# AMC 10
# MOCK TEST 1
# Solution Book

# Number Theory

ThrivingScholars

1. When the expression $\dfrac{(2^2-1)\times(3^2-1)\times(4^2-1)\times(5^2-1)}{(2\times3)\times(3\times4)\times(4\times5)\times(5\times6)}$ is simplified, which of the following is obtained?

A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{5}$ E $\frac{1}{6}$

**Solution** **D**

Using the standard factorization $n^2 - 1 = (n-1)(n+1)$, we have

$$\begin{aligned}\frac{(2^2-1)\times(3^2-1)\times(4^2-1)\times(5^2-1)}{(2\times3)\times(3\times4)\times(4\times5)\times(5\times6)} &= \frac{(1\times3)\times(2\times4)\times(3\times5)\times(4\times6)}{(2\times3)\times(3\times4)\times(4\times5)\times(5\times6)}\\ &= \frac{1\times2\times3\times4}{2\times3\times4\times5}\\ &= \frac{1}{5}.\end{aligned}$$

**For investigation**

Write the following expressions in simplified form.

(a) $\dfrac{(2^2-1)\times(3^2-1)\times(4^2-1)\times(5^2-1)\times(6^2-1)}{(2\times3)\times(3\times4)\times(4\times5)\times(5\times6)\times(6\times7)}$,

(b) $\dfrac{(2^2-1^2)\times(3^2-2^2)\times(4^2-3^2)\times(5^2-4^2)}{1\times3\times5\times7}$.

(c) $\dfrac{(3^4-1)\times(4^4-1)\times(5^4-1)\times(6^4-1)}{(2\times4)\times(3\times5)\times(4\times6)\times(5\times7)}$.

2. What is the value of $\sqrt{\frac{2023}{2+0+2+3}}$?

A 13 B 15 C 17 D 19 E 21

**Solution** **C**

$2+0+2+3=7$ and $2023 \div 7 = 289 = 17^2$. Therefore

$$\sqrt{\frac{2023}{2+0+2+3}} = \sqrt{17^2} = 17.$$

**For investigation**

- Which is the smallest integer $n$ with $n > 2023$ which has the property that $n$ is divisible by the sum of its digits?
- Note that since $2023 \div 7 = 17^2$, it follows that $2023 = 7 \times 17^2$ and that 7 and 17 are primes.

  Which is the smallest integer $n$ with $n > 2023$ which can be expressed as $p \times q^2$, where $p$ and $q$ are different primes?
- Which is the smallest integer $n$ with $n > 2023$ which can be expressed as $p \times q^2$, where $p$ and $q$ are different primes, and $p$ is the sum of the digits of $n$?

3. One of the following numbers is prime. Which is it?

A $2017 - 2$    B $2017 - 1$    C $2017$    D $2017 + 1$    E $2017 + 2$

Solution **C**

We see that:

$2017 - 2 = 2015$ and 2015 is a multiple of 5. So $2017 - 2$ is not prime.

$2017 - 1 = 2016$ and 2016 is a multiple of 2. So $2017 - 1$ is not prime.

$2017 + 1 = 2018$ and 2018 is a multiple of 2. So $2017 + 1$ is not prime.

$2017 + 2 = 2019$ and 2019 is a multiple of 3. So $2017 + 2$ is not prime.

We are told that one of the given options is prime. We may therefore deduce that the remaining option, 2017, is prime.

For investigation

- In the context of the SMC it is safe to assume that the information given in the question is correct. Therefore, having shown that $2017 - 2$, $2017 - 1$, $2017 + 1$ and $2017 + 2$ are not prime, we may deduce that 2017 is prime.

  However, without this assumption, if we wish to show that 2017 is prime, we need to check that there are no prime factors of 2017 which are less than 2017.

  Which is the largest prime $p$ that we need to check is not a factor of 2017 in order to show that 2017 is prime?

- One way to see that 2019 is a multiple of 3 is by noting that the sum of its digits, $2 + 0 + 1 + 9 = 12$, is a multiple of 3.

  Why does this test for divisibility by 3 work?

- (a) Which is the least positive integer $n$ such that either $2017 - n$ or $2017 + n$ is prime?

  (b) Which is the least positive integer $n$ such that both $2017 - n$ and $2017 + n$ are prime?

4. A ***twip*** is a very short unit of length, derived from imperial units, and is equal to approximately 0.000018 metres. A ***league*** is a long unit of length which is equal to approximately 4800 metres.

Roughly how many twips are there in a league?

A 270 000 000 B 27 000 000 C 2 700 000 D 270 000 E 27 000

SOLUTION **A**

The number of twips in a league is $\frac{4800}{0.000018}$. Now

$$\frac{4800}{0.000018} = \frac{4\,800\,000\,000}{18} = \frac{800\,000\,000}{3} \approx \frac{810\,000\,000}{3} = 270\,000\,000.$$

FOR INVESTIGATION

The length of a cricket pitch is one *chain*, which is 22 yards. A *light year* is the distance that light travels (through a vacuum) in one year. Light travels at around 186 000 miles per second in a vacuum. There are 1760 yards in one mile.

Approximately, how many cricket pitches can be fitted into a light year?

5. For how many positive integer values of $n$ is $n^2 + 2n$ prime?

A 0 B 1 C 2 D 3 E more than 3

SOLUTION **B**

We have $n^2 + 2n = n(n + 2)$. Therefore $n^2 + 2n$ is divisible by $n$. Hence, for $n^2 + 2n$ to be prime, $n$ can only have the value 1.

When $n = 1$, we have $n^2 + 2n = 3$, which is prime.

Therefore there is just one positive integer value of $n$ for which $n^2 + 2n$ is prime.

6. When $4^5 \times 5^4$ is correctly calculated, how many digits are there in the answer?

A 4 B 6 C 10 D 16 E 20

**Solution** **B**

We have

$$4^5 \times 5^4 = (2^2)^5 \times 5^4 = 2^{10} \times 5^4 = 2^6 \times (2^4 \times 5^4) = 2^6 \times 10^4 = 64 \times 10\,000 = 640\,000.$$

Therefore there are 6 digits in the correct answer.

**For investigation**

How many digits are there in the answer when the number $8^3 \times 5^8$ is calculated correctly?

7. Pablo has 100 identical small cubes. He uses some of them to build the largest possible solid cube. How many of the small cubes are left over?

A 16 B 27 C 36 D 73 E 92

Solution **C**

We need to find the largest integer $n$ with $n^3 \leq 100$. Then the largest possible solid cube would use $n^3$ small cubes, leaving Pablo with $100 - n^3$ unused small cubes.

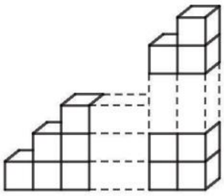
We see that $4^3 = 64$ which is less than 100 but $5^3 = 125$ is not. So the largest possible cube that Pablo could build uses 64 small cubes with $100 - 64 = 36$ left over.

For investigation

Pablo next uses some of his 100 small cubes to build a staircase, as shown.

He builds the staircase with largest possible number of steps.

How many of the small cubes are left over?



8. Which is the largest prime factor of $3^8 - 1$?

A 41 B 37 C 31 D 29 E 23

**Solution** **A**

Using the difference of two squares factorization $x^2 - y^2 = (x - y)(x + y)$, we have

$$
\begin{aligned}
3^8 - 1 &= (3^4 - 1)(3^4 + 1) \\
&= (3^2 - 1)(3^2 + 1)(3^4 + 1) \\
&= (3 - 1)(3 + 1)(9 + 1))(81 + 1) \\
&= 2 \times 4 \times 10 \times 82 \\
&= 2 \times 2^2 \times (2 \times 5) \times (2 \times 41) \\
&= 2^5 \times 5 \times 41.
\end{aligned}
$$

Because 41 is a prime, we deduce that the largest prime factor of $3^8 - 1$ is 41.

**For investigation**

Find the largest prime factor of the integers

(a) $2^8 - 1$, (b) $2^{16} - 1$ and (c) $5^6 - 1$.

9. Sonia writes down three 2-digit numbers whose sum is 46. The first number is prime, the second is square and the third is even.

What is the even number?

A 10 B 12 C 14 D 16 E 18

**Solution** **A**

All the 2-digit primes are odd numbers. So Sonia writes down an odd prime, a square and an even number whose sum is the even number 46. It follows that the square is also an odd number because an even number cannot be the sum of two even numbers and an odd number.

The only odd 2-digit square that is less than 46 is 25. We deduce that the square is 25. Therefore the sum of the prime and the even number is $46 - 25$, that is, 21.

The only two 2-digit numbers with sum 21 are 10 and 11. Hence the prime is 11 and the even number is 10.

**For investigation**

Sonia next writes down three 2-digit numbers whose sum is 47. The first number is prime, the second is square and the third is even.

How many possibilities are there for the even number?

10. A palindromic number is one where the digits read the same forwards as backwards, such as 123 321.

What is the hundreds digit of the largest six-digit palindromic number that is divisible by 18?

A 9 B 7 C 5 D 3 E 1

**Solution** **E**

Let the six-digit palindromic number be '$pqrrqp$'.

For this number to be divisible by 18, it needs to be divisible by 2 and by 9. To be divisible by 2, the digit $p$ must be even. For the number to be as large as possible, $p$ has to be as large as possible. Therefore we try $p = 8$. This makes the number '$8qrrq8$'.

A number is divisible by 9 precisely when the sum of its digits is divisible by 9. [See Problem 9.4.] Therefore $2(8 + q + r)$ needs to be divisible by 9. Hence $8 + q + r$ needs to be divisible by 9. Again, we want $q$ to be as large as possible . So we try $q = 9$. With this choice for $q$, we require that $8 + 9 + r$ is divisible by 9. Therefore $8 + r$ must be divisible by 9. Hence $r = 1$.

It follows that the number '$8qrrq8$' is 891198. Therefore this is the largest six-digit palindromic number that is divisible by 18. The hundreds digit of this number is 1.

**For investigation**

- Which is the smallest six-digit palindromic number which is divisible by 18?
- Which are the smallest and largest six-digit palindromic numbers which are divisible by 45?
- Which is the largest seven-digit palindromic number which is divisible by 18?
- In the solution above we have used the fact that

  A positive integer is divisible by 9 precisely when the sum of its digits is divisible by 9.

  Prove that this is correct.
- One way to answer Problem 9.4 is to prove the following more general fact.

  A positive integer and the sum of its digits have the same remainder when divided by 9.

  Prove that this is correct.
- The solution also uses the fact that, for every positive integer $n$,

  $$n \text{ is divisible by } 18 \iff n \text{ is divisible by 2 and } n \text{ is divisible by 9.}$$

  Explain why this is correct.
- Is it true that, for every positive integer $n$,

  $$n \text{ is divisible by } 24 \iff n \text{ is divisible by 4 and } n \text{ is divisible by 6 ?}$$

11. Which of the following is equal to $25 \times 15 \times 9 \times 5.4 \times 3.24$?

A $3^9$ B $3^{10}$ C $3^{11}$ D $3^{14}$ E $3^{17}$

SOLUTION **B**

COMMENTARY

The last thing we want to do here is to do the multiplications to work out the value of $25 \times 15 \times 9 \times 5.4 \times 3.24$, and then to factorize the answer.

Instead, we write the decimals as fractions, then factorize the individual numbers, and do some cancellation.

We have $25 = 5^2$, $15 = 3 \times 5$, $9 = 3^2$, $5.4 = \frac{54}{10} = \frac{27}{5} = \frac{3^3}{5}$ and $3.24 = \frac{324}{100} = \frac{81}{25} = \frac{3^4}{5^2}$. Therefore,

$$25 \times 15 \times 9 \times 5.4 \times 3.24 = 5^2 \times (3 \times 5) \times 3^2 \times \frac{3^3}{5} \times \frac{3^4}{5^2}.$$

We can now cancel the factors $5^2$ and 5 in the numerator and denominator to deduce that

$$\begin{aligned} 25 \times 15 \times 9 \times 5.4 \times 3.24 &= 3 \times 3^2 \times 3^3 \times 3^4 \\ &= 3^{10}. \end{aligned}$$

12. What are the last two digits of $7^{2018}$?

A 07 B 49 C 43 D 01 E 18

**Solution** **B**

**Commentary**

Clearly, we cannot answer this question by fully evaluating $7^{2018}$ and then looking to see which are its last two digits.

Instead, we make use of the fact that the last two digits of a product $a \times b$ is determined just by the last two digits of $a$ and $b$, and then we look for a pattern.

It is convenient to introduce some notation for the last two digits of an integer. There is no standard notation for this (but see 8.1, below). For the purpose of this question we use the notation $[n]$ for the number consisting of the last two digits of the integer $n$. For example, $[12345] = 45$.

Using this notation we can express the fact we use by the equation

$$[m \times n] = [[m] \times [n]].$$

We then have

$$\begin{aligned}
[7^1] &= [7] = 07\\
[7^2] &= [49] = 49\\
[7^3] &= [7^2 \times 7] = [[7^2] \times [7]] = [49 \times 7] = [343] = 43\\
[7^4] &= [7^3 \times 7] = [[7^3] \times [7]] = [43 \times 7] = [301] = 01\\
[7^5] &= [7^4 \times 7] = [[7^4] \times [7]] = [01 \times 7] = [7] = 07.
\end{aligned}$$

Each term in the sequence giving the last two digits of $7^n$ for $n = 1, 2, 3\ldots$ depends only on the previous term. Hence as 07 has reoccurred we can deduce that the sequence consisting of the values of $[7^n]$ is made up of repeating cycle of length 4, and so is

$$07, \quad 49, \quad 43, \quad 01, \quad 07, \quad 49, \quad 43, \quad 01, \quad \ldots.$$

Because $2018 = 504 \times 4 + 2$ it follows that $[7^{2018}]$ is the second pair of digits in this cycle, namely, 49.

**For investigation**

- Show that the formula $[m \times n] = [[m] \times [n]]$ is correct.

  **Note**

  The number $[m]$ is the remainder when $m$ is divided by 100. If you are familiar with the language of modular arithmetic, you will see that $[m] \equiv m (\text{mod } 100)$.

- What are the last two digits of $9^{2018}$?
- What are the last two digits of $3^{2018}$?
- Prove (by Mathematical Induction) that for every positive integer $n$, the last two digits of $7^{4n+2}$ are 49.[To find out about *Mathematical Induction* go to https://nrich.maths.org/4718]

13. What is $25^2 - 24^2 - 23^2 + 22^2$?

A 0    B 1    C 2    D 3    E 4

**Solution** **E**

You are not allowed to use a calculator in the SMC. So when, as here, you are faced with a question that seems to involve a lot of arithmetic, you should look for a method that doesn't involve a lot of calculation. Often, finding a quick method is the point of the question.

The expression $25^2 - 24^2$ in this question should bring to mind the very useful factorization $x^2 - y^2 = (x - y)(x + y)$.

This is the key to answering this question without a lot of numerical work.

We have

$$\begin{aligned} 25^2 - 24^2 - 23^2 + 22^2 &= (25^2 - 24^2) - (23^2 - 22^2) \\ &= (25 - 24)(25 + 24) - (23 - 22)(23 + 22) \\ &= 1 \times 49 - 1 \times 45 \\ &= 49 - 45 \\ &= 4. \end{aligned}$$

**For investigation**

- What is the value of $2025^2 - 2024^2 - 2023^2 + 2022^2$?
- What is the value of $n^2 - (n-1)^2 - (n-2)^2 + (n-3)^2$?

14. What is the value of $99(0.\dot{4}\dot{9} - 0.\dot{4})$?

A 5 B 4 C 3 D 2 E 1

**Solution** **A**

The standard method for converting a recurrent decimal to a fraction shows that

$$0.\dot{4}\dot{9} = \frac{49}{99} \quad \text{and} \quad 0.\dot{4} = \frac{4}{9}.$$

[Problem 8.1 asks you to check this.]

Therefore

$$\begin{aligned} 99(0.\dot{4}\dot{9} - 0.\dot{4}) &= 99\left(\frac{49}{99} - \frac{4}{9}\right) \\ &= 99\left(\frac{49}{99} - \frac{44}{99}\right) \\ &= 99\left(\frac{5}{99}\right) \\ &= 5. \end{aligned}$$

**For investigation**

- Show that $0.\dot{4} = \frac{4}{9}$ and $0.\dot{4}\dot{9} = \frac{44}{99}$.
- Express the recurring decimal $0.\dot{2}3\dot{4}$ as a fraction in its lowest terms.
- Write the solution to the equation

$$x + 0.0\dot{7} = 0.\dot{1}\dot{3}$$

as a recurring decimal.
- Prove that every recurring decimal may be expressed in the form $\frac{p}{q}$ where $p$ and $q$ are integers, with $q > 0$.

15. Which of these numbers is the largest?

A $2^{5000}$ B $3^{4000}$ C $4^{3000}$ D $5^{2000}$ E $6^{1000}$

**Solution** **B**

We have

$$2^{5000} = (2^5)^{1000} = 32^{1000},$$
$$3^{4000} = (3^4)^{1000} = 81^{1000},$$
$$4^{3000} = (4^3)^{1000} = 64^{1000},$$

and

$$5^{2000} = (5^2)^{1000} = 25^{1000}.$$

Since $6 < 25 < 32 < 64 < 81$, it follows that $6^{1000} < 25^{1000} < 32^{1000} < 64^{1000} < 81^{1000}$.

Therefore $6^{1000} < 5^{2000} < 2^{5000} < 4^{3000} < 3^{4000}$.

Hence, of the given numbers, it is $3^{4000}$ that is the largest.

**For investigation**

Which of these numbers is the largest?
(a) $2^{7000}$, (b) $3^{6000}$, (c) $4^{5000}$, (d) $5^{4000}$, (e) $6^{3000}$.

16. What is the largest prime factor of $106^2 - 15^2$?

A 3 | B 7 | C 11 | D 13 | E 17

Solution **D**

*Note*: It is not a good idea to attempt to calculate $106^2$ and $15^2$, then do a subtraction and finally attempt to factorize the resulting answer.

Instead we make use of the standard factorization of the difference of two squares:

$$x^2 - y^2 = (x - y)(x + y).$$

We have

$$\begin{aligned} 106^2 - 15^2 &= (106 - 15)(106 + 15) \\ &= 91 \times 121 \\ &= 7 \times 13 \times 11 \times 11. \end{aligned}$$

Therefore the prime factorization of $106^2 - 15^2$ is $7 \times 11^2 \times 13$, from which we see that its largest prime factor is 13.

For investigation

What is the largest prime factor of $300^2 - 3^2$?

17. Which of the following five values of $n$ is a counterexample to the statement in the box below?

For a positive integer $n$, at least one of $6n - 1$ and $6n + 1$ is prime.

A 10 B 19 C 20 D 21 E 30

Solution **C**

A counterexample to the statement in the box is a value of $n$ for which it is not true that at least one of $6n - 1$ and $6n + 1$ is prime. That is, it is a value of $n$ for which neither $6n - 1$ nor $6n + 1$ is prime.

We set out the values of $6n - 1$ and $6n + 1$ for $n = 10, 19, 20, 21$ and 30 in the following table.

| $n$ | $6n - 1$ | $6n + 1$ |
|---|---|---|
| 10 | 59 | 61 |
| 19 | 113 | **115** |
| 20 | **119** | **121** |
| 21 | **165** | 167 |
| 30 | 179 | 181 |

The values of $6n - 1$ and $6n + 1$ that are not prime are shown in bold.

We therefore see that for $n = 20$, neither $6n - 1$ nor $6n + 1$ is prime. Therefore $n = 20$ provides the required counterexample.

For investigation

- Check that, of the numbers that occur in the table above, 59, 61, 113, 167, 179 and 181 are prime, and that 115, 119, 121 and 165 are not prime.
- Show that if $p$ is a prime number other than 2 or 3, then there is a positive integer $n$ such that $p$ is either equal to $6n - 1$ or $6n + 1$.

18. For how many integer values of $k$ is $\sqrt{200 - \sqrt{k}}$ also an integer?

A 11 B 13 C 15 D 17 E 20

**Solution** **C**

The number $\sqrt{200 - \sqrt{k}}$ is an integer if, and only if, the number $200 - \sqrt{k}$ is a square.

Now $0 \leq 200 - \sqrt{k} \leq 200$. There are 15 squares in this range, namely $n^2$ for integer values of $n$ with $0 \leq n \leq 14$.

We have that $200 - \sqrt{k} = n^2$, if, and only if $k = (200 - n^2)^2$.

Hence there are 15 integer values of $k$ for which $\sqrt{200 - \sqrt{k}}$ is an integer, namely, $k = (200 - n^2)^2$ for $0 \leq n \leq 14$.

19. The number $16! \div 2^k$ is an odd integer. Note that $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.

What is the value of $k$?

A 9 B 11 C 13 D 15 E 17

**Solution** **D**

The number $k$ that we seek is such that $16! = 2^k \times q$, where $q$ is an odd integer. Thus $k$ is the power of 2 that occurs in the prime factorization of $16!$.

**Method 1**

To find the highest power of 2 that is a factor of $16!$ we need only consider the even numbers that occur in the product $1 \times 2 \times \cdots \times 15 \times 16$ that gives the value of $16!$

In the following table we give the powers of 2 that divide each of these factors.

| factor | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 |
|---|---|---|---|---|---|---|---|---|
| power of 2 | 1 | 2 | 1 | 3 | 1 | 2 | 1 | 4 |

The power of 2 in the prime factorization of $16!$ is the sum of the powers in the second row of this table. That is, it is $1 + 2 + 1 + 3 + 1 + 2 + 1 + 4 = 15$.

Hence, the required value of $k$ is 15.

**Method 2**

*Note*:
In this method we use the formula, usually attributed to the French mathematician Adrian-Marie Legendre (1752-1833), for the highest power of a prime $p$ that divides $n!$. This formula makes use of the *floor* function which we first explain.

We define the *floor* of $x$, written as $\lfloor x \rfloor$, to be the largest integer that is not larger than $x$.

For example, $\lfloor 2\frac{6}{7} \rfloor = 2$, $\lfloor 4.275 \rfloor = 4$, $\lfloor \pi \rfloor = 3$, $\lfloor 7 \rfloor = 7$, $\lfloor 0.43 \rfloor = 0$ and $\lfloor -5.23 \rfloor = -6$.

According to Legendre's formula, the highest power of the prime $p$ that divides $n!$ is given by

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots.$$

At first sight, the formula involves an infinite sum. However, if $p^k > n$, we have $0 < \dfrac{n}{p^k} < 1$ and hence $\left\lfloor \dfrac{n}{p^k} \right\rfloor = 0$. Therefore only a finite number of terms in the above sum are non-zero. The number of non-zero terms in this sum is the largest integer $k$ for which $p^k \leq n$.

20. One of the following is the largest square that is a factor of 10!. Which one?

Note that, $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$

A $(4!)^2$ B $(5!)^2$ C $(6!)^2$ D $(7!)^2$ E $(8!)^2$

**Solution** **C**

**Commentary**

If you are familiar with the values of $n!$ for small values of $n$, it is not difficult to spot the answer quite quickly, as in Method 1. If not, a more systematic method is to work out the prime factorization of 10!, as in Method 2.

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**Method 1**

We have

$$\begin{aligned} 10! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \\ &= (1 \times 2 \times 3 \times 4 \times 5 \times 6) \times 7 \times (8 \times 9 \times 10) \\ &= 6! \times 7 \times (8 \times 9 \times 10) \\ &= 6! \times 7 \times 720 \\ &= 6! \times 7 \times 6! \\ &= (6!)^2 \times 7. \end{aligned}$$

It follows that $(6!)^2$ is the largest square that is a factor of 10!.

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**Method 2**

We have

$$\begin{aligned} 10! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \\ &= 1 \times 2 \times 3 \times 2^2 \times 5 \times (2 \times 3) \times 7 \times 2^3 \times 3^2 \times (2 \times 5) \\ &= 2^8 \times 3^4 \times 5^2 \times 7 \\ &= (2^4 \times 3^2 \times 5)^2 \times 7. \end{aligned}$$

It follows that $2^4 \times 3^2 \times 5$ is the largest square that is a factor of 10!. Now $2^4 \times 3^2 \times 5 = 720 = 6!$ and therefore $(6!)^2$ is the largest square that is a factor of 10!.

**For investigation**

- Which is the largest square that is a factor of (a) 12!, and (b) 14! ?
- List the values of $n!$ for all positive integers $n$ with $n \leq 10$.

21. The highest common factors of all the pairs chosen from the positive integers $Q$, $R$ and $S$ are three different primes.

What is the smallest possible value of $Q + R + S$?

A 41 B 31 C 30 D 21 E 10

SOLUTION **B**

We use the notation HCF$(X, Y)$ for the highest common factor of the two integers $X$ and $Y$.

Suppose that HCF$(Q, R) = a$, HCF$(Q, S) = b$ and HCF$(R, S) = c$, where $a$, $b$ and $c$ are three different primes.

It follows that both $a$ and $b$ are factors of $Q$. Therefore the smallest possible value of $Q$ is $ab$. Likewise, the smallest possible values of $R$ and $S$ are $ac$ and $bc$, respectively.

We seek the smallest possible value of $Q + R + S$, that is, of $ab + ac + bc$, where $a$, $b$ and $c$ are different primes. To do this we choose the values of $a$, $b$ and $c$ to be the three smallest primes, that is 2, 3 and 5, in some order.

Because $ab + ac + bc$ is symmetric in $a$, $b$ and $c$, the order does not matter. With $a = 2$, $b = 3$ and $c = 5$, we have

$$ab + ac + bc = 2 \times 3 + 2 \times 5 + 3 \times 5 = 6 + 10 + 15 = 31.$$

We deduce that the smallest possible value of $Q + R + S$ is 31.

FOR INVESTIGATION

- Show that if, in the above solution, $a$, $b$ and $c$ are given the values 2, 3 and 5 in some other order, then the value of $ab + ac + bc$ is again 31.
- Show that if, in the above solution, any of 2, 3 and 5 is replaced by a prime larger than 5, then the resulting value of $Q + R + S$ is greater thsn 31.
- The highest common factors of all the pairs chosen from the positive integers $Q$, $R$, $S$ and $T$ are six different primes.

  What is the smallest possible value of $Q + R + S + T$?

22. For how many positive integers $n$ does $1+2+\cdots+n$ evenly divide $6n$?

A) 3 B) 4 C) 6 D) 5 E) 8

SOLUTION **D**

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

We need $\frac{n(n+1)}{2} \mid 6n$, i.e. $n(n+1) \mid 12n$.
Because $\gcd(n,n+1)=1$, the factor $n+1$ must divide $12$. Thus $n+1 \in \{1,2,3,4,6,12\}$, giving $n \in \{1,2,3,5,11\}$.
There are $5$ such $n$, so the correct choice is $\boxed{D}$.

23. Jeroen writes a list of 2019 consecutive integers. The sum of his integers is 2019.

What is the product of all the integers in Jeroen's list?

A $2019^2$ B $\frac{2019 \times 2020}{2}$ C $2^{2019}$ D 2019 E 0

**Solution** **E**

In a list of 2019 consecutive *positive* integers, at least one of them will be greater than or equal to 2019, and therefore the sum of these integers will be greater than 2019. So the integers in Jeroen's list are not all positive.

The sum of 2019 *negative* integers is negative and therefore cannot be equal to 2019. So the integers in Jeroen's list are not all negative.

We deduce that Jeroen's list of consecutive integers includes both negative and positive integers.

Because the integers in the list are consecutive it follows that one of them is 0.

Therefore the product of all the numbers in Jeroen's list is 0.

**For investigation**

- Note that we were able to answer this question without finding a list of 2019 consecutive integers with sum 2019. Show that there is just one list of 2019 consecutive integers whose sum is 2019, and find it.
- Find all the lists of at least two consecutive integers whose sum is 2019.
- Investigate which positive integers can be expressed as the sum of two or more consecutive positive integers.

24. How many pairs of real numbers $(x, y)$ satisfy the simultaneous equations $x^2 - y = 2022$ and $y^2 - x = 2022$?

A infinitely many    B 1    C 2    D 3    E 4

**Solution** **E**

By subtracting the equation $y^2 - x = 2022$ from the equation $x^2 - y = 2022$, we obtain

$$x^2 - y^2 + x - y = 0. \qquad (1)$$

The left hand side of equation (1) factorizes to give

$$(x - y)(x + y + 1) = 0. \qquad (2)$$

It follows that either $x - y = 0$ or $(x + y + 1) = 0$. That is, either $y = x$ or $y = -x - 1$.

When $y = x$ both equations of the question are equivalent to the equation $x^2 - x = 2022$. We can rewrite this equation as

$$x^2 - x - 2022 = 0. \qquad (3)$$

Equation (3) is a quadratic equation of the form $ax^2 + bx + c = 0$, with $a = 1$, $b = -1$ and $c = -2022$. Therefore $b^2 - 4ac = (-1)^2 + 4 \times 2022$ which is greater than 0.

It follows that equation (3) has two distinct real number solutions, say $x_1$ and $x_2$. It follows that the two pairs of real numbers $(x_1, x_1)$ and $(x_2, x_2)$ satisfy the simultaneous equations of the question.

It can be checked that when $y = -x - 1$ both equations of the question are equivalent to the equation

$$x^2 + x - 2021 = 0. \qquad (4)$$

Equation (4) is a quadratic equation of the form $ax^2 + bx + c = 0$, with $a = 1$, $b = 1$ and $c = -2021$. Therefore $b^2 - 4ac = 1^2 + 4 \times 2021$ which is greater than 0.

It follows that equation (4) has two distinct real number solutions, say $x_3$ and $x_4$. It follows that the two pairs of real numbers $(x_3, -x_3 - 1)$ and $(x_4, -x_4 - 1)$ satisfy the simultaneous equations of the question.

In these last two solutions $y \neq x$ and therefore they are different from the first two solutions.

We can therefore conclude that there are four pairs of real numbers that satisfy the simultaneous equations of the question.

**For investigation**

- Check that when $y = -x - 1$ both equations (1) and (2) are equivalent to the equation $x^2 + x - 2021 = 0$.
- Consider the curves corresponding to the equations $x^2 - y = 2022$ and $y^2 - x = 2022$.
  (a) What is the geometrical relationship between these two curves?
  (b) Sketch the two curves.
  (c) Check that the two curves meet in four distinct points.
- Explain why $b^2 - 4ac > 0$ is the condition for the quadratic equation $ax^2 + bx + c = 0$ to have two distinct real number solutions.

25. When written out in full, the number $(10^{2020}+2020)^2$ has 4041 digits.

What is the sum of the digits of this 4041-digit number?

A 9 B 17 C 25 D 2048 E 4041

**Solution** **C**

Using the expansion $(x+y)^2 = x^2 + 2xy + y^2$, we have

$$\begin{aligned}(10^{2020}+2020)^2 &= (10^{2020})^2 + 2\times 10^{2020} \times 2020 + 2020^2 \\ &= 10^{4040} + 4040\times 10^{2020} + 4080400. \qquad (1)\end{aligned}$$

We now note that when we add the three terms $10^{4040}$, $4040\times 10^{2002}$ and 408040, no two non-zero digits occur in the same column:

$$\begin{array}{rr} & 100\ldots\ldots\ldots\ldots0000\ldots\ldots\ldots\ldots0000000 \\ + & 4040\ldots\ldots\ldots\ldots0000000 \\ + & 4080400 \\ \hline & 100\ldots\ldots\ldots\ldots4040\ldots\ldots\ldots\ldots4080400 \end{array}$$

It follows that the non-zero digits in the final answer are exactly the non-zero digits in the three terms in (1) above.

Therefore the sum of the digits in the number $(10^{2020}+2020)^2$ is

$$1+4+4+4+8+4 = 25.$$

**Commentary**

The exact number of 0s in the final answer is not important. What matters is that the non-zero digits in the three terms $10^{4040}$, $4040\times 10^{2020}$ and 4080400 are in different columns, and so, when they are added, the non-zero digits in the final answer are exactly the non-zero digits in these terms. However, you are asked to work out the number of 0s in Problem 21.1 below.

**For investigation**

Find the positive integers $m$ and $n$ so that

$$(10^{2020}+2020)^2 = 1\overbrace{000\ldots000}^{m}404\overbrace{000\ldots000}^{n}4080400.$$