

Notes for the Physics GRE

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1 Introduction

Hello, potential physics Ph.D. student! If you are reading these notes, chances are you are preparing to take the physics GRE subject test (pGRE). This exam, administered by the Educational Testing Service (ETS), consists of 100 multiple choice problems in 170 minutes. Students score one point for every correct answer and lose a quarter of a point for every incorrect answer, so arbitrary guessing is costly. The raw score out of 100 is converted to a scaled score out of 990, which roughly corresponds to a raw score of 85 or above and a percentile of 95.

The test is designed to measure the student's physical intuition and education, but in practice it also measures a student's calculation speed: most students who take the test are unable to complete it in the time allowed. Hence, you do not want to have to waste time on the exam re-deriving formulas and should try to commit as much of your undergraduate physics education to memory as possible. In these notes, I will present some of the most prolific formulas found on the pGRE, along with some general tips for taking the exam. The rest is in your hands. Good luck!

2 General Advice

These are the best tips I can give to someone preparing for the pGRE.

1. **Take the test seriously.** ETS isn't going to give you any extra points because the name of your undergraduate institution is "Harvard," "Stanford," "MIT," "Princeton," "Cornell," etc. The test might not be as difficult or rigorous as the exams you're accustomed to, but taking it lightly is a big mistake (and one that some of the smartest and most talented undergrads make every year). The pGRE is probably about as important as the rest of the exams you've taken in your undergraduate career combined: treat it as such. If you want to study theoretical physics at one of these top schools, you should probably shoot for a score of 900 or better.
2. **Take every available practice exam, and do so under testing conditions.** ETS should mail you a practice booklet when you register for the pGRE, and you can find a set of four other practice tests on the Ohio State Physics web page. You should take all five of these exams under testing conditions (i.e. in 170 minute intervals). As Vince Lombardi said, "Practice does not make perfect. Only perfect practice makes perfect." Don't waste your time by doing a few practice problems here and there: approach the practice test in the same manner you approach the real test, and get accustomed to 170 minutes of furious thought.
3. **Start studying early.** I started studying for the October physics GRE subject test the day that I finished the math GRE subject test in April. This is probably a tad extreme, but you should probably start studying

at least 2 months before the exam. A good strategy would be to take the first practice test 2 months before the exam to get an idea of what you need to learn, the second practice test 1 month before the exam, the third 2 weeks before the exam, the fourth 1 week before the exam, and the fifth 2 days before the exam.

4. **Set a consistent sleep schedule of at least 8-9 hours per night.** Anyone who has taken Psych 101 at Cornell with Professor James Maas can tell you that the average undergrad needs 9.25 hours of sleep per night to be well-rested. Furthermore, it is critical to develop a consistent sleep schedule. This can be a problem for many aspiring physics grad students because the pGRE is administered at 8:30am, so you will probably have to wake up at 7:30am or so. For the week before the exam, you should really try to get on an 11pm to 7:30am sleep schedule (as tough as that might be for a physics party animal such as yourself). Whatever you do, don't stay up late studying for the exam.
5. **Take your own, handwritten notes.** Actually writing formulas down improves your ability to remember them. While the notes presented here are nice as a guide, you should create your own notes, even if they are just paraphrases of mine.
6. **Use dimensional analysis.** A decent fraction of the problems on the pGRE can be solved using dimensional analysis. Before starting a 5-minute calculation, pause to see if you can determine the answer based solely on its dimensionality.

3 Classical Mechanics (20%)

Newton's laws:

$$F = ma, \tag{1}$$

$$F_{12} = -F_{21}. \tag{2}$$

Newton's gravitational law:

$$\vec{F} = \frac{GMm}{r^2} \hat{r} \tag{3}$$

where r is the distance between two objects of mass M and m , respectively, and G is the gravitational constant. On earth, $g = \frac{GM}{r^2} = 9.8m \cdot s^{-2}$. Kepler's laws of planetary motion:

1. The orbit of every planet is an ellipse with the Sun at one of the two foci.
2. A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

3. The square of the orbital period T of a planet is directly proportional to the cube of the semi-major axis a of its orbit.

In mathematical terms, the third law is

$$T^2 \propto a^3. \quad (4)$$

This can be seen, for circular motion, from

$$\frac{GMm}{r^2} = \frac{mv^2}{r} = m\omega^2 r = \frac{mr(2\pi)^2}{T^2}. \quad (5)$$

An elliptical orbit may be parametrized by the radius r and angle θ by

$$r = \frac{p}{1 + \epsilon \cos \theta}, \quad (6)$$

where ϵ is the eccentricity of the orbit.

$$\begin{aligned} \epsilon = 0 &\Rightarrow \text{circle} \\ \epsilon < 1 &\Rightarrow \text{ellipse} \\ \epsilon = 1 &\Rightarrow \text{parabola} \\ \epsilon > 1 &\Rightarrow \text{hyperbola.} \end{aligned} \quad (7)$$

The relation between escape velocity v_e and circular orbit velocity v_c :

$$v_e = \sqrt{2}v_c. \quad (8)$$

For a spring of spring constant k ,

$$\vec{F} = -k\vec{x}, \quad (9)$$

$$V = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2. \quad (10)$$

Thus, $\omega = \sqrt{\frac{k}{m}}$, and the solution to the equation of motion is harmonic motion,

$$x(t) = A \sin(\omega t + \phi). \quad (11)$$

For springs of spring constant k_i in parallel, the spring constants sum,

$$k = \sum k_i. \quad (12)$$

For springs in series, the reciprocals sum,

$$\frac{1}{k} = \sum \frac{1}{k_i}. \quad (13)$$

If two free masses m_1 and m_2 are connected by a spring k , we use the reduced mass,

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad (14)$$

so

$$\omega = \sqrt{\frac{k}{\mu}}. \quad (15)$$

Note that if $m_1 = m_2 = m$, then $\mu = m/2$.

The Lagrangian L of a (classical) system is

$$L = T - V \quad (16)$$

where $T(q_i, \dot{q}_i)$ is the kinetic energy ($T = \frac{1}{2}m\dot{x}_i^2$, in cartesian coordinates) and $V(q_i)$ is the potential energy. The Euler-Lagrange equations are,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \quad (17)$$

These give rise to the equations of motion. If our system is multidimensional, we get one equation of motion for each coordinate x_i . The energy of the system is defined as,

$$E = \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) - L. \quad (18)$$

Furthermore, one can show that the Euler-Lagrange equation yields,

$$\frac{dE}{dt} = - \frac{\partial L}{\partial t} \quad (19)$$

Note that if L has no explicit time dependence, $\frac{\partial L}{\partial t} = 0$, then energy is conserved in the system. The Lagrangian is related to the Hamiltonian H of the system by,

$$H = \left(\sum_i p_i \dot{q}_i \right) - L. \quad (20)$$

Here, q_i is the canonical coordinate, and $p_i = \frac{\partial L}{\partial \dot{q}_i}$ is the conjugate canonical momentum. Note that if $\frac{\partial L}{\partial \dot{q}_i} = 0$, then p_i is conserved by the Euler-Lagrange equation. After computing the Hamiltonian in terms of $q_i, \dot{q}_i$, and p_i , you should be able to substitute p_i in for $\dot{q}_i$ and get $H = H(q_i, p_i)$. Hamilton's equations are,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad (21)$$

$$\dot{p}_i = - \frac{\partial H}{\partial q_i}. \quad (22)$$

We may define the Poisson bracket between two functions $f(q_i, p_i, t)$ and $g(q_i, p_i, t)$ by

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i}. \quad (23)$$

With this, Hamilton's equation generalizes to,

$$\frac{d}{dt} f = \{f, H\} + \frac{\partial f}{\partial t}. \quad (24)$$

Geometry	I
Solid disc/cylinder, radius R	$\frac{1}{2}MR^2$
Hollow hoop/cylinder, radius R	MR^2
Solid sphere, radius R	$\frac{2}{5}MR^2$
Hollow sphere, radius R	$\frac{2}{3}MR^2$

Table 1: Moments of inertia for some common objects of mass M . Note that the axes of the disc/hoop/cylinder are the ones through the center of ring/cylinder.

The moment of inertia I for a rigid body around an axis is

$$I = \sum m_i R_i^2, \quad (25)$$

where the sum is over all the point masses m_i and R_i is the distance from the mass to the axis. The kinetic energy for a body rotating about its center of mass is

$$T = T_{\text{rotation}} + T_{\text{translation}} = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2, \quad (26)$$

where ω is the angular speed of rotation. The angular momentum of a rotating body is

$$L = I\omega. \quad (27)$$

The moments of inertia for some common geometries are shown in Table 1.

In an elastic collision, both total energy and total momentum are conserved. If the objects have initial velocities v_i^1 and v_i^2 along the x-direction, then they will have final velocities v_f^1 and v_f^2 that satisfy,

$$v_i^1 - v_i^2 = v_f^2 - v_f^1. \quad (28)$$

In an inelastic collision, such as when two globs collide and stick together, only total momentum is conserved, energy is not.

For an object sliding on a surface with friction, the force of friction is proportional to the normal force exerted orthogonal to the surface,

$$F_{\text{fr}} = \mu F_N. \quad (29)$$

If the object is not moving, use the coefficient of static friction, $\mu = \mu_S$. If it is moving, use the coefficient of kinetic friction, μ_K . In general, $\mu_S > \mu_K$.

4 Electromagnetism (18%)

Under Franklin's model of electricity, electric charge is always conserved. There are 2 kinds of charge, positive and negative, which are quantized. The force obeys an inverse-square law, $F \propto \frac{1}{r^2}$.

In conductors, electric charges move freely. Insulators do not readily transport charge. Semiconductors are in-between.

Coulomb's law:

$$F = \frac{kQq}{r^2}, \quad (30)$$

where $k = \frac{1}{4\pi\epsilon_0}$ and ϵ_0 is the permittivity of free space. Electric field from point charge:

$$E = F/q = \frac{kQ}{r^2} \hat{r}. \quad (31)$$

Total E-field = vector sum of E-fields,

$$\vec{E}_{\text{Tot}} = \sum \vec{E}_{\text{fields}}. \quad (32)$$

This becomes an integral for a continuous charge distribution,

$$\vec{E} = k \int \frac{dq}{r^2} \hat{r} = k \int \frac{\rho dV}{r^2} \hat{r}. \quad (33)$$

Rules for drawing E-field lines:

1) Begin on positive charges, end at negative—can begin/end at infinity if the total net charge $\neq 0$.

2) No two field lines can cross or touch.

Electric flux through a surface S :

$$\begin{aligned} \Phi &= \int_S \vec{E} \cdot d\vec{A} \\ &= EA, \text{ if the electric field and surface are orthogonal} \\ &= EA \cos \theta, \text{ if } \theta \text{ is the angle between the electric field and surface normal.} \end{aligned} \quad (34)$$

For a closed surface S , we have Gauss's Law:

$$\Phi = \int_S \vec{E} \cdot d\vec{A} = \frac{q_{\text{in}}}{\epsilon_0} = 4\pi k q_{\text{in}}, \quad (35)$$

where q_{in} is the charge enclosed by the surface.

A conductor in electrostatic equilibrium satisfies:

1. $\vec{E} = 0$ inside the conductor.
2. Any charge resides on the surface.
3. The $\vec{E}$ field just outside the surface is perpendicular to the surface with magnitude $\frac{q/A}{\epsilon_0} = \frac{\rho}{\epsilon_0}$.
4. Charge accumulates at sharp points (a.k.a. points with the smallest radius of curvature).

Change in potential energy for a particle of charge q_0 moving from point A to point B in an electric field $\vec{E}(\vec{x})$:

$$\Delta U = -q_0 \int_A^B \vec{E} \cdot d\vec{s}. \quad (36)$$

Here, the $d\vec{s}$ implies that this is a path integral. For the same process, the change in electrostatic potential:

$$\Delta V = \frac{\Delta U}{q_0} = - \int_A^B \vec{E} \cdot d\vec{s}. \quad (37)$$

If the E-field is uniform, $\vec{E}(\vec{x}) = \vec{E}_0$,

$$\Delta V = \vec{E}_0 \cdot \vec{d}, \quad (38)$$

where $\vec{d}$ is the displacement vector.

An equipotential surface is a surface with points of constant V . The E-field lines are always perpendicular to these equipotential surfaces. For a point charge q moving around near another point charge Q :

$$F = \frac{kQq}{r^2}. \quad (39)$$

The potential energy:

$$U = \frac{kQq}{r}. \quad (40)$$

The field generated by Q :

$$E = \frac{kQ}{r^2}. \quad (41)$$

The electrostatic potential due to Q :

$$V = \frac{kQ}{r}. \quad (42)$$

More generally,

$$\vec{E} = -\vec{\nabla}V. \quad (43)$$

For a spherical charge distribution, the radial component of the electric field is given by,

$$E_r = -\frac{\partial V}{\partial r}. \quad (44)$$

The electrostatic potential due to several different charges is a sum,

$$V = k \sum_i \frac{q_i}{r_i}. \quad (45)$$

In the continuous limit, this becomes an integral

$$V = k \int \frac{dq}{r}. \quad (46)$$

A grounded plane induces an “image” charge, which is opposite the actual charge in both position and sign. For instance, if the yz -plane is grounded and there is a charge Q at position $(1, 0, 0)$, then an image charge $-Q$ will be produced

at $(-1, 0, 0)$. A charged particle moving in this space will feel the electric field from both the actual charge and the image charge.

The electric displacement field in a material:

$$D = \epsilon_0 E + P = \epsilon_0(1 + \chi)E \equiv \epsilon_0 \kappa E, \quad (47)$$

Here, κ is called the “relative permittivity” or “dielectric constant,” and P is the polarization,

$$P = \epsilon_0 \chi E, \quad (48)$$

where χ is the electric susceptibility.

For a (parallel-plate) capacitor of capacitance C , with charge Q on each plate, voltage drop V ,

$$Q = CV. \quad (49)$$

Hence, the electric field inside the parallel-plate capacitor is

$$E = \frac{V}{d} = \frac{Q}{Cd}. \quad (50)$$

For capacitors C_i in parallel, the capacitances sum:

$$C = \sum C_i. \quad (51)$$

For capacitors in series, their reciprocals sum:

$$\frac{1}{C} = \sum \frac{1}{C_i}. \quad (52)$$

The work done in charging a capacitor = the potential energy U stored in the capacitor, which is given by,

$$U = \frac{1}{2} CV^2 = \frac{Q^2}{2C} = \frac{1}{2} QV. \quad (53)$$

When a dielectric material is inserted between the plates of a capacitor, the capacitance is multiplied by a factor κ = dielectric constant,

$$C_{\text{dielectric}} = \kappa C_0. \quad (54)$$

When two charges of charge q , $-q$, respectively, are separated by a distance $2a$, the electric dipole moment is given by,

$$\vec{p} = 2q\vec{a}, \quad (55)$$

where the vector $\vec{a}$ points from the negative charge to the positive charge. The torque on an electric dipole in a uniform E-field is,

$$\vec{\tau} = \vec{p} \times \vec{E}. \quad (56)$$

The potential energy of the electric dipole is,

$$U = -\vec{p} \cdot \vec{E}. \quad (57)$$

Geometry	Capacitance
Isolated sphere, charge Q , radius R	$C = 4\pi\epsilon_0 R$
Parallel plate capacitor, area A , separation d	$C = \epsilon_0 \frac{A}{d}$

Table 2: Geometry and Capacitance.

The capacitance for two important geometric setups can be found in Table 2.

With a dielectric, $\kappa > 1$ always, and

$$E_{\text{dielectric}} = \frac{1}{\kappa} E_{\text{free}}, \quad (58)$$

$$V_{\text{dielectric}} = \frac{1}{\kappa} V_{\text{free}}. \quad (59)$$

The bound charge density on plates is related to the free charge density by

$$\rho_{\text{bound}} = -\left(\frac{\kappa - 1}{\kappa}\right)\rho_{\text{free}}. \quad (60)$$

Gauss's Law with dielectrics:

$$\Phi = \frac{Q}{\kappa\epsilon_0}. \quad (61)$$

The electric current I in a conductor:

$$I = \frac{dQ}{dt} = nqv_d A, \quad (62)$$

where n is the number density of charge carriers, q is the charge per carrier, v_d is the drift velocity, and A is the cross-sectional area of the conductor (usually a wire). The current density:

$$J = nqv_d = \frac{I}{A}. \quad (63)$$

Current density is proportional to the E-field in the conductor:

$$\vec{J} = \sigma \vec{E}, \quad (64)$$

where σ is the conductivity.

$$\text{Resistivity} = \rho = \frac{1}{\sigma}. \quad (65)$$

A material is said to obey Ohm's Law if and only if its conductivity σ is independent of the applied field. The resistance R of a conductor is related to the current I through it and the voltage drop V across it,

$$V = IR. \quad (66)$$

Resistivity is further related to resistance,

$$R = \rho \frac{l}{A} \quad (67)$$

where l is the length of the conductor and A is the cross-sectional area of the conductor. Resistivity is linearly dependent upon temperature:

$$\rho = \rho_0[1 + \alpha(T - T_0)]. \quad (68)$$

Here, α is called the temperature coefficient of resistivity.

In the classical model of electron conduction in a metal, electrons $\sim$ molecules of a gas:

$$\vec{v}_d = \frac{e\vec{E}}{m}\tau, \quad (69)$$

where τ is the average time between collisions with atoms of the metal, e is the electron charge, and m is the electron mass. The resistivity ρ obeys

$$\rho = \frac{m}{ne^2\tau}. \quad (70)$$

Here, n is the number density of free electrons.

The power dissipated by a resistor:

$$P = IV = I^2R = \frac{V^2}{R}. \quad (71)$$

The EMF of a battery is the voltage across terminals of the battery when the current is 0. For resistors in series, resistances R_i sum,

$$R = \sum R_i. \quad (72)$$

For resistors in parallel, reciprocals sum,

$$\frac{1}{R} = \sum \frac{1}{R_i}. \quad (73)$$

Note that this is opposite to the rule for capacitors.

Kirchoff's Rules for circuits:

1. The sum of the currents going into a junction = the sum of the currents coming out of the junction.
2. The sum of potential differences across any closed loop = 0.

If a capacitor (capacitance C) is charged with a battery of EMF $\mathcal{E}$ through a resistance R , the charge q on the capacitor and the current I in the circuit vary according to,

$$q(t) = Q[1 - e^{-\frac{t}{RC}}], \quad (74)$$

$$I(t) = \frac{\mathcal{E}}{R}e^{-\frac{t}{RC}}. \quad (75)$$

Here, $Q = C\mathcal{E}$ is the maximum charge on the capacitor. The quantity RC is often called the time constant of the circuit. If the capacitor with charge Q is discharged through a resistance R ,

$$q(t) = Qe^{-\frac{t}{RC}}, \quad (76)$$

$$I(t) = I_0 e^{-\frac{t}{RC}}, \quad (77)$$

where $I_0 = \frac{Q}{RC}$ is the initial current.

Force on a charged particle moving with velocity $\vec{v}$ in a magnetic field B :

$$\vec{F} = q\vec{v} \times \vec{B}. \quad (78)$$

Force on a wire or other conductor of length L carrying current $\vec{I}$ in a uniform magnetic field:

$$\vec{F} = L(\vec{I} \times \vec{B}), \quad (79)$$

or, in infinitesimal form,

$$d\vec{F} = Id\vec{s} \times \vec{B}. \quad (80)$$

Magnetic moment μ of a current loop carrying a current I :

$$\vec{\mu} = I\vec{A}. \quad (81)$$

Here, $\vec{A}$ points through the center of the loop in a direction determined by the right-hand rule; put your right thumb along the direction of the current, and curl your fingers through the center of the hoop. The direction they point is the right (no pun intended) one.

Torque on a current loop in a magnetic field:

$$\vec{\tau} = \vec{\mu} \times \vec{B}. \quad (82)$$

A charged particle of charge q , mass m in a uniform magnetic field B with an initial velocity v perpendicular to the field will henceforth experience circular motion with radius

$$r = \frac{mv}{qB}. \quad (83)$$

Hence,

$$\omega = \frac{qB}{m}, \quad (84)$$

since $\vec{v} = \vec{\omega} \times \vec{r}$.

The Biot-Savart Law:

$$d\vec{B} = k_m \frac{Id\vec{s} \times \hat{r}}{r^2} = k_m \frac{Id\vec{s} \times \vec{r}}{r^3}. \quad (85)$$

$k_m = \mu_0/4\pi = 10^{-7}$ Tesla m A⁻¹. μ_0 is the permeability of free space. In integral form:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times \hat{r}}{r^2}. \quad (86)$$

Given two parallel wires separated by a distance a , each carrying a current I_i , the force per unit length f between the wires is

$$f = \mu_0 \frac{I_1 I_2}{2\pi a}. \quad (87)$$

If the currents are in the same direction, the force is attractive. If they are in opposite directions, the force is repulsive.

Ampere's Law:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I (+\mu_0 I_d). \quad (88)$$

Here, the term in parentheses is Maxwell's correction term, and $I_d = \epsilon_0 \frac{d\Phi_E}{dt}$. The loop around which we are integrating should be taken to encircle the wire/conductor of interest. For an infinitely long solenoid, this gives an interior magnetic field of,

$$B = \mu_0 I n \quad (89)$$

where $n = N/l$ is the number of turns per unit length. Outside the infinitely long solenoid, the magnetic field vanishes.

For a long, current-carrying wire of radius R ,

$$\begin{aligned} B &= \mu_0 I \frac{1}{2\pi r}, & r \geq R \\ &= \mu_0 I r \frac{1}{2\pi R^2}, & r < R. \end{aligned} \quad (90)$$

Magnetic flux through a surface:

$$\Phi_B = \int_S \vec{B} \cdot \vec{A}. \quad (91)$$

For uniform $\vec{B}$, planar $\vec{A}$,

$$\Phi_B = BA \cos(\theta). \quad (92)$$

For a closed surface, Gauss's law for magnetism:

$$\Phi_B = \int_S \vec{B} \cdot \vec{A} = 0. \quad (93)$$

This is assuming that there are no magnetic monopoles in nature, which should be your working assumption for the pGRE.

Given a magnetic field B_0 and a material, the magnetic field in the material is:

$$B = B_0 + B_m = B_0 + \mu_0 M = \mu_0 (H + M), \quad (94)$$

where H is the magnetic field strength and M is the magnetization of the substance. For para/dia-magnetic materials,

$$M = \chi H, \quad (95)$$

where χ is the magnetic susceptibility. $\chi < 0$ for diamagnetic materials, $\chi > 0$ for paramagnetic materials, and $\chi \gg 1$ for ferromagnetic materials. The magnetic permeability μ_m is defined according to

$$\mu_m = \mu_0(1 + \chi). \quad (96)$$

Faraday's Law:

$$\oint \vec{E} \cdot d\vec{s} = \mathcal{E} = -\frac{d\Phi_B}{dt}. \quad (97)$$

The minus sign in this equation is so important that it gets a name of its own: Lenz's law. $\mathcal{E}$ is the EMF produced in the loop of wire by the time-varying magnetic flux through it. For a bar of length l moving in a magnetic field B with velocity v perpendicular to B , a motional EMF is produced,

$$\mathcal{E} = -Blv. \quad (98)$$

The Lorentz Force Law:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \quad (99)$$

Maxwell's Equations:

$$\int \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0} \quad (100)$$

$$\int \vec{B} \cdot d\vec{A} = 0 \quad (101)$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \quad (102)$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt}. \quad (103)$$

The EMF through an inductor of inductance L :

$$\mathcal{E} = -L \frac{dI}{dt}. \quad (104)$$

For a coil of wire with N turns carrying current I with magnetic flux Φ_B ,

$$L = \frac{N\Phi_B}{I}. \quad (105)$$

For a solenoid in particular,

$$L = \frac{\mu_0 N^2 A}{l}, \quad (106)$$

where A is the cross-sectional area and l is the length of the solenoid.

When the battery is turned on in an L-R circuit:

$$I(t) = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}), \quad \tau = L/R. \quad (107)$$

When the battery is turned off:

$$I(t) = \frac{\mathcal{E}}{R} e^{-t/\tau}. \quad (108)$$

The energy stored in the magnetic field of an inductor:

$$U = \frac{1}{2} L I^2. \quad (109)$$

The energy per unit volume in the B-field region of the inductor:

$$u_B = \frac{B^2}{2\mu_0}. \quad (110)$$

The current and charge of an LC-circuit oscillate in time:

$$Q(t) = Q_{\max} \cos(\omega t + \phi) \quad (111)$$

$$I(t) = -\omega Q_{\max} \sin(\omega t + \phi). \quad (112)$$

Here, $\omega = \frac{1}{\sqrt{LC}}$ and the phase ϕ depends upon the initial state of the circuit. The energy U stored in an LC-circuit:

$$U = U_C + U_L = \frac{Q_{\max}^2}{2C} \cos^2(\omega t + \phi) + \frac{L I_{\max}^2}{2} \sin^2(\omega t + \phi). \quad (113)$$

Note that U is constant in time. In an RLC-circuit, we get damped harmonic motion for I , Q .

In AC-circuits,

$$\mathcal{E} = V_{\max} \sin(\omega t). \quad (114)$$

If only a resistor, I is in phase with V , and

$$I_{\text{rms}} = I_{\max}/\sqrt{2}, \quad (115)$$

$$V_{\text{rms}} = V_{\max}/\sqrt{2}. \quad (116)$$

For RLC in series, inductive reactance $\chi_L = \omega L$, capacitive reactance $\chi_C = \frac{1}{\omega C}$, impedance

$$Z = \sqrt{R^2 + (\chi_L - \chi_C)^2}. \quad (117)$$

Voltage and current are out of phase by ϕ , where

$$\tan \phi = \frac{\chi_L - \chi_C}{R}. \quad (118)$$

The average power produced by the generator of RLC AC-circuit is

$$P_{\text{avg}} = I_{\text{rms}} V_{\text{rms}} \cos \phi = I_{\text{rms}}^2 R, \quad (119)$$

and the energy is dissipated as heat in the resistor. For this general RLC circuit,

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z}. \quad (120)$$

The current in the RLC AC-circuit reaches a maximum at the resonant frequency,

$$\omega = \omega_0 = \frac{1}{\sqrt{LC}}. \quad (121)$$

Electromagnetic waves obey the wave equation (in 1-D),

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}, \quad (122)$$

$$\frac{\partial^2 B}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2}. \quad (123)$$

Thus, the solutions are

$$E = E_{\max} \cos(kx - \omega t), \quad (124)$$

$$B = B_{\max} \cos(kx - \omega t). \quad (125)$$

The speed of light is related to μ_0 and ϵ_0 by

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}. \quad (126)$$

Also,

$$c = \frac{|E|}{|B|}. \quad (127)$$

And,

$$c = \nu \lambda = \frac{\omega \lambda}{2\pi} = \omega/k. \quad (128)$$

The rate of flow of energy corssing a unit area is given by the Poynting vector,

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}. \quad (129)$$

Radiation pressure:

$$P = \frac{|S|}{c}. \quad (130)$$

Furthermore,

$$S_{\text{av}} = \frac{E_{\max} B_{\max}}{2\mu_0}. \quad (131)$$

At cavity walls,

$$E_{\parallel} = B_{\perp} = 0. \quad (132)$$

During reflection by a conducting surface, the E-field gets a shift of π , but the B-field does not.

The characteristic impedance of a transmission line is given by,

$$Z_0 = \sqrt{\frac{L}{C}}, \quad (133)$$

where L is the inductance per unit length and C is the capacitance per unit length. The speed of the wave on this transmission line is given by,

$$c = \frac{1}{\sqrt{LC}}. \quad (134)$$

5 Optics and Waves (9%)

Waves propagate in the direction orthogonal to the wave fronts. When light hits a surface, it may be:

1. Partially or totally reflected.
2. Scattered in random directions.
3. Partially transmitted by refraction.
4. Partially absorbed in either media.

Specular reflection is the name for regular, geometric reflections. Diffuse is the name for irregular reflections from jagged surfaces. We concentrate on specular reflections.

Law of reflection: angle of incidence = angle of reflection. These angles are measured from the normal to the surface of reflection. Index of refraction:

$$n \equiv \frac{c}{v}. \quad (135)$$

Here, c is the vacuum speed of light and v is the speed of light in the medium. Note that $n \geq 1$ for all materials, and $n_{\text{air}} \approx 1$. Snell's Law relates the angle of incidence and the angle of refraction:

$$n_i \sin \theta_i = n_f \sin \theta_f. \quad (136)$$

Here, θ_i, θ_f denote the initial and final angles with respect to the surface normal, respectively. n_i and n_f denote the indices of refraction for the initial and final materials, respectively. Note that if

$$\frac{n_i \sin \theta_i}{n_f} > 1, \quad (137)$$

then no value of θ_f will satisfy Snell's Law. In this case, we get evanescent waves leaking into the final region. This phenomenon is called "total internal reflection."

The variation of refractive index n with wavelength λ is called dispersion. In general, as λ increases, n decreases slightly. This is why we see rainbows.

Concave and convex mirrors have different properties, as do concave and convex lenses. For the convex mirror, the image is virtual, upright, and smaller than the object. Think "objects in mirror are closer than they appear." For concave mirrors, the image is real, inverted, and smaller if you are behind the focal point and virtual, upright, and enlarged if you are closer than the focal point (see Figure 1). For both concave and convex spherical mirrors, the focal length (distance from the mirror to the point where light rays focus) is half the radius of curvature of the mirror. The mirror equation for spherical mirrors:

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}, \quad (138)$$

where f is the focal length, d_o is the object distance, and d_i is the image distance. The magnification equation:

$$M = \frac{h_i}{h_o} = -\frac{d_i}{d_o}, \quad (139)$$

where h_i is the image height and h_o is the object height. The sign convention is:

- $d_o > 0$ if object is in front (real).
- $d_o < 0$ if object is in back (virtual).
- $d_i > 0$ if image is in front (real).
- $d_i < 0$ if image is in back (virtual).
- $f > 0$ for concave.
- $f < 0$ for convex.
- $M > 0$ for upright image.
- $M < 0$ for inverted image.

We will soon see that the sign conventions are different for lenses.

For a lens of diameter D , focal length f , the “f-number” or “relative aperture” is given by,

$$\text{f-number} = \frac{f}{D}. \quad (140)$$

Increasing f-number means increasing light-gathering power. The “numerical aperture” is given by

$$\text{n.a.} = n \sin \alpha, \quad (141)$$

where n is the refractive index of the medium between the object and the lens and α is the half-angle of the limiting ray. The image brightness is inversely proportional to both the square of the f-number and the square of the numerical aperture. For a thin lens,

$$\frac{1}{f} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right). \quad (142)$$

Here, f is the focal length, n is the index of refraction of the lens, R_1 is the radius of curvature for the lens nearest the object, R_2 is the radius of curvature for the lens farther from the object. Analogously to the mirror equation,

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}. \quad (143)$$

But the sign conventions have changed:

- $d_o > 0$ if object is in front (real).

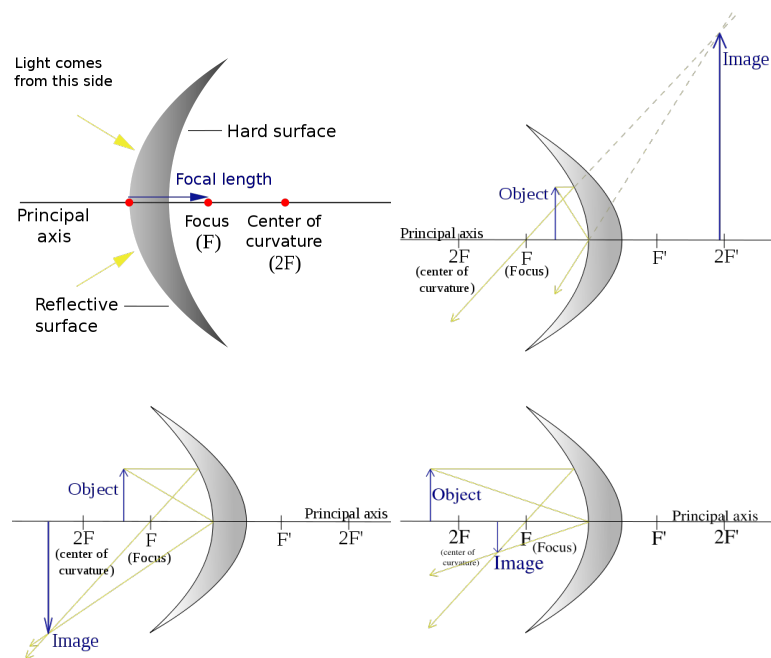


Figure 1: Convex (top left) and concave mirrors. Note the position of the image relative to the object for each case. Source: Wikipedia.

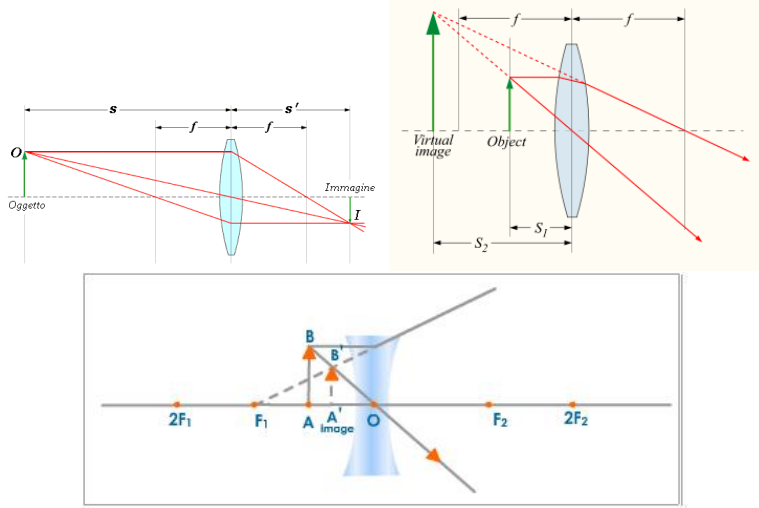


Figure 2: Convex (top) and concave (bottom) lenses. Note the position of the image relative to the object for each case. Source: http://physics20p1.blogspot.com/2011_12_01_archive.html and <http://images.tutorvista.com/content/refraction-light/>.

- $d_O < 0$ if object is in back (virtual).
- $d_i > 0$ if image is in back (real).
- $d_i < 0$ if image is in front (virtual).
- $R_1 > 0$ if center of lens is in back.
- $R_1 < 0$ if center of lens is in front.
- $R_2 > 0$ if center of lens is in back.
- $R_2 < 0$ if center of lens is in front.

Rather than trying to remember all of these sign conventions, it is probably easier to remember the images of Figure 2. When two thin lenses are in contact,

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}. \quad (144)$$

When light is (at least approximately) perpendicularly incident on a medium, the fraction of light reflected is,

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2, \quad (145)$$

and the fraction of light transmitted is,

$$T = 1 - R = \frac{4n_1n_2}{(n_1 + n_2)^2}. \quad (146)$$

Here, n_1 is the index of refraction of the original medium, n_2 is the index of refraction of the final medium. If light incident on a surface is polarized in such a way that its E-field is in the plane of the path formed by the incident and reflected ray (p -polarized), then there exists an angle of incidence θ_B for which the reflection coefficient $R_p = 0$ and all of the light is transmitted. θ_B is called Brewster's angle. No such Brewster's angle exists if the incident material is a metal or semiconductor.

The wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u. \quad (147)$$

The solution to the 1-dimensional case is

$$u(x, t) = A \sin(kx - \omega t) + B \cos(kx - \omega t), \quad (148)$$

where $\omega/k = c$. When two waves on a string meet each other, the resulting amplitude is the sum of the amplitudes for each individual wave. Two sine waves going in the same direction give interference. For two waves $\sin(kx - \omega t)$ and $\sin(kx - \omega t + \phi)$, we get constructive interference for $\phi = 0$ (in-phase) and destructive interference for $\phi = \pi$ (out-of-phase). Two sine waves heading in opposite directions give standing waves,

$$u = A \sin(kx - \omega t) + A \sin(kx + \omega t) = 2A \sin(kx) \cos(\omega t). \quad (149)$$

For these standing waves, the points of 0 amplitude ($kx = n\pi$) are called “nodes” and the points of maximum amplitude ($kx = \pi/2 + n\pi$) are called “antinodes.” Two waves traveling in the same direction with different frequencies but the same speed give “beats,”

$$\begin{aligned} u(x, t) &= A \sin(k_1 x - \omega_1 t) + A \sin(k_2 x - \omega_2 t) \\ &= 2A \cos((k_1 - k_2)x/2 - (\omega_1 - \omega_2)t/2) \sin((k_1 + k_2)x/2 - (\omega_1 + \omega_2)t/2). \end{aligned} \quad (150)$$

The “beat frequency” is the difference of the individual frequencies,

$$\nu_{\text{beat}} = \nu_1 - \nu_2. \quad (151)$$

Characteristic impedance Z of a string:

$$Z = \mu c, \quad (152)$$

where μ is the mass per unit length of the string and c is the wave speed. When a wave on the string meets a hard, fixed boundary, there is a phase shift of π . When it meets a soft, free boundary, there is no phase shift (see Figure 3).

When a wave moves from a place of low density to high density, the reflected wave experiences a π phase shift. The transmitted wave feels no shift. When

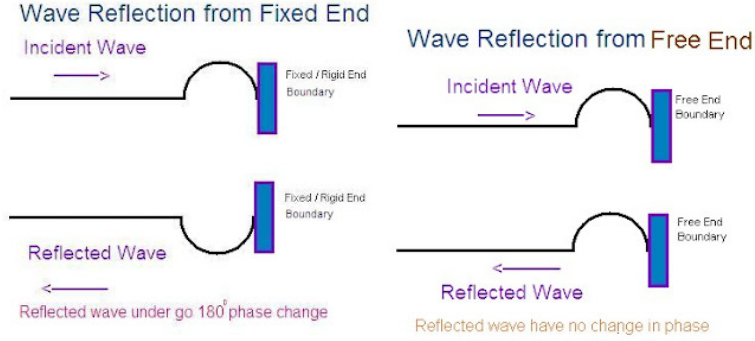


Figure 3: Fixed (left) and free (right) boundary. Note the phase shift in the first case. Source: <http://snvphysics.blogspot.com/2011/03/waves.html>.

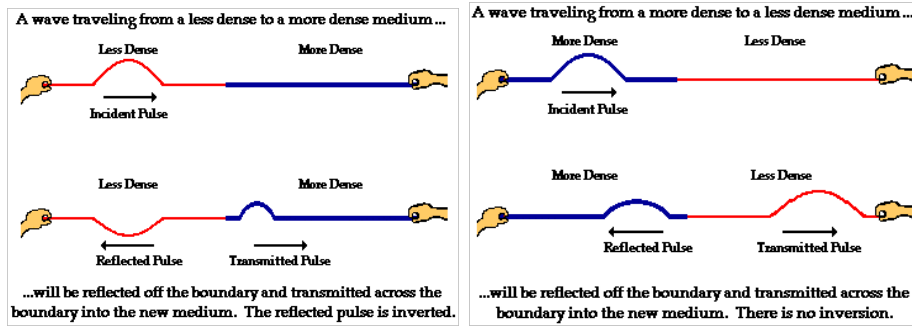


Figure 4: Waves move from less to more dense regions (left) and more to less dense regions (right). Note the phase shift in the first case. Source: <http://www.physicsclassroom.com/class/waves/u10l3a.cfm>.

a wave moves from high density to low density, neither the reflected nor the transmitted wave feels a π phase shift (see Figure 4). The amplitudes of the waves are related by

$$A_{\text{incident}} = A_{\text{transmitted}} - A_{\text{reflected}}. \quad (153)$$

The sign convention (see Figure 4) is + for above, – for below.

For single-slit diffraction, minima occur at,

$$d \sin \theta_n = n\lambda, \quad n = 1, 2, 3, \dots, \quad (154)$$

where d is the width of the slit, θ_n is the angular location of the minima, and λ is the wavelength of the light. For the double-slit experiment, maxima occur at

$$d \sin \theta_n = n\lambda, \quad n = 0, 1, 2, \dots, \quad (155)$$

where d is now the distance between slits. This equation also applies to diffraction gratings. For Bragg diffraction,

$$2d \sin \theta = \lambda. \quad (156)$$

For a film of thickness t ,

$$2t = \lambda/2 \Rightarrow \text{constructive interference}, \quad (157)$$

$$2t = \lambda \Rightarrow \text{destructive interference}. \quad (158)$$

For a telescope,

$$\text{Magnification} = \frac{f_O}{f_e}, \quad (159)$$

where f_O is the focal length of the objective lens and f_e is the focal length of the eyepiece. The distance between the objective lens and the eyepiece is $f_O + f_e$. The aperture formula:

$$\theta = 1.22\lambda/d. \quad (160)$$

Here, θ is the angular aperture, d is the diameter of the lens, and λ is the wavelength.

The E-field for a light wave traveling in the z-direction can be written as,

$$\vec{E}(\vec{r}, t) = (A_x \cos(kz - \omega t), A_y \cos(kz - \omega t + \phi), 0). \quad (161)$$

If $\phi = 0, \pi$, the light is linearly polarized, and the direction depends on the amplitudes, A_x, A_y . If $\phi = \pi/2, -\pi/2$ and $A_x = A_y$, the light is circularly polarized. Otherwise, the light is elliptically polarized. A polarizer allows only certain polarizations of light to pass through. The intensity I of the wave emerging from the polarizer is related to the incoming intensity I_0 and the angle θ between polarization and polarizer by,

$$I = I_0 \cos^2 \theta. \quad (162)$$

If the incoming light is unpolarized, $I = I_0/2$. The B-field is perpendicular to both the direction of propagation and the E-field.

The Doppler effect for sound is,

$$\omega_{\text{obs}} = \omega_{\text{em}} \left(\frac{v + v_r}{v + v_s} \right). \quad (163)$$

Here, v is the speed of sound, v_r is the speed of the receiver, and v_s is the speed of the source. The sign convention can be determined by remembering that moving closer corresponds to a blueshift (greater observed frequency) and moving apart corresponds to a redshift (lesser observed frequency). Note that a blowing wind doesn't affect v_r or v_s , since the time of propagation for consecutive pulses is not changing with time.

Phase velocity is the speed of the wave. Group velocity is the speed of the wave packet. Phase velocity:

$$v_{\text{phase}} = \frac{\omega}{k}. \quad (164)$$

$$v_{\text{group}} = \frac{d\omega}{dk}. \quad (165)$$

For waves on a string, $v_{\text{phase}} = v_{\text{group}}$. For water waves, $v_{\text{phase}} = 2v_{\text{group}}$. For the wavefunction of a free particle in quantum mechanics, $v_{\text{phase}} = \frac{1}{2}v_{\text{group}}$, and v_{group} is the classical velocity of the particle. For de Broglie waves, the de Broglie-Einstein wave equation gives $v_{\text{phase}}v_{\text{group}} = c^2$.

6 Thermodynamics and Statistical Mechanics (10%)

An equilibrium state is one in which all bulk physical properties are uniform throughout the system and don't change with time. State functions are functions that take unique values at each equilibrium state. Adiabatic = adiabatic = there is no interaction between systems. Diathermal = there is a thermal interaction between systems. Thermal equilibrium is transitive (i.e. $A \sim B$, $B \sim C$ implies $A \sim C$). Thermodynamic equilibrium means thermal equilibrium, mechanical equilibrium, and chemical equilibrium.

Isotherms are surfaces of constant T . For an ideal gas,

$$PV = nRT, \quad (166)$$

so isotherms are hyperbolas. A quasistatic process is one that passes through a series of equilibrium states. Reversible processes are quasistatic processes for which no dissipative forces like friction are present.

Bulk modulus:

$$K \equiv -V\left(\frac{\partial P}{\partial V}\right)_T. \quad (167)$$

Volume thermal expansivity:

$$\beta \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P. \quad (168)$$

Linear expansion coefficient:

$$\alpha \equiv \frac{1}{L}\left(\frac{\partial L}{\partial T}\right)_F. \quad (169)$$

Here, F is the tension. In general, $\beta = 3\alpha$. Young's Modulus:

$$Y \equiv \frac{L}{A}\left(\frac{\partial F}{\partial L}\right)_T. \quad (170)$$

Compressibility:

$$\kappa = \frac{1}{K}. \quad (171)$$

Force is related to pressure and area by $F = PdA$, so

$$dW = -PdV \quad (172)$$

for a reversible process. Thus, the total work done on a system is,

$$W = - \int_{V_1}^{V_2} PdV. \quad (173)$$

The adiabatic free expansion is not reversible, and there is no work done on the system and no change in entropy. For a wire:

$$dW = Fdx. \quad (174)$$

The first law of thermodynamics:

$$\Delta U = W + Q, \quad (175)$$

where U is the internal energy of the system, W is the work done on the system, and Q is the heat added to the system. For a reversible process,

$$dU = dQ - PdV. \quad (176)$$

The heat capacity:

$$C = \lim_{\Delta T \rightarrow 0} Q/\Delta T = dQ/dT. \quad (177)$$

The specific heat $c = C/m$. C_V is the heat capacity at constant volume,

$$C_V = dQ_V/dT = (\frac{\partial U}{\partial T})_V. \quad (178)$$

C_P is the heat capacity at constant pressure,

$$C_P = dQ_P/dT = (\frac{\partial H}{\partial T})_P. \quad (179)$$

Here, $H = U + PV$ is the enthalpy. γ is defined by,

$$\gamma = \frac{C_P}{C_V} > 1. \quad (180)$$

For an ideal gas,

$$C_P = C_V + nR. \quad (181)$$

For a monotomic ideal gas,

$$U = \frac{3}{2}nRT, \quad C_V = \frac{3}{2}nR, \quad \gamma = \frac{5}{3}. \quad (182)$$

For an adiabatic free expansion,

$$W = 0, \quad Q = 0, \quad \Delta U = 0. \quad (183)$$

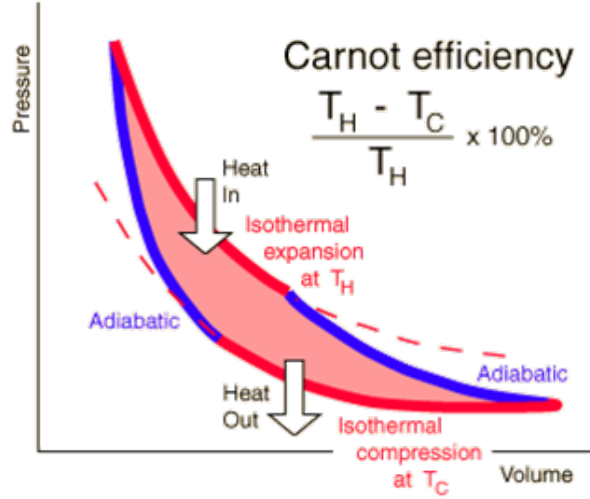


Figure 5: The Carnot cycle. Source: <http://hyperphysics.phy-astr.gsu.edu/hbase/thermo/carnot.html>.

For an ideal gas, $U = U(T)$ is a function of temperature only. In reality, all gases cool slightly during expansion, due to intermolecular attraction potential energy increasing with expansion. For an adiabatic,

$$PV^\gamma = \text{const.} \quad (184)$$

For a Van der Waals gas,

$$(P + \frac{a}{v^2})(V - nb) = nRT, \quad (185)$$

where a and b are constants and $v = V/n$ is the molar volume.

Figure 5 depicts the Carnot cycle. The top and bottom curves are isotherms in which heat enters at temperature T_H and exits and temperature T_C , respectively. The side curves are adiabatics. The work done is the area bounded by the four curves. The efficiency of the engine:

$$\eta = \frac{W}{Q_H} = 1 - \frac{Q_C}{Q_H} = 1 - \frac{T_C}{T_H}. \quad (186)$$

The second law of thermodynamics says that 100% efficiency is impossible. More precisely, it is impossible to construct a device that, operating in a cycle, will produce no effect other than the extraction of heat from a single body at uniform temperature and perform an equivalent amount of work. Carnot's theorem says that the Carnot engine is the most efficient engine between any two reservoirs. The heat entering in an isotherm is,

$$Q = nRT \ln\left(\frac{V_{\text{end}}}{V_{\text{start}}}\right). \quad (187)$$

Running the Carnot cycle backward gives a refrigerator with efficiency,

$$\eta_R = \frac{Q_C}{W} = \frac{Q_C}{Q_H - Q_C} = \frac{T_C}{T_H - T_C}. \quad (188)$$

The efficiency of a heat pump is,

$$\eta_H = \frac{Q_H}{W} = \frac{Q_H}{Q_H - Q_C} = \frac{T_H}{T_H - T_C}. \quad (189)$$

The Clausius equation:

$$\oint \frac{dQ}{T_0} \leq 0. \quad (190)$$

Here, T_0 is the temperature of the auxiliary reservoir, and equality holds if and only if the cycle is reversible. Change in entropy is defined by,

$$\Delta S \equiv \int_i^f \frac{dQ_R}{T}, \quad (191)$$

where the path is reversible. For any reversible process,

$$dQ_R = TdS. \quad (192)$$

For an isothermal free expansion,

$$\Delta S = nR \ln\left(\frac{V_f}{V_i}\right). \quad (193)$$

In general,

$$\frac{dQ}{T} \leq dS, \quad (194)$$

and equality holds if and only if the process is reversible. For a thermally isolated system,

$$\Delta S \geq 0, \quad (195)$$

with equality holding if and only if the process is reversible. The central equation of thermodynamics:

$$TdS = dU + PdV, \quad (196)$$

which holds for a reversible process. For an ideal gas, of molar entropy s ,

$$s = c_V \ln T + R \ln v + s_0. \quad (197)$$

In general,

$$S = k_B \ln \Omega, \quad (198)$$

where k_B is Boltzmann's constant and Ω is the thermodynamic probability, or the number of microstates giving rise to this macrostate. The specific heat capacity in terms of the entropy is,

$$C_V = T \left(\frac{\partial S}{\partial T} \right)_V, \quad (199)$$

$$C_P = T\left(\frac{\partial S}{\partial T}\right)_P. \quad (200)$$

The Helmholtz function:

$$F = U - TS. \quad (201)$$

For a process in which the end point temperatures are the same as the surroundings, the maximum work obtainable equals the decrease in F . F at a minimum is the condition for thermodynamic equilibrium for a system held at constant volume. Gibbs function:

$$G = H - TS. \quad (202)$$

G at a minimum is the condition for thermodynamic equilibrium for a system in contact with a heat and pressure reservoir. For a chemical reaction at temperature T_0 ,

$$\Delta H - T_0 \Delta S \leq 0. \quad (203)$$

Maxwell's relations:

$$\left(\frac{\partial V}{\partial S}\right)_P = \left(\frac{\partial T}{\partial P}\right)_S, \quad (204)$$

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V, \quad (205)$$

$$-\left(\frac{\partial S}{\partial P}\right)_T = \left(\frac{\partial V}{\partial T}\right)_P, \quad (206)$$

$$-\left(\frac{\partial T}{\partial V}\right)_S = \left(\frac{\partial P}{\partial S}\right)_V. \quad (207)$$

These can be remembered by going around the wheel twice in opposite directions from different starting points,

$$\begin{array}{ccc} & - & S \\ P & & V \\ & T & \end{array} \quad (208)$$

The mean free path of gas molecules is

$$l = \frac{1}{\pi d^2 n_v \sqrt{2}}, \quad (209)$$

where d is the diameter of the molecule and n_v is the number density. The collision frequency is,

$$\sqrt{2}\pi d^2 n_v \bar{v} = \frac{\bar{v}}{l}, \quad (210)$$

where $\bar{v}$ is the average speed. The pressure exerted by N molecules of an ideal gas is,

$$P = \frac{2}{3} \frac{N}{V} \left(\frac{1}{2} m \bar{v}^2\right). \quad (211)$$

The temperature is,

$$\frac{3k_B T}{2} = \left(\frac{1}{2} m \bar{v}^2\right). \quad (212)$$

The partition function:

$$Z = \sum_j g_j e^{-\frac{E_j}{k_B T}}, \quad (213)$$

where the sum runs over all microstates and g_j is the degeneracy of the j th microstate. The probability of being found in state j is,

$$Pr(j) = \frac{g_j e^{-\frac{E_j}{k_B T}}}{Z}. \quad (214)$$

The pressure is,

$$P = k_B T \ln Z. \quad (215)$$

The entropy is,

$$S = \frac{\partial}{\partial T} (T k_B \ln Z). \quad (216)$$

7 Quantum Mechanics (12%)

Time-dependent Schrodinger Equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V \Psi. \quad (217)$$

Time-independent Schrodinger Equation:

$$E\psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi. \quad (218)$$

The wavefunction Ψ can be factored into a spatial and time component,

$$\Psi(\vec{x}, t) = \psi(\vec{x})\varphi(t). \quad (219)$$

For a stationary state,

$$\varphi(t) = e^{-iEt/\hbar}. \quad (220)$$

The squared wavefunction gives a probability distribution,

$$\int |\Psi(\vec{x}, t)|^2 d^3x = 1. \quad (221)$$

Thus, the expectation value of $\vec{x}$ is

$$\langle \vec{x} \rangle = \int \vec{x} |\Psi(\vec{x}, t)|^2 d^3x. \quad (222)$$

The expectation value of momentum $\vec{p}$ is

$$\langle \vec{p} \rangle = \int -i\hbar \Psi^* \nabla \Psi. \quad (223)$$

From here, we see the expressions for the operators $\hat{x}$ and $\hat{p}$,

$$\hat{x} = x, \quad (224)$$

$$\hat{p} = \frac{\hbar}{i} \nabla. \quad (225)$$

For a general operator $\hat{Q}$,

$$\langle Q \rangle = \int d^3x \Psi^* \hat{Q} \Psi \equiv \langle \Psi | \hat{Q} | \Psi \rangle. \quad (226)$$

Note that this 3-dimensional formalism simplifies to the 1-dimensional case in the obvious way—simply replace the integral measure d^3x with dx and replace ∇ with $\frac{\partial}{\partial x}$. I will now switch over to 1-d.

The de Broglie wavelength λ of a particle with momentum p :

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{p}. \quad (227)$$

The Heisenberg uncertainty principle:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}. \quad (228)$$

Stationary states ψ_n are states with definite total energy E_n . The general solution to the time-dependent Schrodinger equation is,

$$\Psi(x, t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}, \quad (229)$$

where the $\{\psi_n\}$ form an orthonormal basis of eigenstates of the Hamiltonian. c_n can be loosely thought of as “how much of ψ_n is in Ψ .” These constants satisfy

$$\sum_{n=1}^{\infty} |c_n|^2 = 1, \quad (230)$$

$$c_n = \langle \psi_n | \Psi \rangle, \quad (231)$$

$$\sum_{n=1}^{\infty} E_n |c_n|^2 = \langle \hat{H} \rangle = \langle E \rangle. \quad (232)$$

$\hat{Q}$ is called Hermitian if

$$\langle \hat{Q} \psi | \psi \rangle = \langle \psi | \hat{Q} \psi \rangle. \quad (233)$$

Observables are represented by Hermitian operators because Hermitian operators have only real eigenvalues. The spectrum of eigenvalues for an operator may be either discrete or continuous. If it is discrete, eigenfunctions belonging to distinct eigenvalues are orthogonal, and you may construct a complete, orthonormal basis of eigenfunctions provided it is Hermitian.

The conserved probability current density is given by

$$\vec{j} = \text{Re}(\psi^* \frac{\hbar}{im} \nabla \psi). \quad (234)$$

The momentum space wavefunction $\Phi(p, t)$ is related to the position space wavefunction by Fourier transform,

$$\Phi(p, t) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar} \Psi(x, t) dx, \quad (235)$$

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{ipx/\hbar} \Phi(p, t) dp. \quad (236)$$

In momentum space,

$$\hat{x} = i\hbar \frac{\partial}{\partial p} \quad (237)$$

$$\hat{p} = p. \quad (238)$$

The commutator of two operators:

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}. \quad (239)$$

For momentum and position,

$$[\hat{x}, \hat{p}] = i\hbar. \quad (240)$$

The generalized uncertainty principle says,

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2. \quad (241)$$

Note that the expectation value of the commutator of two Hermitian operators is always imaginary, so this is indeed non-negative. It is true that

$$\frac{d}{dt} \langle Q \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \langle \frac{\partial \hat{Q}}{\partial t} \rangle. \quad (242)$$

Assuming that $\hat{Q}$ has no explicit time-dependence yields the time-energy uncertainty principle:

$$\sigma_H \sigma_Q \geq \frac{\hbar}{2} \left| \frac{d\langle Q \rangle}{dt} \right|. \quad (243)$$

As long as we are dealing with finite values of the potential V , the wavefunction ψ is C^1 continuous (i.e. ψ is continuous, $\frac{\partial \psi}{\partial x}$ is continuous. Contrariwise, for the infinite square well,

$$\begin{aligned} V(x) &= 0, & 0 \leq x \leq a \\ &= \infty, & \text{otherwise,} \end{aligned} \quad (244)$$

the time-independent Schrodinger equation becomes

$$\frac{d^2 \psi}{dx^2} = -k^2 \psi, \quad k = \sqrt{2mE}/\hbar. \quad (245)$$

This yields solutions

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, 3, \dots, \quad (246)$$

with corresponding energy spectrum

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}, \quad n = 1, 2, 3, \dots \quad (247)$$

These solutions satisfy the orthogonality condition,

$$\int_0^a \psi_m^* \psi_n dx = \delta_{mn}. \quad (248)$$

And,

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{2\pi x}{a}\right) \Psi(x, 0) dx. \quad (249)$$

The quantum harmonic oscillator has potential

$$V = \frac{1}{2} kx^2 = \frac{1}{2} m\omega^2 x^2, \quad (250)$$

where $\omega = \sqrt{\frac{k}{m}}$. To solve the Schrodinger equation with this potential, we introduce creation and annihilation operators,

$$\hat{a}_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}} (\mp ip + m\omega x). \quad (251)$$

The Hamiltonian may then be rewritten,

$$\hat{H} = \hbar\omega \left(a_- a_+ - \frac{1}{2}\right). \quad (252)$$

The ground state is a Gaussian:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2} \quad (253)$$

The set of eigenstates can then be built up from the ground state ψ_0 by applying the creation operator,

$$\psi_n(x) = \frac{1}{\sqrt{n!}} (a_+)^n \psi_0(x). \quad (254)$$

The spectrum is evenly spaced,

$$E_n = (n + 1/2) \hbar\omega, \quad n = 0, 1, 2, \dots \quad (255)$$

The annihilation operator kills the ground state,

$$a_- \psi_0 = 0. \quad (256)$$

For the finite square well, there is always at least one bound state. This is not true for the spherical finite square well, which need not have a bound state.

To solve the 3-d Schrodinger equation with a central potential, we use separation of variables $\psi(r, \theta, \phi) = R(r)Y(\theta, \phi)$ and arrive at the radial and angular equations,

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - \frac{2mr^2}{\hbar^2} (V(r) - E) = l(l+1), \quad (257)$$

$$\frac{1}{Y} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial Y}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right) = -l(l+1). \quad (258)$$

The solutions to the angular equation are spherical harmonics,

$$Y_l^m(\theta, \phi). \quad (259)$$

For $m = 0$, these spherical harmonics have no ϕ dependence. The solution to the radial equation depends on the potential. For the spherical square well, the solutions are Bessel functions. For the hydrogen atom potential, see §8 on atomic physics.

Classically, angular momentum is given by $\vec{L} = \vec{r} \times \vec{p}$. In quantum mechanics, the operators of angular momentum similarly obey

$$\hat{L} = \hat{r} \times \hat{p}. \quad (260)$$

These operators obey commutation relations,

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y. \quad (261)$$

The eigenvalues of $\hat{L}_z$ are $m\hbar$, where $m = -l, \dots, +l$. The spherical harmonics are eigenfunctions of both $\hat{L}_z$ and $\hat{L}^2$,

$$\hat{L}^2 Y_l^m = l(l+1)\hbar^2 Y_l^m, \quad (262)$$

$$\hat{L}_z Y_l^m = m\hbar Y_l^m. \quad (263)$$

In spherical coordinates,

$$\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}. \quad (264)$$

$\hat{L}^2$ commutes with each of the individual components of angular momentum,

$$[\hat{L}^2, \hat{L}_i] = 0. \quad (265)$$

Spin operators $\hat{S}_i$ obey the same commutation relations,

$$[\hat{S}_i, \hat{S}_j] = i\hbar \epsilon_{ijk} \hat{S}_k, \quad (266)$$

$$[\hat{S}^2, \hat{S}_i] = 0. \quad (267)$$

For eigenstates $|sm\rangle$ of $\hat{S}^2, \hat{S}_z$,

$$\hat{S}^2 |sm\rangle = s(s+1)\hbar^2 |sm\rangle, \quad (268)$$

$$\hat{S}_z|sm\rangle = m\hbar|sm\rangle. \quad (269)$$

Here, $s = 0, 1/2, 1, 3/2, \dots$, $m = -s, -s+1, \dots, s-1, s$. Π^0 mesons have spin $s = 0$, electrons have spin $1/2$, photons have spin 1 , and gravitons have spin 2 . Bosons have integer spin and fermions have half-integer spin.

Spin $1/2$ is the most important case. We pick a basis and define spinors,

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (270)$$

where χ_+ corresponds to spin up and χ_- corresponds to spin down. Then

$$\hat{S}^2\chi_{\pm} = \frac{3}{4}\hbar^2\chi_{\pm}, \quad (271)$$

$$\hat{S}_z\chi_{\pm} = \pm\frac{\hbar}{2}\chi_{\pm}. \quad (272)$$

The spin operators are related to the Pauli matrices in this basis, $\hat{S}_i = \frac{\hbar}{2}\sigma_i$, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (273)$$

The eigenspinors of $\hat{S}_x$ in this basis are,

$$\chi_{\pm}^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \quad (274)$$

The magnetic dipole moment for an electron in a magnetic field:

$$\vec{\mu} = \gamma\vec{S}, \quad (275)$$

where γ is the gyromagnetic ratio. The Hamiltonian for this system is

$$\hat{H} = -\hat{\mu} \cdot \vec{B} = -\gamma\vec{B} \cdot \hat{S}. \quad (276)$$

When you have two spin $1/2$ particles A and B , the spins don't sum in the obvious way. Instead,

$$|11\rangle = |\frac{1}{2} \frac{1}{2}\rangle_A |\frac{1}{2} \frac{1}{2}\rangle_B \quad (277)$$

$$|10\rangle = |\frac{1}{2} \frac{1}{2}\rangle_A |\frac{1}{2} - \frac{1}{2}\rangle_B + |\frac{1}{2} - \frac{1}{2}\rangle_A |\frac{1}{2} \frac{1}{2}\rangle_B \quad (278)$$

$$|1-1\rangle = |\frac{1}{2} \frac{1}{2}\rangle_A |\frac{1}{2} - \frac{1}{2}\rangle_B \quad (279)$$

form the triplet states, and

$$|00\rangle = |\frac{1}{2} \frac{1}{2}\rangle_A |\frac{1}{2} - \frac{1}{2}\rangle_B - |\frac{1}{2} - \frac{1}{2}\rangle_A |\frac{1}{2} \frac{1}{2}\rangle_B \quad (280)$$

is the singlet state. For two distinguishable particles,

$$\psi(r_1, r_2) = \psi_A(r_1)\psi_B(r_2). \quad (281)$$

But, for indistinguishable particles,

$$\psi(r_1, r_2) = N[\psi_A(r_1)\psi_B(r_2) \pm \psi_B(r_1)\psi_A(r_2)]. \quad (282)$$

Here, N is a normalization constant, the $+$ corresponds to bosons, and the $-$ corresponds to fermions. Note that for fermions, if $\psi_A = \psi_B$, then the whole expression vanishes. This is the Pauli Exclusion Principle: two fermions cannot occupy the same state. Identical bosons tend to be somewhat closer together than distinguishable particles, and identical fermions tend to be farther apart. This is called an “exchange” force. For hydrogen H_2 , the bonding state is the antisymmetric spin-singlet state, hence the spatial part is symmetric, hence electrons are closer together in the middle, pulling the protons inward and creating a covalent bond.

Perturbation theory for a Hamiltonian $H = H^0 + \lambda H'$ is done in a different way depending on whether the spectrum of energies is degenerate or nondegenerate. In the nondegenerate case, we expand in the small parameter λ ,

$$(H^0 + \lambda H')[\psi_n^0 + \lambda \psi_n^1 + \lambda^2 \psi_n^2 + \dots] = (E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \dots)[\psi_n^0 + \lambda \psi_n^1 + \lambda^2 \psi_n^2 + \dots]. \quad (283)$$

This can be solved to give, to first order,

$$E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle, \quad (284)$$

$$\psi_n^1 = \sum_{m \neq n} \frac{\langle \psi_m^0 | H' | \psi_n^0 \rangle}{E_n^0 - E_m^0} \psi_m^0. \quad (285)$$

To second order,

$$E_n^2 = \sum_{m \neq n} \frac{|\langle \psi_m^0 | H' | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}. \quad (286)$$

In the degenerate case, in which each $\{\psi_i\}$ has the same energy, if we define the matrix

$$W_{ij} = \langle \psi_i^0 | H' | \psi_j^0 \rangle, \quad (287)$$

then the adjusted energies are, to first order, the eigenvalues of W .

Finally, we examine blackbody radiation. For light, $m = \pm 1$ but $m \neq 0$. The most probable occupation number for photons is

$$N_\omega = \frac{d_k}{e^{\frac{\hbar\omega}{kT}} - 1}, \quad (288)$$

where $d_k = \frac{V}{\pi^2 c^3} \omega^2 d\omega$, and V is the volume of the blackbody. The energy density in the frequency range $d\omega$ is $\rho(\omega)d\omega$, where

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3 (e^{\frac{\hbar\omega}{kT}} - 1)}. \quad (289)$$

The total energy density is

$$\frac{E}{V} = \sigma T^4. \quad (290)$$

8 Atomic Physics (10%)

The potential for the hydrogen atom:

$$V(r) = \frac{-e^2}{4\pi\epsilon_0} \frac{1}{r}. \quad (291)$$

The effective potential is defined by,

$$V_{\text{eff}} = V + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2}. \quad (292)$$

The energy spectrum for eigenfunctions of the Hamiltonian:

$$E_n = - \left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2} \equiv \frac{E_1}{n^2}. \quad (293)$$

The Bohr radius:

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} = 0.529 \times 10^{-10} m. \quad (294)$$

Here, $E_1 = -13.6$ eV. In general, for hydrogen-like atoms, we replace $e^2 \rightarrow Ze^2$, where Z is the central charge. The mass m appearing in these equations is actually the reduced mass μ for the system. Hence, if the proton in the center is replaced by a positron, then $m = 0.5m_e$. The ground state wavefunction has $l = m = 0$, $n = 1$, and

$$\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}. \quad (295)$$

The energy of the photon released when an electron drops from an initial energy state to a final energy state is the difference in energies,

$$E_\gamma = E_i - E_f = -13.6 \text{ eV} \left(\frac{1}{n_i^2} - \frac{1}{n_f^2} \right). \quad (296)$$

In terms of the wavelength of the photon, this becomes

$$\frac{1}{\lambda} = R \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right), \quad (297)$$

where R is the Rydberg constant. The Paschen series corresponds to $n_f = 3$ and is in the IR range. The Balmer series corresponds to $n_f = 2$ and is in the visible range. The Lyman series corresponds to $n_f = 1$ and is in the UV range.

From the above analysis, we would expect helium to have a ground state energy of $8E_1 = -109$ eV. Once we include for shielding of the nucleus, however, we find a true value of -79 eV. The ground state of helium is parahelium, which means a spin singlet state with both electrons in the $n = 1$ state. Orthohelium is the spin triplet state, which has an antisymmetric spatial part, so one electron is in the $n = 1$ state and one is in the $n = 2$ state. In general, the energies corresponding to the same n values are lower for the spin triplet state than for

the singlet state, because exchange forces lead the electrons to be farther apart in the triplet state.

Knowing the n, l, m values of an electron tells you the orbital of the electron. There are two electrons per orbital, one for spin up and one for spin down. There are n^2 wavefunctions for any particular n , hence $2n^2$ electrons per n . We have $n = 1, 2, 3, \dots$, $l = 0, \dots, n - 1$, and $m = -l, -l + 1, \dots, l - 1, l$. $l = 0$ is given the letter “s,” $l = 1$ is given the letter “p,” $l = 2$ is “d,” then f, g, h , (skip j) $i, k, l, \dots$ As l increases, the electrons screen the nucleus more effectively. So, the state with the lowest energy and most tightly bound electron is $l = 0$. As n increases, Z increases, so there is more pull on the electrons.

The electron configuration of carbon, which has 6 electrons, is $(1s)^2(2s)^2(2p)^2$. In general, the orbitals fill up in the order of 1s, 2s, 2p, 3s, 3p, 4s, 3d, 4p, 5s, 4d, 5p, 6s, 4f, 5d, 6p, 7s, and so on. The symbol people made up for the state of an atom is,

$$^{2S+1}L_J, \quad (298)$$

where $\vec{J} = \vec{L} + \vec{S}$ is the total angular momentum. For instance,

$$H = {}^2S_{1/2}, \quad He = {}^1S_0. \quad (299)$$

For two electrons, $L = |l_1 - l_2|, \dots, l_1 + l_2$, $S = |s_1 - s_2|, \dots, s_1 + s_2$, and $J = |L - S|, \dots, L + S$.

Hund's Rules tell electrons how to fill up the orbitals. They are,

1. The state with the highest total spin will have the lowest energy.
2. If more than one state has that spin, the ground state is one with the greatest energy.
3. If a subshell is less than half-filled, the one with lowest J is the ground state. If it is more than half-filled, the one with the highest J is the ground state.

Selection rules tell which electron transitions are allowed. For dipoles:

$$\Delta J = 0, \pm 1, \quad J \neq 0 \rightarrow 0, \quad \Delta m_J = 0, \pm 1. \quad (300)$$

For electric dipole transitions,

$$\Delta l = \pm 1. \quad (301)$$

For magnetic dipole transitions,

$$\Delta l = 0. \quad (302)$$

Furthermore, if $\Delta S = 0$, then for an electric dipole transition

$$\Delta L = 0, \pm 1, \quad L \neq 0 \rightarrow 0, \quad (303)$$

and for a magnetic dipole,

$$\Delta L = 0. \quad (304)$$

If $\Delta S = \pm 1$, then $\Delta L = 0, \pm 1, \pm 2$.

The fine structure constant is given by $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$. The fine structure splitting of the hydrogen atom, which is due to coupling of the spin of the electron to the orbit of the electron as well as relativistic effects, is handled using perturbation theory and gives an $O(\alpha^2)$ correction to the energy eigenvalues. For this correction, we must use eigenstates of total angular momentum $\hat{J}^2$ and $\hat{J}_z$ rather than eigenstates of $\hat{L}_z$ and $\hat{S}_z$. Taking the fine structure into account always lowers the energy values, with lower j corresponding to lower energy. To be explicit,

$$E_{nj} = \frac{E_1}{n^2} \left[1 + \frac{\alpha^2}{n^2} \left(\frac{n}{j + 1/2} - 3/4 \right) \right]. \quad (305)$$

Note that $j = |l - 1/2|, |l + 1/2|$, so each energy level for $l > 0$ is split in two by the fine structure. The Lamb shift gives an $O(\alpha^3)$ correction to the energies, and the hyperfine splitting gives an $O(\alpha^4)$ correction. Hyperfine splitting is due to coupling of the electron spin with the proton spin, and is positive for the spin triplet state, negative for the spin singlet state:

$$\begin{aligned} E_{hf}^1 &= \frac{g_p \hbar^4}{3m_p m_e^2 c^2 a_0^4}, & \text{triplet} \\ &= -\frac{g_p \hbar^4}{m_p m_e^2 c^2 a_0^4}, & \text{singlet.} \end{aligned} \quad (306)$$

Here, $g_p \approx 5.59$ for the proton's gyromagnetic ratio.

The Stark effect describes the physics of an atom in an electric field $\hat{H}' = eE\hat{z}$. For a hydrogen atom in the ground state, the first-order correction to the energy vanishes. The second-order correction is,

$$E_1^2 = \sum_{n \neq 1, l, m} \frac{|\langle nlm | eEz | 100 \rangle|^2}{E_1 - E_n}. \quad (307)$$

The Zeeman effect describes the physics of an atom in a magnetic field, $H'_z = -(\vec{\mu}_l + \vec{\mu}_s) \cdot \vec{B}$, where

$$\vec{\mu}_s = -\frac{e}{m} \vec{S}, \quad \vec{\mu}_l = -\frac{e}{2m} \vec{L}. \quad (308)$$

For the weak-field case, we want eigenstates of $\hat{J}^2$ and $\hat{J}_z$, and

$$E_z^1 = \frac{e}{2m} \vec{B} \cdot \langle \vec{L} + 2\vec{S} \rangle = \frac{e}{2m} \vec{B} \cdot \langle \vec{J} \rangle \left[1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)} \right]. \quad (309)$$

This splits the ground state into two levels for $m_j = \pm 1/2$. For the strong-field case, we want eigenstates of $\hat{L}_z$ and $\hat{S}_z$, and

$$E'_z = \mu_B B (m_l + 2m_s), \quad (310)$$

where $\mu_B \equiv \frac{e\hbar}{2m}$ is the Bohr magneton.

As we head left and down on the periodic table, atomic size increases, ionization energy (energy required to remove the outermost electron) decreases, and electron affinity (ease of gaining electrons) decreases.

9 Special Relativity (6%)

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = \Delta s'^2 \quad (311)$$

is a Lorentz-invariant distance. We may define $\gamma = \sqrt{\frac{1}{1-(v/c)^2}}$ for a particle moving at speed v . Time dilation:

$$\tau = \gamma t. \quad (312)$$

Here τ is the proper time, which is the time measured by a clock in the particle's rest frame. Remember that “moving clocks appear to tick slowly.”

Length contraction:

$$L' = \frac{L}{\gamma}. \quad (313)$$

Here, L is the proper length of the object, as measured in its rest frame. Remember that “moving rods appear shorter.” Note that length contraction occurs only along the direction of motion: perpendicular distances are unaltered.

Lorentz transformation for motion at velocity v :

$$t' = \gamma(t \pm \frac{vx}{c^2}), \quad (314)$$

$$x' = \gamma(x \pm vt). \quad (315)$$

Note that the $\pm$ is the same for both time and position. To figure out which sign should be used, consider the Newtonian limit for position. This tells you the sign for time.

Einstein's famous formula generalizes to

$$E = \gamma mc^2 = \gamma E_{\text{rest}} = \sqrt{p^2 c^2 + m^2 c^4}. \quad (316)$$

To lowest order in v , this gives the Newtonian limit,

$$E = mc^2 + \frac{1}{2}mv^2. \quad (317)$$

The four-vector for momentum:

$$p^\nu = (E/c, p^x, p^y, p^z). \quad (318)$$

Here, $p_i = \gamma m v_i$. The four-force vector:

$$F_\nu = \frac{dp_\nu}{d\tau}. \quad (319)$$

Simultaneity is relative to the observer. A and B are simultaneous events with respect to observer O if light rays emitted from A and B will reach O at the same time. If $\Delta x'$ is the distance between events in the simultaneous frame, then

$$\Delta t = \gamma(v/c^2)\Delta x' \quad (320)$$

gives the time difference between the events in a frame moving with velocity v in the x' direction relative to the simultaneous frame.

Two parallel velocities v and u sum according to

$$s = \frac{v \pm u}{1 \pm vu}, \quad (321)$$

in units where $c = 1$. Note that this formula maintains the speed limit of 1 in these units.

The relativistic Doppler effect for motion parallel to the propagation of light:

$$\omega_{\text{obs}} = \omega_{\text{em}} \sqrt{\frac{1 \pm v/c}{1 \mp v/c}}. \quad (322)$$

The sign can be determined in the usual way—if the observer and emitter are heading toward each other, there is a blueshift, and $\omega_{\text{obs}} > \omega_{\text{em}}$. If they are moving apart, the light is redshifted, and $\omega_{\text{obs}} < \omega_{\text{em}}$.

10 Miscellaneous (15%)

This section involves both “laboratory methods” and “specialized topics” such as nuclear and particle physics.

One important rule of thumb: the weak force/interaction is just about always the culprit. This is because the weak force doesn’t conserve lepton flavor. So whenever the question asks, “Which force is responsible for this decay process?” you should probably answer “weak force.”

Compton scattering is the process by which an incoming light ray is scattered by an electron. The change in wavelength is,

$$\lambda_f - \lambda_i = \frac{2\pi\hbar}{m_e c} (1 - \cos\theta), \quad (323)$$

where $\theta = 0$ corresponds to no change in direction of the light ray. The photoelectric effect describes the release of an electron from a metal when struck by a photon. Einstein’s equation relating the maximum kinetic energy of the released electron K_{max} to the stopping potential V_0 to the frequency ν of the photon is,

$$K_{\text{max}} = eV_0 = h\nu - \phi, \quad (324)$$

where ϕ is the work function. Some strange facts about the photoelectric effect that helped lead to a better understanding of quantum mechanics were:

1. Rate of ejection is proportional to intensity of light.
2. There is a minimum frequency below which no photoelectron emission occurs.
3. The maximum kinetic energy of the electron is independent of intensity.

During solid formation, energy levels of outer shell electrons get split up into many closely-packed energy levels and form bands. The current-carrying electrons in the conduction band are called free electrons. In insulators, the conduction band and valence band are very far apart. In conductors, the two bands overlap, and electrons move freely in a solid like an ideal gas. In n-type semiconductors, dopant atoms provide extra conduction electrons. In p-type semiconductors, dopant atoms steal extra conduction electrons. The Fermi energy for a p-type semiconductor is lower and closer to the valence energy band. The Fermi energy for an n-type semiconductor is higher and closer to the conduction energy band. p-type and n-type semiconductors can be brought together, creating a depletion region. In p-type semiconductors, dopant atoms steal extra conduction electrons. Semiconductors involving dopants are called extrinsic semiconductors, whereas intrinsic semiconductors are free from impurities.

If there is no electric field applied, free electrons move in random directions. If an external E-field is applied, the average drift velocity is,

$$v_d = \frac{-eEt}{m}, \quad (325)$$

where t is the average scattering time. The Wiedemann-Franz law relates the thermal conductivity κ , the electrical conductivity σ , the Lorenz number L , and the temperature T ,

$$\frac{\kappa}{\sigma} = LT. \quad (326)$$

The Wiedemann-Franz law fails at low T .

Superconductivity is the phenomenon of zero electrical resistance occurring in some materials below a characteristic energy. The Meissner effect is the complete expulsion of a magnetic field from a superconductor during its transition to the superconducting state ($\chi_M = -1$). There are two types of superconductors, type I involve a first-order phase transition and type II involve a second-order phase transition. All high temperature superconductors are type II, and are usually made of metal alloys or ceramics. Cooper pairs are formed when two electrons (or other fermions) bind together at low T to become bosons. A pair is bound if and only if the energy is lower than the Fermi energy. Cooper pairs are responsible for superconductivity. In conventional superconductors, the pair is due to electron-phonon interactions (interaction with ionic lattice).

In quantum chromodynamics (QCD), momentum, energy, and charge are conserved. Lepton flavor is conserved, where the six flavors of leptons are tauon, electron, muon, and one each for their corresponding neutrinos. The number of leptons and the number of baryons are conserved, and quarks have baryonic number $1/3$. Antiparticles have equal magnitude but opposite sign of flavor numbers. The weak interaction does not conserve each of the six flavor numbers, but the number in each family (muon, tauon, electron) is conserved.

When a charge accelerates, it radiates, with Poynting vector

$$\vec{S}(R, \theta) = \frac{q^2}{4\pi c^3 R^2} \sin^2 \theta |\ddot{\vec{x}}| \hat{n}, \quad (327)$$

where θ is the angle with respect to the direction of propagation. The total power is,

$$P = \frac{2}{3} \frac{q^2 |\ddot{\vec{x}}|^2}{c^3}. \quad (328)$$

The Auger effect describes the process by which an electron from a higher energy level drops into a lower energy level and kicks another electron out. Beta decay happens via the weak interaction,

$$\text{energy} + p \rightarrow n + e^+ + \nu_e, \quad (329)$$

or,

$$n \rightarrow p + e^- + \bar{\nu}_e. \quad (330)$$

Electron capture:

$$\text{energy} + p + e^- \rightarrow n + \nu_e. \quad (331)$$

Protons consists of two up quarks and one down quark, while neutrons consist of one up and two downs.

If a U-tube is filled with water on the right and mercury on the left, then the height of the mercury L , the difference in heights between the left and right arm d , the density of water ρ_w , and the density of mercury ρ_m are related by,

$$\frac{\rho_w}{\rho_m} = \frac{L}{L + d}. \quad (332)$$

For cosmic microwave background radiation, distance is proportional to temperature, which is proportional to redshift.

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