, 2, 2, 1 \ldots$ The problem asks for the sum of eight consecutive terms in the sequence. Because there are eight numbers in the repeating pattern, we just need to find the sum of the numbers in the sequence, which is $\boxed{\textbf{(D)}\ 9}$

Question 9

Chloé chooses a real number uniformly at random from the interval $[0, 2017]$. Independently, Laurent chooses a real number uniformly at random from the interval $[0, 4034]$. What is the probability that Laurent's number is greater than Chloé's number?

$\textbf{(A)}\ \frac{1}{2} \qquad \textbf{(B)}\ \frac{2}{3} \qquad \textbf{(C)}\ \frac{3}{4} \qquad \textbf{(D)}\ \frac{5}{6} \qquad \textbf{(E)}\ \frac{7}{8}$

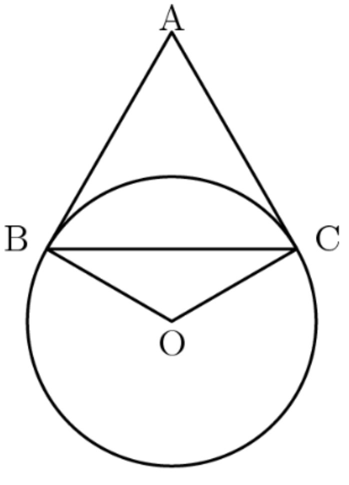
Answer C

Denote "winning" to mean "picking a greater number". There is a $\frac{1}{2}$ chance that Laurent chooses a number in the interval $[2017, 4034]$. In this case, Chloe cannot possibly win, since the maximum number she can pick is 2017. Otherwise, if Laurent picks a number in the interval $[0, 2017]$, with probability $\frac{1}{2}$, then the two people are symmetric, and each has a $\frac{1}{2}$ chance of winning. Then, the total probability is: $\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2} = \boxed{\textbf{(C)}\ \frac{3}{4}}$.

Question 10

Sides $\overline{AB}$ and $\overline{AC}$ of equilateral triangle ABC are tangent to a circle at points B and C respectively. What fraction of the area of $\triangle ABC$ lies outside the circle?

(A) $\frac{4\sqrt{3}\pi}{27} - \frac{1}{3}$ **(B)** $\frac{\sqrt{3}}{2} - \frac{\pi}{8}$ **(C)** $\frac{1}{2}$ **(D)** $\sqrt{3} - \frac{2\sqrt{3}\pi}{9}$ **(E)** $\frac{4}{3} - \frac{4\sqrt{3}\pi}{27}$



Answer E

Let the radius of the circle be r, and let its center be O. Since $\overline{AB}$ and $\overline{AC}$ are tangent to circle O, then $\angle OBA = \angle OCA = 90^\circ$, so $\angle BOC = 120^\circ$. Therefore, since $\overline{OB}$ and $\overline{OC}$ are equal to r, then (pick your favorite method) $\overline{BC} = r\sqrt{3}$. The area of the equilateral triangle is $\frac{(r\sqrt{3})^2\sqrt{3}}{4} = \frac{3r^2\sqrt{3}}{4}$, and the area of the sector we are subtracting from it is $\frac{1}{3}\pi r^2 - \frac{1}{2} \cdot r\sqrt{3} \cdot \frac{r}{2} = \frac{\pi r^2}{3} - \frac{r^2\sqrt{3}}{4}$. The area outside of the circle is $\frac{3r^2\sqrt{3}}{4} - \left(\frac{\pi r^2}{3} - \frac{r^2\sqrt{3}}{4}\right) = r^2\sqrt{3} - \frac{\pi r^2}{3}$. Therefore, the answer is

$$\frac{r^2\sqrt{3} - \frac{\pi r^2}{3}}{\frac{3r^2\sqrt{3}}{4}} = \boxed{\textbf{(E)}\ \frac{4}{3} - \frac{4\sqrt{3}\pi}{27}}$$

Question 11

Real numbers x, y, and z satisfy the inequalities $0 < x < 1$, $-1 < y < 0$, and $1 < z < 2$. Which of the following numbers is necessarily positive?

$\textbf{(A)}\ y + x^2 \qquad \textbf{(B)}\ y + xz \qquad \textbf{(C)}\ y + y^2 \qquad \textbf{(D)}\ y + 2y^2 \qquad \textbf{(E)}\ y + z$

Answer E

Notice that $y + z$ must be positive because $|z| > |y|$. Therefore the answer is $\boxed{\textbf{(E)}\ y + z}$.

The other choices:

$\textbf{(A)}$ As x grows closer to 0, x^2 decreases and thus becomes less than y.

$\textbf{(B)}$ x can be as small as possible ($x > 0$), so xz grows close to 0 as x approaches 0.

$\textbf{(C)}$ For all $-1 < y < 0$, $|y| > |y^2|$, and thus it is always negative.

$\textbf{(D)}$ The same logic as above, but when $-\frac{1}{2} < y < 0$ this time.

Question 12

Supposed that x and y are nonzero real numbers such that $\frac{3x+y}{x-3y}=-2$.

What is the value of $\frac{x+3y}{3x-y}$?

(A) -3 **(B)** -1 **(C)** 1 **(D)** 2 **(E)** 3

Answer D

Rearranging, we find $3x+y=-2x+6y$, or $5x=5y \implies x=y$.
Substituting, we can convert the second equation into
$\frac{x+3x}{3x-x}=\frac{4x}{2x}=\boxed{\textbf{(D) } 2}$.

Question 13

The lines with equations $ax - 2y = c$ and $2x + by = -c$ are perpendicular and intersect at $(1, -5)$. What is c?

$\textbf{(A)}\ -13 \qquad \textbf{(B)}\ -8 \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 8 \qquad \textbf{(E)}\ 13$

Answer E

Writing each equation in slope-intercept form, we get $y = \frac{a}{2}x - \frac{1}{2}c$ and $y = -\frac{2}{b}x - \frac{c}{b}$. We observe the slope of each equation is $\frac{a}{2}$ and $-\frac{2}{b}$, respectively. Because the slope of a line perpendicular to a line with slope m is $-\frac{1}{m}$, we see that $\frac{a}{2} = -\frac{1}{-\frac{2}{b}}$ because it is given that the two lines are perpendicular. This equation simplifies to $a = b$.

Because $(1, -5)$ is a solution of both equations, we deduce $a \times 1 - 2 \times -5 = c$ and $2 \times 1 + b \times -5 = -c$. Because we know that $a = b$, the equations reduce to $a + 10 = c$ and $2 - 5a = -c$. Solving this system of equations, we get $c = \boxed{\textbf{(E)}\ 13}$

Question 14

At Typico High School, 60% of the students like dancing, and the rest dislike it. Of those who like dancing, 80% say that they like it, and the rest say that they dislike it. Of those who dislike dancing, 90% say that they dislike it, and the rest say that they like it. What fraction of students who say they dislike dancing actually like it?

$\textbf{(A)}\ 10\%\qquad\textbf{(B)}\ 12\%\qquad\textbf{(C)}\ 20\%\qquad\textbf{(D)}\ 25\%\qquad\textbf{(E)}\ 33\frac{1}{3}\%$

Answer D

$60\% \cdot 20\% = 12\%$ of the people that claim that they like dancing actually dislike it, and $40\% \cdot 90\% = 36\%$ of the people that claim that they dislike dancing actually dislike it. Therefore, the answer is

$$\frac{12\%}{12\% + 36\%} = \boxed{\textbf{(D)}\ 25\%}.$$

Question 15

The mean, median, and mode of the 7 data values $60, 100, x, 40, 50, 200, 90$ are all equal to x. What is the value of x?

(A) 50 **(B)** 60 **(C)** 75 **(D)** 90 **(E)** 100

Answer D

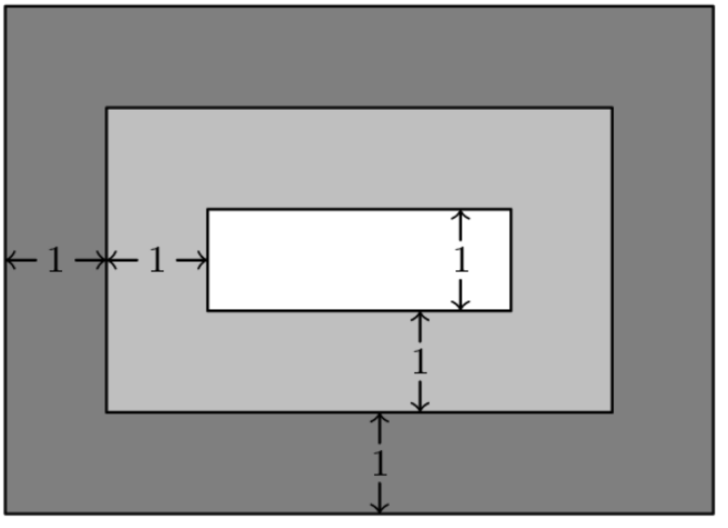
Since x is the mean,

$$\begin{aligned} x &= \frac{60+100+x+40+50+200+90}{7} \\ &= \frac{540+x}{7}. \end{aligned}$$

Therefore, $7x = 540 + x$, so $x = \boxed{\textbf{(D) } 90}$.

Question 16

A rug is made with three different colors as shown. The areas of the three differently colored regions form an arithmetic progression. The inner rectangle is one foot wide, and each of the two shaded regions is 1 foot wide on all four sides. What is the length in feet of the inner rectangle?



$\textbf{(A) } 1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 4 \qquad \textbf{(D) } 6 \qquad \textbf{(E) } 8$

Answer B

Let the length of the inner rectangle be x.

Then the area of that rectangle is $x \cdot 1 = x$.

The second largest rectangle has dimensions of $x+2$ and 3, making its area $3x+6$. The area of the second shaded area, therefore, is $3x+6-x = 2x+6$.

The largest rectangle has dimensions of $x+4$ and 5, making its area $5x+20$. The area of the largest shaded region is the largest rectangle minus the second largest rectangle, which is $(5x+20)-(3x+6) = 2x+14$.

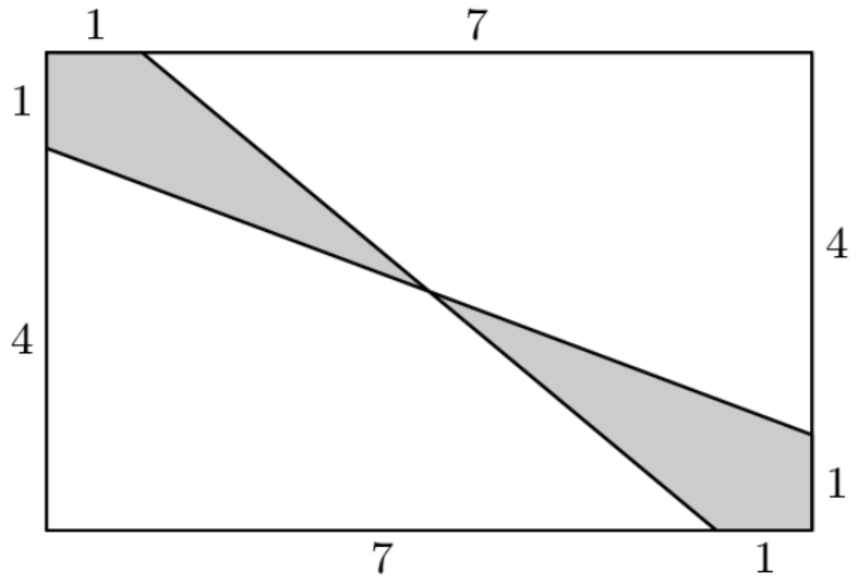
The problem states that $x, 2x+6, 2x+14$ is an arithmetic progression, meaning that the terms in the sequence increase by the same amount each term.

Therefore,

$$(2x+6)-(x) = (2x+14)-(2x+6) \implies x+6=8 \implies x=2$$

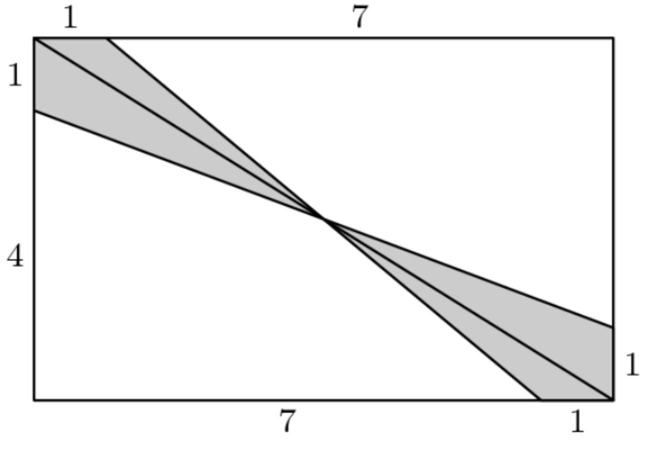
Question 17

Find the area of the shaded region.



(A) $4\frac{3}{5}$ **(B)** 5 **(C)** $5\frac{1}{4}$ **(D)** $6\frac{1}{2}$ **(E)** 8

Answer D



The bases of these triangles are all 1, and by symmetry, their heights are 4, $\frac{5}{2}$, 4, and $\frac{5}{2}$. Thus, their areas are 2, $\frac{5}{4}$, 2, and $\frac{5}{4}$, which add to the area of the shaded region, which is $\boxed{6\frac{1}{2}}$.

Question 18

Last year Isabella took 7 math tests and received 7 different scores, each an integer between 91 and 100, inclusive. After each test she noticed that the average of her test scores was an integer. Her score on the seventh test was 95. What was her score on the sixth test?

$\textbf{(A)}\ 92 \qquad \textbf{(B)}\ 94 \qquad \textbf{(C)}\ 96 \qquad \textbf{(D)}\ 98 \qquad \textbf{(E)}\ 100$

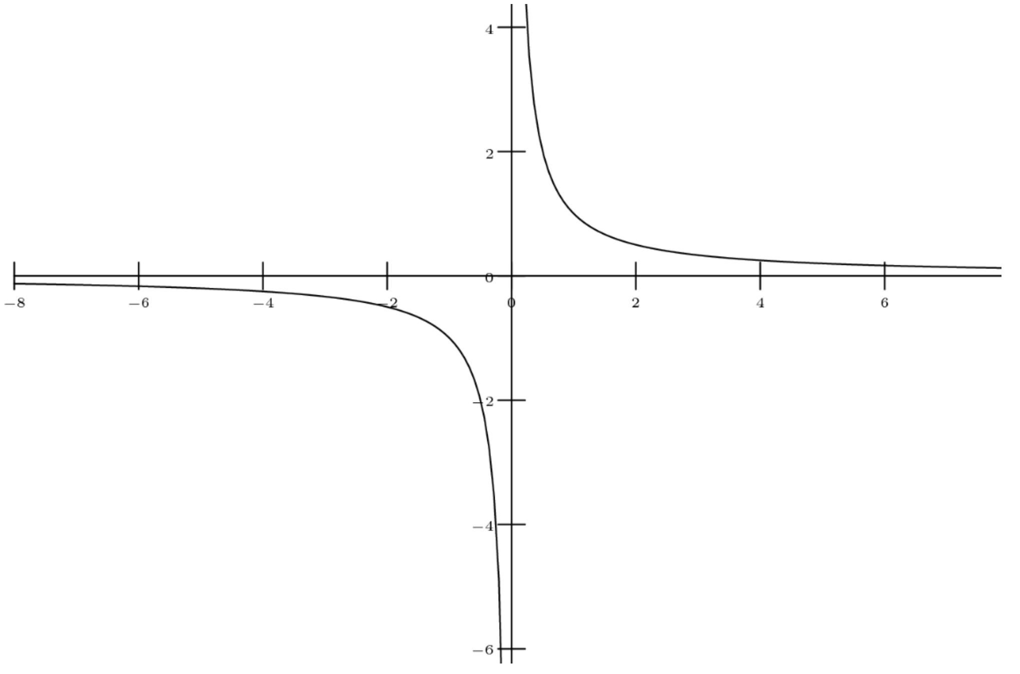
Answer E

First, we find the largest sum of scores which is $100+99+98+97+96+95+94$ which equals $7(97)$. Then we find the smallest sum of scores which is $91+92+93+94+95+96+97$ which is $7(94)$. So the possible sums for the 7 test scores so that they provide an integer average are $7(97)$, $7(96)$, $7(95)$ and $7(94)$ which are $679, 672, 665,$ and 658 respectively. Now in order to get the sum of the first 6 tests, we subtract 95 from each sum producing $584, 577, 570,$ and 563. Notice only 570 is divisible by 6 so, therefore, the sum of the first 6 tests is 570. We need to find her score on the $6th$ test so we have to find which number will give us a number divisible by 5 when subtracted from 570. Since 95 is the $7th$ test score and all test scores are distinct that only leaves $\boxed{\textbf{(E)}\ 100}$.

Question 19

The vertices of an equilateral triangle lie on the hyperbola $xy = 1$, and a vertex of this hyperbola is the centroid of the triangle. What is the square of the area of the triangle?

$\textbf{(A)}\ 48 \qquad \textbf{(B)}\ 60 \qquad \textbf{(C)}\ 108 \qquad \textbf{(D)}\ 120 \qquad \textbf{(E)}\ 169$



Answer C

WLOG, let the centroid of $\triangle ABC$ be $I = (-1, -1)$. The centroid of an equilateral triangle is the same as the circumcenter. It follows that the circumcircle must intersect the graph exactly three times. Therefore, $A = (1, 1)$, so $AI = BI = CI = 2\sqrt{2}$, so since $\triangle AIB$ is isosceles and $\angle AIB = 120^\circ$, then by the Law of Cosines, $AB = 2\sqrt{6}$. Alternatively, we can use the fact that the circumradius of an equilateral triangle is equal to $\frac{s}{\sqrt{3}}$. Therefore, the area of the triangle is $\frac{(2\sqrt{6})^2\sqrt{3}}{4} = 6\sqrt{3}$, so the square of the area of the triangle is $\boxed{\textbf{(C)}\ 108}$.

Question 20

For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

$\textbf{(A)} -9009 \qquad \textbf{(B)} -8008 \qquad \textbf{(C)} -7007 \qquad \textbf{(D)} -6006 \qquad \textbf{(E)} -5005$

Answer C

Solution 1.1

A faster ending to Solution 1 is as follows.

$$\begin{aligned} f(1) &= (1+p)(1^3 + a \cdot 1^2 + 1 + 10) \\ &= (91)(-77) \\ &= (7)(13)(11)(-7) = (1001)(-7) \\ &= \boxed{\textbf{(C)} -7007}. \end{aligned}$$

Solution 1.2

Also a faster ending to Solution 1 is as follows.

To find $f(1)$ we just need to find the sum of the coefficients which is $1 + 1 - 8009 + 100 + 900 = \boxed{\textbf{(C)}\ , -7007}$.