AMC 10
MOCK TEST 2
Solution Book

Number Theory

ThrivingScholars

1. What is two-fifths of the recurring decimal $0.\dot{2}\dot{5}$?

A $0.\dot{1}$ B $0.0\dot{1}$ C $0.\dot{0}\dot{1}$ D 0.10 E $0.\dot{1}\dot{0}$

SOLUTION **E**

METHOD 1

One-fifth of $0.\dot{2}\dot{5}$ is $0.0\dot{5}$. Hence two-fifths of $0.\dot{2}\dot{5}$ is $0.\dot{1}\dot{0}$.

METHOD 2

Since $0.\dot{2}\dot{5} = \frac{25}{99}$, it follows that that two-fifths of $0.\dot{2}\dot{5} = \frac{2}{5} \times \frac{25}{99} = \frac{10}{99} = 0.\dot{1}\dot{0}$.

FOR INVESTIGATION

- Express two-thirds of $0.\dot{0}\dot{6}$ both as a recurring decimal and as a fraction.
- Check that $0.\dot{2}\dot{5} = \frac{25}{99}$.
- Express $\frac{17}{99}$ as a recurring decimal.
- Express the recurring decimal $0.\dot{1}2\dot{3}$ as a fraction.

2. What is the smallest prime which is the sum of five different primes?

A 39 B 41 C 43 D 47 E 53

Solution **C**

If five different primes include 2, they consist of 2 and four odd primes. Hence their sum is even and hence is not prime.

Therefore five different primes whose sum is also prime cannot include 2. Hence they are five odd primes.

The sum of the first five odd primes is $3 + 5 + 7 + 11 + 13 = 39$, which not prime.

The smallest prime greater than 13 is 17. If we replace 13 by 17 we obtain the sum $3 + 5 + 7 + 11 + 17 = 43$, which is prime.

It follows that the smallest prime that is the sum of five different primes is 43.

For investigation

Which is the smallest prime which is the sum of six different primes?

3. The positive integer 2018 is the product of two primes.

What is the sum of these two primes?

A 1001 B 1010 C 1011 D 1100 E 1101

SOLUTION **C**

The prime factorization of 2018 is given by $2018 = 2 \times 1009$. The sum of the two prime factors is $2 + 1009 = 1011$.

NOTE

There is no very quick way to check that 1009 is prime, as this would involve checking that no prime ≤ 29 is a divisor. [Why is this enough?] Fortunately, the question tells us that 2018 is the product of two primes. We can conclude from this that the second factor 1009 is prime.

4. Last year, an earthworm from Wigan named Dave wriggled into the record books as the largest found in the UK. Dave was 40 cm long and had a mass of 26 g.

What was Dave's mass per unit length?

A 0.6 g/cm B 0.65 g/cm C 0.75 g/cm D 1.6 g/cm E 1.75 g/cm

Solution **B**

Dave's mass per unit length is $\frac{26}{40}$ g/cm. We have

$$\frac{26}{40} = \frac{26}{10 \times 4} = \frac{2.6}{4} = 0.65.$$

Therefore Dave's mass per unit length is 0.65 g/cm.

5. On a Monday, all prices in Isla's shop are 10 % more than normal. On Friday all prices in Isla's shop are 10 % less than normal. James bought a book on Monday for £5.50. What would be the price of another copy of this book on Friday?

A £5.50 B £5.00 C £4.95 D £4.50 E £4.40

SOLUTION **D**

The book cost £5.50 on a Monday after prices have risen by 10 %. Therefore the price before this increase was £5. It follows that on a Friday, after prices have been cut by 10 %, the price of the book has fallen by 50p to £4.50.

6. Which of the following is not a multiple of 5?

A $2019^2 - 2014^2$ B $2019^2 \times 10^2$ C $2020^2 \div 101^2$
D $2010^2 - 2005^2$ E $2015^2 \div 5^2$

SOLUTION **E**

Using the factorization of the difference of two squares, we see that $2019^2 - 2014^2 = (2019 - 2014)(2019 + 2014) = 5 \times 4033$, and hence is a multiple of 5.

$2019^2 \times 10^2$ is a multiple of 10^2, that is, a multiple of 100, and hence is a multiple of 5.

$2020^2 \div 101^2 = (20 \times 101)^2 \div 101^2 = 20^2 = 400$ and so is a multiple of 5.

Because 2010^2 and 2005^2 are both multiples of 5, their difference, $2010^2 - 2005^2$, is also a multiple of 5.

However, $2015^2 \div 5^2 = \left(\frac{2015}{5}\right)^2 = 403^2$ and, because 403 is not a multiple of 5, 403^2 is not a multiple of 5.

Hence the correct answer is option E.

FOR INVESTIGATION

Prove the following facts about divisibility by 5 that are used in the above solution.

(a) For all positive integers m and n, if m and n are multiples of 5, then $m - n$ is a multiple of 5.

(b) For all positive integers m, if m is not a multiple of 5, then m^2 is not a multiple of 5.

Which of the statements in 4.1 are true when ‘5’, is replaced by ‘6’?

Which of the statements in 4.1 are true when ‘5’, is replaced by ‘20’?

For which integers d is it true that for all positive integers m, if m is not a multiple of d, then m^2 is not a multiple of d?

7. What is the difference between one-third and 0.333?

A 0 B $\frac{3}{1000}$ C $\frac{1}{3000}$ D $\frac{3}{10000}$ E $\frac{1}{30000}$

Solution **C**

Expressed as fractions, one-third is $\frac{1}{3}$ and 0.333 is $\frac{333}{1000}$. Therefore, the difference between one-third and 0.333 is

$$\frac{1}{3} - \frac{333}{1000} = \frac{1000}{3000} - \frac{999}{3000} = \frac{1}{3000}.$$

For investigation

(a) What is the difference between one-third and 0.3333 ?

(b) What is the least positive k such that the difference between one-third and $0.\overbrace{33\ldots33}^{k}$ is less than 10^{-6} ?

What is the difference between $\frac{22}{7}$ and 3.141 ?

What is the difference between 1 and $0.\dot{9}$?

ThrivingScholars

8. What is the remainder when 1234×5678 is divided by 5?

A 0 B 1 C 2 D 3 E 4

Solution **C**

The units digit of 1234×5678 is the same as the units digit of 4×8, and hence is 2. Therefore the remainder when 1234×5678 is divided by 5 is 2.

For investigation

- What is the remainder when 1234×5678 is divided by 3?
- What is the remainder when 1234×5678 is divided by 11?
- The integer m has remainder 3 when it is divided by 11. The integer n has remainder 4 when divided by 11.

  What is the remainder when mn is divided by 11?

9. The sum of four consecutive primes is itself prime.

What is the largest of the four primes?

A 37 B 29 C 19 D 13 E 7

Solution **E**

The sum of four odd primes is an even number greater than 2, and therefore is not a prime. Therefore the four consecutive primes whose sum is prime includes the only even prime 2.

It follows that the four consecutive primes are 2, 3, 5 and 7. Their sum is 17 which is a prime. The largest of these four consecutive primes is 7.

For investigation

Which is the smallest prime that is the sum of five consecutive primes?

10. What is the value of $\frac{4^{800}}{8^{400}}$?

A $\frac{1}{2^{400}}$ B $\frac{1}{2^{200}}$ C 1 D 2^{200} E 2^{400}

Solution **E**

We have

$$\begin{aligned}\frac{4^{800}}{8^{400}} &= \frac{(2^2)^{800}}{(2^3)^{400}}\\ &= \frac{2^{1600}}{2^{1200}}\\ &= 2^{1600-1200}\\ &= 2^{400}.\end{aligned}$$

For investigation

For which integer n is $\frac{27^{900}}{9^{2700}} = 3^n$?

11. What is $\sqrt{123454321}$?

A 1111111 B 111111 C 11111 D 1111 E 111

SOLUTION **C**

Note: You could not be expected to be able to calculate the value of $\sqrt{123454321}$ without the use of a calculator. So you need to find some other way to select the correct option. We use a method based on the size of the number 123454321.

We have

$$10^8 < 123454321 < 10^{10}.$$

Therefore

$$\sqrt{10^8} < \sqrt{123454321} < \sqrt{10^{10}},$$

that is,

$$10^4 < \sqrt{123454321} < 10^5.$$

The correct option is therefore the only one that is between 10^4 and 10^5. Therefore, of the given options, it is 11111 that equals $\sqrt{123454321}$.

FOR INVESTIGATION

The answer given above assumes that one of the given options is correct.

Verify that $11111 = \sqrt{123454321}$ by checking that $11111^2 = 123454321$.

12. The positive integer k satisfies the equation $\sqrt{2} + \sqrt{8} + \sqrt{18} = \sqrt{k}$.

What is the value of k?

A 28 B 36 C 72 D 128 E 288

SOLUTION **C**

Because $8 = 2^2 \times 2$ and $18 = 3^2 \times 2$, we have $\sqrt{8} = 2\sqrt{2}$ and $\sqrt{18} = 3\sqrt{2}$. Therefore

$$\begin{aligned}\sqrt{2} + \sqrt{8} + \sqrt{18} &= \sqrt{2} + 2\sqrt{2} + 3\sqrt{2}\\ &= 6\sqrt{2}\\ &= \sqrt{6^2 \times 2}\\ &= \sqrt{72}.\end{aligned}$$

Therefore $k = 72$.

13. When evaluated, which of the following is not an integer?

A 1^{-1} B $4^{-\frac{1}{2}}$ C 6^0 D $8^{\frac{2}{3}}$ E $16^{\frac{3}{4}}$

SOLUTION **B**

We have

$$1^{-1} = \frac{1}{1} = 1,$$

$$4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2},$$

$$6^0 = 1,$$

$$8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2 = 2^2 = 4,$$

and

$$16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = (\sqrt[4]{16})^3 = 2^3 = 8.$$

We therefore see that only the expression of option B gives a number which is not an integer when it is evaluated.

FOR INVESTIGATION

- The standard convention is that for $x \neq 0$, we take 1 as the value of x^0. Why is this convention a sensible one to use?
- The standard convention is that 0^0 represents the number 1. Why is this a sensible convention?
- In the answer to Question 8 we have used the fact that when p and q are positive integers, and $x > 0$, the convention is that $x^{\frac{p}{q}}$ means $(\sqrt[q]{x})^p$. Why do we adopt this convention?

14. The prime factorization of 2024 is $2^3 \times 11 \times 23$.

How many two-digit numbers are factors of 2024?

A 2 B 4 C 6 D 7 E 8

SOLUTION **D**

Written in full we have

$$2024 = 2 \times 2 \times 2 \times 11 \times 23.$$

The factors of 2024 are 1 and the numbers obtained by multiplying together some, or all, of the numbers that occur in this product.

We see from the factorization that 11 and 23 are two-digit factors of 2024.

Since $11 \times 23 = 253$ no product which includes both 11 and 23 produces a two-digit factor of 2024. However, we can obtain two-digit factors of 2024 by multiplying 11 or 23 by 2, 2^2 or 2^3 provided that the resulting product has two digits.

In this way we obtain the two-digit factors

$$\begin{aligned} 2 \times 11 &= 22, \\ 2^2 \times 11 &= 44, \\ 2^3 \times 11 &= 88, \\ 2 \times 23 &= 46 \end{aligned}$$

and

$$2^2 \times 23 = 92.$$

Hence, altogether there are 7 two-digit factors of 2024, namely, 11, 22, 23, 44, 46, 88 and 92.

FOR INVESTIGATION

- Find all the three-digit factors of 2024.
- Find all the two-digit factors of 10 000.

15. How many of the numbers 6, 7, 8, 9, 10 are factors of the sum $2^{2024} + 2^{2023} + 2^{2022}$?

A 1 B 2 C 3 D 4 E 5

Solution **B**

For convenience, we put $S = 2^{2024} + 2^{2023} + 2^{2022}$.

We have

$$S = 2^{2024} + 2^{2023} + 2^{2022} = 2^{2022}(2^2 + 2^1 + 1) = 2^{2022}(4 + 2 + 1) = 2^{2022} \times 7.$$

It follows that 7 is a factor of S. Also, since $8 = 2^3$, 8 is a factor of 2^{2022} and hence it is a factor of S. On the other hand, 3 and 5 are neither factors of 2^{2022} nor factors of 7. So they are not factors of S. It follows that 6 and 9 which are multiples of 3 are not factors of S. Similarly, 10 is a multiple of 5 and hence it is not a factor of S.

We therefore see that just two of the numbers 6, 7, 8, 9 and 10 are factors of S, namely 7 and 8.

For investigation

10.1 Which is the largest prime factor of $2^{2024} + 2^{2023} + 2^{2022} + 2^{2021}$?

11. Wenlu, Xander, Yasser and Zoe make the following statements:

Wenlu: "Xander is lying." Xander: " Yasser is lying."

Yasser: "Zoe is telling the truth." Zoe: "Wenlu is telling the truth."

What are the possible numbers of people telling the truth?

A 1 or 2 B 1 or 3 C 2 D 2 or 3 E 3

Solution **B**

Suppose Wenlu is telling the truth. Then Xander is lying. Therefore Yasser is telling the truth. Hence Zoe is telling the truth, and this agrees with the fact that Wenlu is telling the truth.

Hence it is possible that Wenlu is telling the truth. We have seen that in this case Wenlu, Yasser and Zoe are telling the truth.

Suppose Wenlu is lying. Then Xander is telling the truth. Therefore Yasser is lying. Hence Zoe is lying, and this agrees with the fact that Wenlu is lying.

Hence it is possible that Wenlu is lying. We have seen that in this case only Xander is telling the truth.

Therefore the number of people telling the truth is either 1 or 3.

For investigation

Wenlu says "Xander is lying", Xander say "Yasser is telling the truth", Yasser says "Zoe is lying" and Zoe says "Wenlu is telling the truth".

How many of them could be telling the truth?

16. Which one of the following expressions is a square number for each positive integer n?

A $n + 1$

B $n(n + 1) + 1$

C $n(n + 1)(n + 2) + 1$

D $n(n + 1)(n + 2)(n + 3) + 1$

E $n(n + 1)(n + 2)(n + 3)(n + 4) + 1$

Solution **D**

When $n = 1$, we have

$$\begin{aligned} n + 1 &= 2 \\ n(n + 1) + 1 &= 1 \times 2 + 1 = 3 \\ n(n + 1)(n + 2) + 1 &= 1 \times 2 \times 3 + 1 = 7 \\ n(n + 1)(n + 2)(n + 3) + 1 &= 1 \times 2 \times 3 \times 4 + 1 = 25 \\ \text{and} \quad n(n + 1)(n + 2)(n + 3)(n + 4) + 1 &= 1 \times 2 \times 3 \times 4 \times 5 + 1 = 121. \end{aligned}$$

Of these values, only 25 and 121 are squares. So the correct answer is either D or E. To decide between these we put $n = 2$. This gives

$$\begin{aligned} n(n + 1)(n + 2)(n + 3) + 1 &= 2 \times 3 \times 4 \times 5 + 1 = 121 \\ \text{and} \quad n(n + 1)(n + 2)(n + 3)(n + 4) + 1 &= 2 \times 3 \times 4 \times 4 \times 6 + 1 = 721 \end{aligned}$$

of which only 121 is a square.

In the context of the SMC we can now conclude that the correct option is D.

In the SMC you are entitled to assume that just one of the given options is correct. Therefore the correct answer can be found by eliminating four of the options, as we have done above. However, this would not be an adequate answer if you were required to give a fully explained answer.

In this case we can show that all the values of the polynomial $n(n + 1)(n + 2)(n + 3) + 1$ are squares, as follows.

$$\begin{aligned} n(n + 1)(n + 2)(n + 3) + 1 &= n(n + 3)(n + 1)(n + 2) + 1 \\ &= (n^2 + 3)(n^2 + 3n + 2) + 1 \\ &= ((n^2 + 3n + 1) - 1)((n^2 + 3n + 1) + 1) + 1 \\ &= (n^2 + 3n + 1)^2 - 1 + 1 \\ &= (n^2 + 3n + 1)^2, \end{aligned}$$

and hence is always a square.

It is true that, in general, if a polynomial has a value which is a square for each positive integer n then this polynomial is the square of another polynomial. The proof of this fact is not straightforward. The above calculation shows the truth of this in one particular case.

For investigation

Prove that there does not exist a positive integer n for which $n(n + 1) + 1$ is a square.

17. p, q, r and s are two-digit primes which between them use all the non-zero digits except 5.

What is the value of $p + q + r + s$?

A 220 B 210 C 200 D 190
E more information needed

Solution **A**

Of the digits 1, 2, 3, 4, 6, 7, 8 and 9, only 1, 3, 7 and 9 can be the units digit of a two-digit prime. This leaves 2, 4, 6 and 8 as the tens digits of the primes.

Therefore the sum of the units digits of the four primes is $1 + 3 + 7 + 9$, which is equal to 20, and the sum of the tens digits is $2 + 4 + 6 + 8$, which is also 20.

Hence the sum of the four primes is $20 \times 10 + 20 \times 1 = 200 + 20 = 220$.

This answer has been obtained without finding the primes p, q, r and s, or indeed even showing that four primes which use all the non-zero digits except 5, exist. You are asked to check in Problem 12.1 that a solution does exist.

For investigation

Find four two-digit primes p, q, r and s which between them use all the non-zero digits except 5.

How many different sets are there of four two-digit primes which between them use all the digits except 5?

Find one three-digit prime and three two-digit primes which between them include all the non-zero digits.

Find two three-digit primes and two two-digit primes which between them use all the ten digits, including 0.

18. What is the sum of the digits of the integer which is equal to $6666666^2 - 3333333^2$?

A 27 B 36 C 45 D 54 E 63

Solution **E**

Using the standard factorization $x^2 - y^2 = (x - y)(x + y)$, we have

$$\begin{aligned} 6666666^2 - 3333333^2 &= (6666666 - 3333333)(6666666 + 3333333) \\ &= 3333333 \times 9999999 \\ &= 3333333 \times (10000000 - 1) \\ &= 3333333 \times 10000000 - 3333333 \times 1 \\ &= 33333330000000 - 3333333 \\ &= 33333326666667. \end{aligned}$$

It may be seen that the number 33333326666667 is written with six 3s followed by one 2, six 6s and one 7. Therefore the sum of its digits is

$$\begin{aligned} (6 \times 3) + 2 + (6 \times 6) + 7 &= 18 + 2 + 36 + 7 \\ &= 63. \end{aligned}$$

For investigation

Let $n = 666\,666\,666^2 - 333\,333\,333^2$.

What is the sum of the digits of n?

(a) Let a be the integer $\overbrace{666\ldots666}^{k}$ which in standard form is written as a string of k 6s, and let b be the integer $\overbrace{333\ldots333}^{k}$ which is written as a string of k 3s.

Find a formula, in terms of k, for the sum of the digits of the integer $a^2 - b^2$.

(b) Find a formula for the sum of the digits of $c^2 - d^2$ where $c = \overbrace{777\ldots777}^{k}$ and $d = \overbrace{222\ldots222}^{k}$.

19. The greatest power of 7 which is a factor of 50! is 7^k $(n! = 1\times2\times3\times4\times\ldots\times(n-1)\times n)$.

What is k?

A 4 B 5 C 6 D 7 E 8

Solution **E**

$50! = 1 \times 2 \times \cdots \times 50$. Because the seven numbers 7, 14, 21, 28, 35, 42 and 49 are divisible by 7, they each contribute 1 to the power of 7 which is a factor 50! In addition, because 49 is divisible by 7^2, it contributes an additional power of 7.

Therefore the highest power of 7 that divides 50! is $7 + 1 = 8$.

For investigation

12.1 Find the greatest power of 3 which is a factor of 50!

The general formula for the greatest power of a prime p which is a factor of $n!$ is

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots. \qquad (1)$$

Here $\lfloor x \rfloor$ is the *integer part* of x which is defined by

$$\lfloor x \rfloor = k \Leftrightarrow k \text{ is the largest integer } \leq x.$$

For example $\left\lfloor \frac{3}{4} \right\rfloor = 0$, $\left\lfloor \frac{22}{7} \right\rfloor = 3$, and $\lfloor \sqrt{5} \rfloor = 2$.

Note that, although the sum in (1) looks infinite, whenever $p^t > n$, we have $0 < \frac{n}{p^t} < 1$ and therefore $\left\lfloor \frac{n}{p^t} \right\rfloor = 0$. Therefore we could replace (1) by the finite sum

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots + \left\lfloor \frac{n}{p^s} \right\rfloor \qquad (2)$$

where s is the largest integer such that $p^s \leq n$.

For example, since $7^3 < 1000$, but $7^4 > 1000$, it follows from (2) that the greatest power of 7 that is a factor of 1000! is

$$\left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{7^2} \right\rfloor + \left\lfloor \frac{1000}{7^3} \right\rfloor = \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{49} \right\rfloor + \left\lfloor \frac{1000}{343} \right\rfloor = 142 + 20 + 2 = 164.$$

Find the greatest power of 2 that is a factor of 100!

Find the greatest power of 11 that is a factor of 1000!

Find the greatest power of 10 that is a factor of 1000!

Explain why the formula (1) given above for the greatest power of a prime p that divides $n!$ is correct.

20. How many different squares are factors of 2025?

A 2 B 3 C 4 D 5 E 6

Solution **E**

One method would be to list *all* the factors of 2025 and then to check which of them are squares. This would take rather a lot of time.

It is quicker to make use of the factorization of integers into primes, This can be used to decide whether an integer is a square.

Before discussing the general case, we give the example of the number 72^2.

The prime factorization of 72 is $2^3 \times 3^2$. It follows that $72^2 = (2^3)^2 \times (3^2)^2 = 2^6 \times 3^4$. Note that the exponents of the primes in this factorization are both even numbers.

In general, suppose that $n = p^a \times q^b \times r^c \times \ldots$, where $p, q, r, \ldots$ are the different primes that are factors of n. Then $n^2 = p^{2a} \times q^{2b} \times r^{2c} \times \ldots$. So the exponents of the primes in the prime factorization of a square are always even integers.

Conversely, an integer with a prime factorization of this form is a square.

Thus the test for whether an integer n is a square is that in the prime factorization of n, the exponent of each prime is an even number.

The factorization of 2025 into primes is given by

$$2025 = 3 \times 3 \times 3 \times 3 \times 5 \times 5, \text{ that is, } 3^4 \times 5^2.$$

Therefore the factors of 2025 that are squares are all the integers of the form $3^a \times 5^b$, where a is an even number in the range $0 \leq a \leq 4$ and b is an even integer in the range $0 \leq b \leq 2$.

Hence there are 3 possible values for a, namely, 0, 2 and 4, and 2 possible values, 0 and 2, for b. This gives $2 \times 3 = 6$ choices for the pair a, b.

Therefore 2025 has 6 square factors.

For investigation

List the 6 square factors of 2025.

How many square factors does the number 2025^2 have?

How many square factors does the number 10! have?[*Note:* $10! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$.]

Explain why, if the prime factorization of n is $n = p^a \times q^b \times r^c \times \ldots$, where $p, q, r, \ldots$ are distinct primes, then n has $(a + 1)(b + 1)(c + 1)\ldots$ different factors.

Suppose that the prime factorization of n is $n = p^a \times q^b \times r^c \times \ldots$, where $p, q, r, \ldots$ are distinct primes.

Give a formula in terms of $a, b, c, \ldots$ for the number of factors of n that are squares.

21. Ayain writes down all the 3-digit numbers consisting of three different odd digits

How many of Ayain's numbers are divisible by 3?

A 42 B 36 C 30 D 24 E 18

Solution **D**

The test for whether an integer is divisible by 3 is that the sum of its digits is divisible by 3. [You are asked to prove this in Problem 12.4.]

There are four sets made up of three of the odd digits 1, 3, 5, 7, 9, whose sum is a multiple of 3. These are $\{1, 3, 5\}$, $\{1, 5, 9\}$, $\{3, 5, 7\}$ and $\{5, 7, 9\}$.

The three digits of each set can be arranged in six ways to form six three digits numbers. Therefore $4 \times 6 = 24$ of Ayain's numbers are divisible by 3.

For investigation

- Make a list, in numerical order of the 24 of Ayain's numbers that are divisible by 3.
- How many of Ayain's numbers are divisible by 5?
- How many of Ayain's numbers are divisible by 11?
- Prove that an integer is divisible by 3 if, and only if, the sum of its digits is divisible by 3.

22. The number $M = 124563987$ is the smallest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number. For example, the 5th and 6th digits of M make the number 63 which is not prime. N is the largest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number.

What are the 5th and 6th digits of N?

A 6 and 3 B 5 and 4 C 5 and 2 D 4 and 8 E 3 and 5

Solution **E**

For convenience, in this solution by a *subnumber* of a positive number n we mean a number formed from consecutive digits of n. For example, the two-digit subnumbers of 1234 are 12, 23 and 34, and the three-digit subnumbers of 1234 are 123 and 234.

Note that this is not standard mathematical terminology, but has been introduced just for the purposes of this question.

We note first that 63 and 93 are the only numbers formed of two different non-zero digits with 3 as the unit digits that are not primes. It follows that, if the digit 3 is not the first digit of the number N, the digit 3 could only occur in N immediately after either the digit 9 or the digit 6.

We construct N by beginning with the largest digit 9, and then use all the other non-zero digits once each by always choosing the largest digit not yet used subject to the condition that 3 comes immediately after 9 or immediately after 6. In this way we obtain the number 987635421.

We now see that in the number 987635421 none of the two-digit subnumbers is a prime. Any larger number using all the digits must either begin 98765... or 98764..., but in each case the 3 would follow either 1,2,4 or 5, and so produce a two-digit subnumber that is a prime.

Therefore the largest number with the required property is $N = 987635421$. It follows that the 5th and 6th digits of N are 3 and 5.

For investigation

- Find the largest positive integer that uses different non-zero digits, and has the property that all its two-digit subnumbers are prime.
- Find the largest positive integer that uses different non-zero digits, and has the property that all its two-digit subnumbers are divisible by 7.
- Find the largest positive integer that uses different non-zero digits, and has the property that all its three-digit subnumbers are divisible by 3.
- Find the largest positive integer that uses different non-zero digits, and has the property that all its three-digit subnumbers are divisible by 11.

23. The integer n is such that $1 < n < 99$. P is the difference between 1 and n, Q is the difference between 99 and n, and R is the difference between P and Q.

For how many values of n is R a prime number?

A 0 B 2 C 4 D 8 E 99

SOLUTION **B**

For n between 1 and 99, the difference between 1 and n is $n - 1$, and the difference between n and 99 is $99 - n$. Thus $P = n - 1$ and $Q = 99 - n$.

The difference between P and Q depends on which is the larger. You are asked, in Problem 15.1, to check that $P < Q$ for $1 < n < 50$ and $Q < P$ for $50 < n < 99$, while $P = Q$ when $n = 50$.

It follows that when $1 < n < 50$, we have $R = Q - P = (99 - n) - (n - 1) = 100 - 2n$ and that when $50 < n < 99$, we have $R = P - Q = (n - 1) - (99 - n) = 2n - 100$. Also, $R = 0$ when $n = 50$.

When n is an integer both $100 - 2n$ and $2n - 100$ are even integers. The only even prime is 2. It follows that R is prime only when either $100 - 2n = 2$ or $2n - 100 = 2$.

The equation $100 - 2n = 2$ has the single solution $n = 49$.

The equation $2n - 100 = 2$ has the single solution $n = 51$.

Therefore R is prime only when n is either 49 or 51.

Therefore there are just 2 values of n for which R is prime.

FOR INVESTIGATION

Check that $P < Q$ for $1 < n < 50$ and $Q < P$ for $50 < n < 99$, while $P = Q$ when $n = 50$.

Note: This problem is the version of Question 15 with the restriction that the integer n is between 1 and 99 dropped.

With P, Q and R as in Question 15, for how many integer values of n is R a prime number?

Note: This problem is the version of Question 15 with the restriction that n is an integer dropped.

With P, Q and R as in Question 15, for how many rational numbers, n, where $1 < n < 99$, is R a prime number?

24. The number 840 can be written as $\frac{p!}{q!}$, where p and q are positive integers less than 10.

What is the value of $p + q$?

Note that, $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.

A 8 B 9 C 10 D 12 E 15

Solution **C**

We note first that, as $\frac{p!}{q!} = 840$, it follows that $p! > q!$ and hence $p > q$. Therefore

$$\frac{p!}{q!} = \frac{1 \times 2 \times \cdots \times q \times (q+1) \times \cdots \times p}{1 \times 2 \times \cdots \times q} = (q+1) \times (q+2) \times \cdots \times (p-1) \times p.$$

Thus $840 = \frac{p!}{q!}$ is the product of the consecutive integers $q + 1, q + 2, \ldots, p - 1, p$, where $p \leq 9$.

Since 840 is not a multiple of 9, $p \neq 9$. Since 840 is a multiple of 7, $p \geq 7$.

Now $5 \times 6 \times 7 \times 8 = 1680 > 840$, while $6 \times 7 \times 8 = 336 < 840$. Hence 840 is not the product of consecutive integers of which the largest is 8.

We deduce that $p = 7$. It is now straightforward to check that

$$840 = 4 \times 5 \times 6 \times 7 = \frac{7!}{3!}.$$

Therefore $p = 7$ and $q = 3$. Hence $p + q = 7 + 3 = 10$.

For investigation

- Find positive integers p and q with $q < p \leq 20$ such that

$$\frac{p!}{q!} = 2730.$$

- Is it possible to find positive integers p and q with $q < p \leq 20$ such that

$$\frac{p!}{q!} = 253\,?$$

25. The positive integer N has 2025 digits. The first digit is a 3. Every two consecutive digits of N form a number that is divisible by either 17 or 23. The units digit of N could either be p or q.

What is the value of $p + q$?

A 3 B 6 C 7 D 9 E 10

Solution **C**

The two-digit multiples of 17 are 17, 34, 51, 68 and 85.

The two-digit multiples of 23 are 23, 46, 69 and 92.

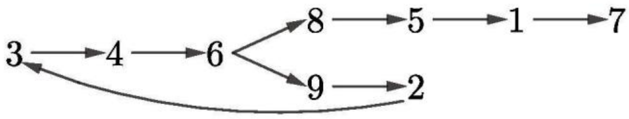
In numerical order, these multiples are 17, 23, 34, 46, 51, 68, 69, 85, 92.

From this we see that in the number N the digit 3 must be followed by 4, and 4 by 6. The digit 6 could be followed by either 8 or 9.

The digit 8 must be followed by 5, 5 by 1 and 1 by 7. The sequence of digits would then end as there is no two-digit multiple of either 17 or 23 with tens digit 7.

The digit 9 must be followed by 2 and 2 by 3, creating a loop.

This is illustrated by the following diagram:



Since $2025 \div 5 = 405$, we see that the 2025 digits of N result either from going 405 times round the loop $3 \to 4 \to 6 \to 9 \to 2$, or from going round this loop 404 times followed by the sequence $3 \to 4 \to 6 \to 8 \to 5$.

Therefore the final digit of N is either 2 or 5.

Hence $p + q = 2 + 5 = 7$.

For investigation

Which digits *cannot* occur in the number N?

Consider those positive integers with 2025 digits in which every two consecutive digits form a number that is divisible by either 17 or 23, but which do not necessarily have first digit 3.

(a) Which digits could be the first digit of such an integer?

(b) For each digit that could be the first digit of such a number, determine which digits could be the units digit.

The positive integer M has 1000 digits.

Every two consecutive digits of M form a number that is divisible by either 13 or 41.

(a) Which digits could be the first digit of M?

(b) Which digits could be the final digit of M?